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Numerical Solution for Classical Optimal Control Problem Governing by Hyperbolic Partial Differential Equation via Galerkin Finite Element-Implicit method with Gradient Projection Method

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Abstract

This paper deals with the numerical solution of the discrete classical optimal control problem (DCOCP) governing by linear hyperbolic boundary value problem (LHBVP). The method which is used here consists of: the GFEIM " the Galerkin finite element method in space variable with the implicit finite difference method in time variable" to find the solution of the discrete state equation (DSE) and the solution of its corresponding discrete adjoint equation, where a discrete classical control (DCC) is given. The gradient projection method with either the Armijo method (GPARM) or with the optimal method (GPOSM) is used to solve the minimization problem which is obtained from the necessary condition for optimality of the DCOCP to find the DCC.An algorithm is given and a computer program is coded using the above methods to find the numerical solution of the DCOCP with step length of space variable $h = 0.1$, and step length of time variable $\Delta t = 0.05$. Illustration examples are given to explain the efficiency of these methods. The results show the methods which are used here are better than those obtained when we used the Gradient method (GM) or Frank Wolfe method (FWM) with Armijo step search method to solve the minimization problem.

Keywords: Numerical classical optimal control, hyperbolic boundary value problem, finite element method, Gradient Projection method, Armijo step search method, Optimal step method.

1. Introduction

Optimal control problems of partial differential equations PDEs have wide applications in many real live problems such as in economic, electromagnetic waves, biology, robotics, dynamical elasticity, medicine, air traffic and many others. The numerical solution of the DCOCP is studied by many researches. The GPARM or GPOSM are used to find the numerical solution of the DCOCP governing either by systems of nonlinear elliptic PDEs as in [1, 2], or by systems of semi linear parabolic PDEs as in [3, 4], or by systems of nonlinear ordinary differential equations (ODEs) as in [5, 6], or by systems of LHBVP so as our previous work [7]. Since the GFEM is one of the most an efficient and fast methods for solving different types of differential equations in general [8, 9], and DCOCP in particular. In fact, the GFEIM is used in [7]. To find the numerical solution for the state of the LHBVP and its adjoint equations while the Gradient method (GM)and Frank Wolfe method (FM) with the Armijo method is used there to solve the minimization problem which is obtained from the necessary condition for optimality of the DCOCP to find the DCC. The our previous work push us to continue in studying the numerical solution for the DCOCP governed by LHBVP via GFEIM but instead of GM or FWM with the Armijo method to find the numerical DCOC, the GPM with both the ARM (GPARM) and the optimal step method (GPOSM) is used to find the numerical DCOP to solve the minimization problem which is obtained from the necessary condition for optimality of the DCOCP to find the DCC An algorithm is given and a computer program is coded in Mat lab software to solve the DCOCP, the results are drawing by figures and show that the GPARM is better than those methods which are used in [7], to solve the minimization problem.

2. Description of the CCOC and the DCOC Problems [7]

2.1. Description of the CCOC Problem

Consider the bounded and open region $\mathrm{K} \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\mathrm{K}$, let $\mathrm{E} = (0,\mathrm{T})$, $0<\mathrm{T}<\infty$, $\rho = \mathrm{K} \times \mathrm{E}$ and $\partial\rho = \partial\mathrm{K} \times [0,T]$. The CCOC of LHBVP is given by:

$$\psi_{tt} + \mathcal{B}(t)\psi = g(\vec{x},t) + \psi + \omega - \omega_d, \text{ in } \rho, \vec{x} = (y,z) \qquad (1)$$

with the BC:

$$\psi(\vec{x},t) = 0, \text{ in } \partial\rho \qquad (2)$$

and the ICs:

$$\psi(\vec{x},0) = \psi^0(\vec{x}), \text{ in } K \qquad (3)$$

$$\psi_t(\vec{x},0) = \psi^1(\vec{x}), \text{ in K} \qquad (4)$$

Where $\psi = \psi_\omega(\vec{x},t) \in C^2(\bar{\rho})$ is the state which corresponds to the continuous classical control (CCC) $\omega = \omega(\vec{x},t) \in L^2(\rho)$, $g = g(\vec{x},t) \in L^2(\rho)$ is a given function and $\mathcal{B}(t)$ is the second-order operator $\mathcal{B}(t)\psi = -\sum_{i,j=1}^{2} \frac{\partial^2\psi}{\partial y_i \partial z_j}$.

The set of the CCCs is $\omega \in \mathrm{W}$, $\mathrm{W} \subset L^2(\rho)$.
where $\mathrm{W} = \{\, \omega \in L^2(\rho) \mid \omega(\vec{x},t) \in \mathrm{U}, \text{ a.e. in } \rho \,\}$, $\mathrm{U} \subset \mathbb{R}^r$ is a convex.

The cost functional is given by

$$G_0(\omega)=\int_\rho [\tfrac{1}{2}(\psi - \psi_d)^2 + \tfrac{1}{2}(\omega - \omega_d)^2] d\vec{x}dt \qquad (5)$$

where $\psi_d = \psi_d(\vec{x},t)$and $\omega_d = \omega_d(\vec{x},t)$ are the desired state and the desired control respectively.

The CCOC problem is to minimize the cost functional (5) subject to $\omega \in \mathrm{W}$.
In this work, the inner product and the norm in $L^2(\mathrm{K})$ it will be indicated by $(.,.)_\mathrm{K}$ and $\|\cdot\|_\mathrm{K}$ respectively, while the norm in Sobolev space $\mathrm{S} = H^1(\mathrm{K})$ by $\|\cdot\|_1$, and the norm in $L^2(\rho)$ by $\|\cdot\|_\rho$.

Now, the weak form (WF) of the problem (1- 4) is given by:

$$(\psi_{tt},\varphi)_\mathrm{K} + b(t,\psi,\varphi) = (g(t),\varphi)_\mathrm{K} + (\psi,\varphi)_\mathrm{K} + (\omega,\varphi)_\mathrm{K} - (\omega_d,\varphi)_\mathrm{K}, \forall\varphi \in \mathrm{S} \qquad (6)$$

with the (ICs)

$$\psi(0) = \psi^0, \text{ in K} \qquad (7)$$

$$\psi_t(0) = \psi^1, \text{ in K} \qquad (8)$$

Where $b(t,\psi,\varphi) = (\nabla\psi, \nabla\varphi)_{\mathrm{K}}$ is a bilinear form which is symmetric, and satisfies the following assumptions, $\forall\psi,\varphi \in \mathrm{S}, t \in \bar{E}$ and for some positive scalars a_1 and a_2.

(i) $|b(t,\psi,\varphi)| \le a_2 \parallel \psi \parallel_1 \parallel \varphi \parallel_1$
(ii) $|b(t,\varphi,\varphi)| \ge a_1 \parallel \varphi \parallel_1^2$

Suppose $\psi_t = \zeta$, and then equations (6-8) can be written by:

$(\zeta_t, \psi)_\Lambda + b(t,\psi,\varphi) = (g(t),\varphi)_{\mathrm{K}} + (\psi,\varphi)_{\mathrm{K}} + (\omega,\varphi)_{\mathrm{K}} - (\omega_d,\varphi)_{\mathrm{K}}, \forall\varphi \in \mathrm{S}$ (6a)

with the (ICs)

$\psi(0) = \psi^0$, in K (7a)

$\zeta(0) = \psi^1$, in K (8a)

2.2. Description of the DCCOCP

The discretization of the CCOC is obtained by using the GFEM. Assume that K is polyhedron domain. For every integer (s), let $\{Z_i^n\}_{i=1}^{D(s)}$ be an admissible regular triangulation of $\overline{\mathrm{K}}$, $\{E_j^s\}_{j=0}^{m(s)-1}$ be a subdivision of the interval $\overline{\mathrm{E}}$ and $\mathrm{S}_s \subset \mathrm{S} = H_0^1(\mathrm{K})$ be the space of continuous piecewise affine mapping in K.

For each $\psi \in \Psi_m$, the discrete state equations (DSEs) of (1-4) is written by (for $j = 0,1,\dots,m-1$)

$$\left(\zeta_{j+1}^s - \zeta_j^s, \varphi\right)_{\mathrm{K}} + \Delta t\, b\left(\psi_{j+1}^s, \varphi\right) = \Delta t\left(g\left(t_j^s\right), \varphi\right)_{\mathrm{K}} + \Delta t\left(\psi_{j+1}^s, \varphi\right)_{\mathrm{K}} + \Delta \mathrm{t}\left(\omega_j^s, \varphi\right)_{\mathrm{K}} - \Delta t(\omega_d(t_j^s), \varphi)_{\mathrm{K}} \quad (9)$$

$$\psi_{j+1}^s - \psi_j^s = \Delta t \zeta_{j+1}^s \quad (10)$$

$$(\psi_0^s, \varphi)_{\mathrm{K}} = (\psi^0, \varphi)_{\mathrm{K}} \quad (11)$$

$$(\zeta_0^s, \varphi)_{\mathrm{K}} = (\psi^1, \varphi)_{\mathrm{K}} \quad (12)$$

Where $\varphi_j^s = \varphi(t_j^s)$, $\zeta_j^s = \zeta(t_j^s) \in \mathrm{S}_s$ for $j = 0,1,\dots,m$.

The discrete cost functional (DCF) $G_0^s(\omega^s)$ is defined by

$$G_0^s(\omega^s) = \Delta t \sum_{j=0}^{m-1} \int_{\mathrm{K}} \frac{1}{2}[(\psi_{j+1}^s - \psi_d)^2 + (\omega_j^s - \omega_d)^2] d\vec{x} \quad (13)$$

The DCOC problem here is to find $\omega^s \in W^s$, such that

$G_0^s(\omega^s) = \min_{\overline{\omega}^s \in W^s} G_0^s(\overline{\omega}^s)$

3. The solution of the DCOCP

This section deals with some theorems and lemmas which are important in the next section they can be proved by using the same techniques which are used in [7].

Theorem 1: For any fixed j ($0 \le j \le m-1$), and $\forall\ \omega^s \in W^s$, the DSEs (9-12) has a unique solution $\psi_{\omega^s}^s = \psi^s = (\psi_0^s, \psi_1^s, \dots, \psi_m^s)$ for sufficiently small Δt.

Theorem 2: The operator $\omega^s \mapsto \psi^s = \psi_{\omega^s}^s$ is continuous.

Lemma 3: If the DCCs ω^s and $\widetilde{\omega}^s$ are bounded in $L^2(\rho)$, ψ_j^s and $\psi_{\varepsilon j}^s = \psi_j^s + \Delta_\varepsilon \psi_j^s$ (with ϵ is a small positive number) are corresponding discrete states solutions to the DCCs ω_j^s and $\omega_{\varepsilon j}^s = \omega_j^s + \varepsilon\Delta\, \omega_j^s$ respectively ($\forall\, j = 1,2,\dots,m$) then:
$\| \Delta_\varepsilon \psi_{\hbar}^s \|_1^2 \le \kappa\varepsilon^2 \| \Delta\, \omega^s \|_\rho^2$ and $\| \Delta_\varepsilon \zeta_{\hbar}^s \|_{\mathrm{K}}^2 \le \kappa\varepsilon^2 \| \Delta\, \omega^s \|_\rho^2$
Or $\| \Delta_\varepsilon \psi_{\hbar}^s \|_1^2 \le \kappa$, and $\| \Delta_\varepsilon \zeta_{\hbar}^s \|_{\mathrm{K}}^2 \le \kappa$

Theorem 4 (Existence of DCOCP): Consider the DCF in equation (13). Assume that U^s is convex and closed. If $G_0^s(\omega^s)$ is coercive, then there exists a discrete classical optimal control.
Theorem 5 (The Necessary conditions for DCOCP): The discrete classical adjoint state $\eta_{\omega^s}^s = \eta^s = (\eta_0^s, \eta_1^s, \dots, \eta_{m-1}^s)$ is given by (for $j = m-1, m-2, \dots, 0$)

$$\left(\phi_{j+1}^s - \phi_j^s, \varphi\right) + \Delta t\, b\left(\eta_j^s, \varphi\right) = \Delta t\left(\eta_j^s, \varphi\right) + \Delta \mathrm{t}\left(\psi_{j+1}^s - \psi_d, \varphi\right) \quad (14)$$

$$\eta_{j+1}^s - \eta_j^s = \Delta \mathrm{t}\, \phi_j^s \quad (15)$$

$$\eta_m^s = \phi_m^s = 0 \quad (16)$$

where $\eta_j^s, \phi_j^s \in \mathrm{S}_s$,($\forall\, j = 0,1, \dots m$).

The discrete directional derivative of G in equation (13) is given by:

$$\begin{aligned}\mathrm{D}G_0^s(\omega^s, \omega'^s - \omega^s) &= \Delta t \sum_{j=0}^{m-1} \left(H_\omega^s\left(t_j^s, \psi_{j+1}^s, \eta_j^s, \omega_j^s\right), \Delta\omega_j^s\right)_{\mathrm{K}} \\ &= \Delta t \textstyle\sum_{j=0}^{m-1} \left(\eta_j^s + \omega_j^s - \omega_d, \Delta\omega_j^s\right)_{\mathrm{K}} \ge 0, \quad \forall \omega^s \in W^s \end{aligned} \quad (17)$$

Where $\omega'^s, \omega^s \in W^s$, $\Delta\omega_j^s = \omega_j'^s - \omega_j^s$ for ($j = 0,1,\dots,m$), and H_ω^s is called the discrete Hamiltonian functional.
Remark: To prove equation (17) ($\forall \omega^s \in W^s$) is equivalent to the minimum principle block wise.

$$\left(\eta_j^s + \omega_j^s - \omega_d, \omega_j^s\right)_{T_i} = \min_{\omega'^s \in W^s} \left(\eta_j^s + \omega_j^s - \omega_d, \omega_j'^s\right)_{T_i}, \forall j = 0,1, \dots m-1 \quad (18)$$

Let $M^m = \{\omega' = \omega_j'^s : \omega_j'^s \in U^m, j = 0,1,\dots,m-1\}$ and $\omega_j'^s = \omega_j^s$, for all j except once(say v) i.e. $\omega_v'^s \ne \omega_v^s$, then $(\eta_v^s + \omega_v^s - \omega_d, \omega_v^s) = \min_{\omega'^s \in W^s} (\eta_v^s + \omega_v^s - \omega_d, \omega_v'^s)$, since v is arbitrary then we have equation (18), and converse is very clear.

4. The GPM Method [10].

The GPM is an iterative method used to find the point that minimizes the problem. The following algorithms describe the GP, GPARM and GPOSM; we will use the norm $\|\cdot\|$ with respect to vector space Q.

4.1. The basic algorithm GPM [10]

Let Q be a Hilbert space, U is a convex subset of an open set $\mathrm{K} \subset \mathbb{R}^2$, $G\colon \mathrm{K} \subset Q \longrightarrow \mathbb{R}$, and let $\sigma \in (0,1)$, $\{\gamma_n\}$ be a sequence with $\gamma_n \in (0,1]$, for each n, and let $\omega_0 \in U$ be an initial classical control,
(1) Find a direction point $\varpi_n \in U$ and set $d_n =: \varpi_n - \omega_n$
(2) Set $\gamma_n := \max\beta^l$ subject to $l \in \{0, 1, 2, \dots\}$
$G(\omega_n + \beta^l d_n) - G(\omega_n) \le \sigma\beta^l \nabla G(\omega_n)^T d_n$.
(3) Set $\omega_{n+1} := \omega_n + \gamma_n d_n$

4.2. Algorithm [10]

Let Q be a Hilbert space, U is a convex subset of an open set $\mathrm{K} \subset \mathbb{R}^2$, $G: \mathrm{K} \subset Q \longrightarrow \mathbb{R}$, and let $\sigma, \lambda \in (0,1)$, $\{\gamma_n\}$ be a sequence with $\gamma_n \in (0,1]$, for each n. $\mu > 0$, and let $\omega_0 \in U$ be an initial control.

Step1: Set $n := 0$, solve the system of the WF (9-12) (the system of the adjoint WF (14-16)) by GFEM to get ψ_n (η_n), and then Calculate $G'(\omega_n)$ from equation (17) and $G(\omega_n)$ from equation (13).

Step 2: Find a direction point $\varpi_n \in U$, (i.e. a direction $\varpi_n - \omega_n$) by applying the GPM as follows: Find the unique $\varpi_n \in U$, such that

$$\xi_n = (G'(\omega_n), \omega - \omega_n) + \frac{\mu}{2} \| \varpi_n - \omega_n \|^2$$
$$= \min_{\varpi \in U} (G'(\omega_n), \varpi - \omega_n) + \frac{\mu}{2} \| \varpi - \omega_n \|^2$$

Step 3: Solve the system of the WF (9-12) to find the new state ψ_n corresponding to the new control ϖ_n.

Step 4: Choose γ_n using one of the following methods:

ARM: Assume an initial value $\beta = \gamma_n \in [0,1]$. Find the $G\big(\omega_n + \beta(\varpi_n - \omega_n)\big)$ from equation (13). If β satisfies the inequality

$$\mathrm{X}_n(\alpha) = G\big(\omega_n + \beta(\varpi_n - \omega_n)\big) - G(\omega_n) \leq \beta\sigma\xi_n$$

we set $\beta \coloneqq \beta/\lambda$, and choose for β_n, the last $\beta \in (0, \infty)$ that satisfies the above inequality. If unsatisfied, set $\beta \coloneqq \beta\lambda$, and choose for β_n the first $\beta \in (0, \infty)$ that satisfies this inequality.

OPSM: Find an $\beta_n \in [0,1]$, such that

$$\big(G'(\omega_n), \varpi_n - \omega_n\big) = \min_{\beta \in [0,1].} (G'(\omega_n), \varpi - \omega_n)$$

Step 5: Set $\omega_{n+1} = \omega_n + \beta(\varpi_n - \omega_n)$, $n := n + 1$ and we go to step 2.

Practically the ARM is faster and is a finite procedure than OPSM. The following examples are solved by using the above algorithm; a computer program in mat lab software version 8.1.0.604 is coded. The solution ψ_n and η_n in step (1) are found using the GFEM with $D = 9, m = 20$, ($\Delta t = 0.05$), the parameters in ARM are taken the value $\sigma = \lambda = 0.5$, and the parameter $\mu = 0.5$ in the GPM.

5. Numerical Examples

Example 1: Consider the following classical optimal control problem (COCP) associated with the linear hyperbolic equation:

$\psi_{tt} - \Delta\psi = g(\vec{x}, t) + \psi + \omega - \omega_d$, in $\rho = \mathrm{K} \times \mathrm{E}$, $\vec{x} = (y, z)$

$\psi(\vec{x}, t) = 0$, in $\partial\rho = \partial\mathrm{K} \times [0, T]$.

and the (ICs)

$\psi(\vec{x}, 0) = 0.5\, yz(1 - y)(1 - z)$, in K

$\psi_t(\vec{x}, 0) = -0.5(yz(y - 1)(z - 1))$, in K

Where $E = [0,1]$, $\mathrm{K} = [0,1] \times [0,1]$, and

$g(\vec{x}, t) = -e^{-t}[y^2 - y + z^2 - z]$

The control constraint is $U = [-0.5,1]$ and the cost functional is given by:

$G_0(\omega) = \int_\rho [\frac{1}{2}(\psi - \psi_d)^2 + \frac{1}{2}(\omega - \omega_d)^2] d\vec{x}dt$,

Where $\psi_d = \psi_d(\vec{x}, t)$ and $\omega_d = \omega_d(\vec{x}, t)$ are the desired state and control and are given by

$\psi_d(\vec{x}, t) = 0.5\, yz(1 - y)(1 - z)e^{-t}$, $\forall(\vec{x}, t) \in \rho$, and

$$\omega_d(\vec{x},t) = \begin{cases} 0 & ,\text{for } 0 \le t \le 0.5 \\ 0.4 & ,\text{for } 0.5 < t \le 1 \end{cases}$$

with the initial control,

$\omega_0(\vec{x},t) = -0.4 + t \ , \forall(\vec{x},t) \in \rho$

Algorithm (5.1) is used here to solve this problem. The following figures are represented the initial control and its corresponding state.

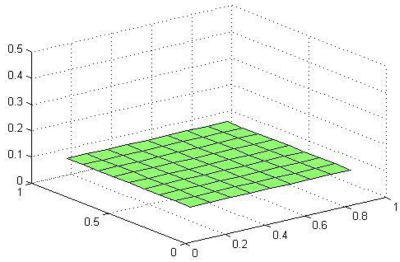

Figure 1. Initial control at t=0.5.

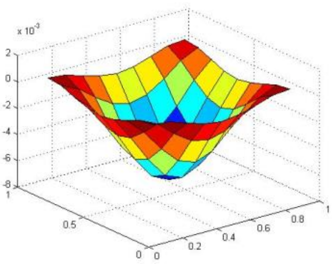

Figure 2. Corresponding state (of initial control).at t= 0.5.

Depending on the above initial control and its corresponding state, we get:

(I) the GPARM, after 6 iterations is used to get the optimal control and its corresponding state, the results show with

$G_0(\omega^s)$=5.9085e-08, $\mathcal{E}_s$ =6.6150e-04, and δ_s=7.6800e-05

where $\mathcal{E}_s$ and δ_s are the discrete maximum error of the state and the control respectively.

The following figures represent the optimal control and its corresponding state which are obtained by using GPARM.

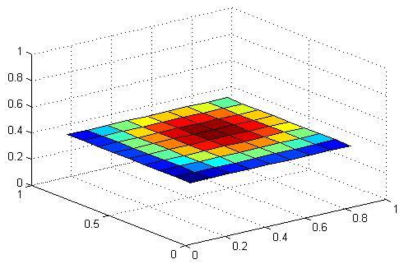

Figure 3. Optimal control at t=0.5.

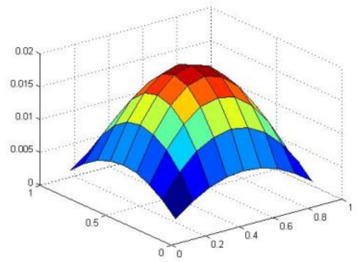

Figure 4. Corresponding state (of optimal control) at t= 0.5.

(II) The GPOSM, after 2 iterations is used to obtain the optimal control and its corresponding state, with:

$G_0(\omega^s)$=5.8424e-08, $\mathcal{E}_s$ =6.6150e-04, and δ_s=7.1644e-05

The following figures are posted by using GPOSM and are represented the optimal control and its corresponding state.

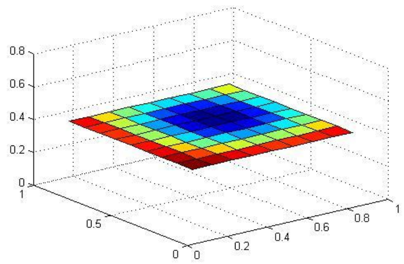

Figure 5. Optimal control at t=0.5.

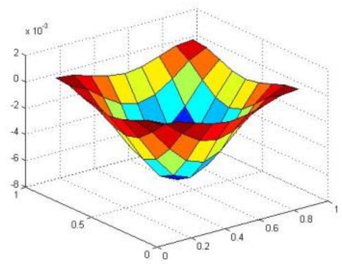

Figure 6. Corresponding state (of initial control) at t=0.5.

Example 2: Consider the COCP, which was considered in example 1, but the control constraint is $U = [-1,2]$, and the desired control is given by

$\omega_d(\vec{x}, t) = -0.5 + 2t$

with the initial control,

$$\omega_0(\vec{x}, t) = \begin{cases} -0.3, & 0 \le t \le 0.3 \\ 0.5, & 0.3 < t \le 0.6 \\ 1.1, & 0.6 < t \le 1 \end{cases}$$

The following figures represent the initial control and its corresponding state.

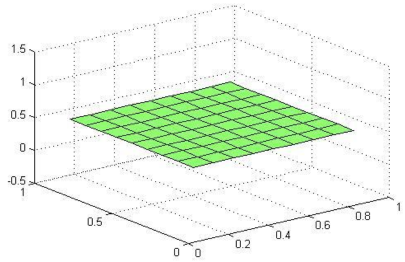

Figure 7. Initial control at t=0.5.

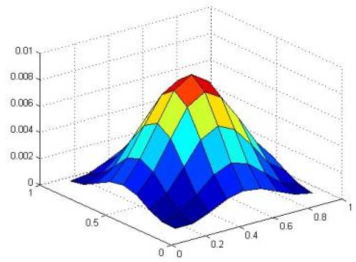

Figure 8. Corresponding state (of initial control).at t=0.5.

(I) In the GPARM, after 9 iterations is used to get the optimal control and corresponding state, the results show with:

$G_0(\omega^s)$=5.9048e-08, $\mathcal{E}_s$ =6.6150e-04, and δ_s=4.0813e-05

The following figures are represented the optimal control and its corresponding state.

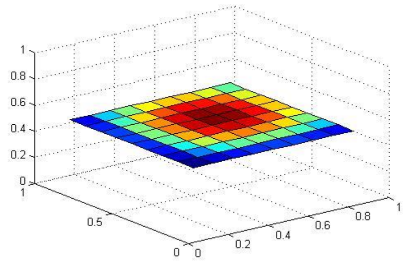

Figure 9. Optimal control at t=0.5.

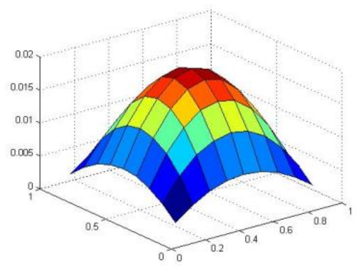

Figure 10. Corresponding state (of optimal control) at t=0.5.

(II) The GPOSM, after 2 iterations is used to get the optimal control and corresponding state, with

$G_0(\omega^s)$=5.8570e-08, $\mathcal{E}_s$ =6.6150e-04, and δ_s=7.4298e-05

The following figures are represented the optimal control and its corresponding state.

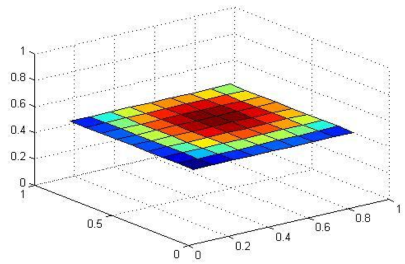

Figure 11. Optimal control at t=0.5.

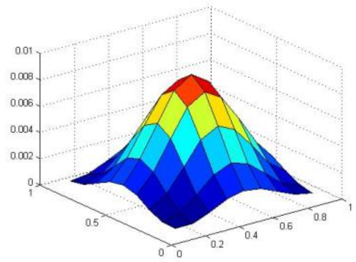

Figure 12. Corresponding state (of optimal control) at t=0.5.

6. Conclusion

The following conclusions can be derived from the obtained results:

(1) the GFEIM which is used here to find the DSE of LHBVP as well as the discrete adjoint equation for the state equation is fast and efficient than other method e.g., the finite difference methods, methods of lines, variational methods which clearly are needed more time and hence have more computations which are Grow up the error.

(2) By the results of the above examples, with (step length of space variable $h = 0.1$, and step length of time $\Delta t = 0.05$) we conclude that:

(I) the GPARM and the GPOSM which are used to solve the minimization problem (the cost functional), are suitable and efficient methods to find the DCOC governed by hyperbolic boundary value problem, with parameters $\mu = 0.5$, $\sigma = 0.5$ and $\lambda = 0.5$.

(II) Although the GPOSM is needed less iterations than GPARM, but the GPARM remain better since it is used for general minimization problems, on the contrary of the GPOSM which is used only for quadratic functional.

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