Ibn Al-Haitham Jour. for Pure & Appl. Sci. 32 (2) 2019

Ibn Al Haitham Journal for Pure and Applied Science

Journal homepage: http://jih.uobaghdad.edu.iq/index.php/j/index

Convergence Comparison of two Schemes for Common Fixed Points with an Application

Salwa Salman Abed **Zahra Mahmood Mohamed Hasan**

Department of Mathematics, College of Education for Pure Sciences Ibn Al-Haitham, University of Baghdad, Baghdad,Iraq.

salwaalbundi@yahoo.com zahramoh1990@gmail.com

Article history: Received 13 Novrmber 2018, Accepted 11 December 2018, Publish May 2019

Doi: 10.30526/32.2.2146

Abstract

Some cases of common fixed point theory for classes of generalized nonexpansive maps are studied. Also, we show that the Picard-Mann scheme can be employed to approximate the unique solution of a mixed-type Volterra-Fredholm functional nonlinear integral equation.

Keywords: Banach space, common fixed point, strong convergence, condition (C_λ).

1. Introduction

Let B be a non-empty subset of a Banach space M. A map T on B is called quasi-nonexpansive [1]. if $F(T) \neq \emptyset$ *and* $\|Ta - b\| \leq \|a - b\|$ for all $a \in B$ *and all* $b \in F(T)$, where $F(T)$ denoted the set of all fixed points of T.

In 2008, Suzuki [2]. introduced a condition on T which is stronger than quasi-nonexpansive and weaker than nonexpansive, called condition (C) and presented some results about fa fixed pointfor such maps.

In 2009, Dhompongsa et al [3]. extended Suzaicr's theorems to the general class of maps in Banach spaces. Garcial-Falset et al [4]. defined two generalization of condition (C), called condition (E_λ) and condition (C_λ) And studied their asymptotic behavior as well as the existence of fixed points. On the other hand, Bruck [5]. introduced a map called firmly nonexpansive map in Banach space. Of course, every firmly nonexpansive is nonexpansive.

To discuss about convergence theorem for two nonexpansive maps S and T on B to itself, Khan and Kim [6]. constricted the following iterative scheme to find a common fixed point of S and T:

$x \in B$

$x_{n+1} = (1 - \alpha_n)Ty_n + \alpha_n Sy_n$

$y_n = (1 - \beta_n)x_n + \beta_n Tx_n \qquad , n \in N$

Where (α_n) *and* $(\beta_n) \in (0,1)$.

This scheme is independent of both Ishikawa scheme and Yao-Chen scheme [6].

In this paper, we prove some convergence theorems for approximating common fixed points of firmly nonexpansive and maps satisfied condition (C_λ).

2. Preliminaries

We will assume throughout this paper that $(M, \|.\|)$ is a uniformly convex Banach space and B is a non-empty closed convex subset of M. For maps $S, T: B \to B$ the set of all fixed points of S and T will be denoted by $F(T, S)$.

A sequences (a_n) in B is called:

Picard-Mann hybrid [7].

$$a_{n+1} = Sb_n$$
$$b_n = (1-\alpha_n)a_n + \alpha_n Ta_n \quad ,\forall\, n \in N \qquad (1)$$

Where $(\alpha_n) \in (0,1)$.

Noor iterative scheme [8]. if

$$z_{n+1} = (1-\alpha_n)z_n + \alpha_n Su_n$$
$$u_n = (1-\beta_n)z_n + \beta_n Tv_n$$
$$v_n = (1-\gamma_n)z_n + \gamma_n Tz_n \quad ,\forall\, n \in N \qquad (2)$$

Where $(\alpha_n), (\beta_n)\ and\ (\gamma_n) \in [0,1]$.

Definition (1) [9]. A map $T:: \to B$ said to be Lipschitz continuous or liLipschitzf $\exists\, K > 0$such that $\|Ta - Tb\| \le K\|a - b\|, \forall\, a, b \in B$.

If $K = 1$, then T is nonexpansive.

Definition (2) [10]. A map $T: B \to B$ is said to satisfying:

1-Condition (C) if $\frac{1}{2}\|a - Ta\| \le \|a - b\| \xrightarrow{yields} \|Ta - Tb\| \le \|a - b\|, \forall\, a, b \in B$.

2-Condition (C_λ) if $\lambda\|a - Ta\| \le \|a - b\| \xrightarrow{yields} \|Ta - Tb\| \le \|a - b\|, \forall a, b \epsilon B\ and\ \lambda \in (0,1)$.

Defintion (3)[5]. A map $T: B \to M$ is said to be firmly nonexpansive map if $\|Ta - Tb\| \le \|(1-t)(Ta - Tb) + t(a - b)\|, \forall\, a, b \in B\ and\ t \ge 0$.

Definition (4)[11]. Two maps are called:

1-Condition (A) if there is a nondecreasing function $g: [0, \infty) \to [0, \infty)\ with\ g(0) = 0, g(i) > 0, \forall\, i \in (0, \infty)$ such that :

Either $\|a - Ta\| \ge g\big(D(a, F)\big)\ or\ \|a - Sa\| \ge g\big(D(a, F)\big), \forall\, a \in B$,

$where\ D(a, F) = inf\{\|a - a^*\|;\ a^* \in F\}\ and\ F = F(T) \cap F(S)$.

2-Condition (I) if $\|a - Tb\| \le \|Sa - Tb\|, \forall\, a, b \in B$.

Definition (5)[12]. A map $T: B \to B$ is called

1-Demiclosed at 0 if $\forall$ sequence $(a_n)\ in\ B$ such that (a_n) converges weakly to (a) and (Ta_n) converges strongly to 0, then $Ta = 0$.

2-Affine if B is convex and

$$T(Ka + (1-K)b) = KT(a) + (1-K)Tb, \forall\, a, b\, \epsilon\, B\ and\ K \in [0,1].$$

Definition (6)[7]. Let (f_n) and (g_n) be two sequences of real numbers that converging to f and g

$$\lim_{n\to\infty} \frac{\|f_n - f\|}{\|g_n - g\|} = 0.$$

Then (f_n) converges faster than (g_n).

Lemma (7)[13]. Let $(\mu_n)^{\infty}{}_{n=0}$ & $(\omega_n)^{\infty}{}_{n=0}$ be nonnegative real sequences satisfying the inequality: $\mu_{n+1} \le (1-\delta_n)\mu_n + \omega_n$

Where $\delta_n \in (0,1), \forall\, n \ge n_0, \sum_{n=1}^{\infty} \delta_n = \infty$ and $\frac{\omega_n}{\delta_n} \to 0$ as $n \to \infty$, $then$ $\lim_{n\to\infty} \mu_n = 0$.

Lemma (8)[10]. Let M be a uniformly convex Banach space and $0 < l \le t_n \le k < 1, \forall\, n \in N$. Suppose that $(a_n) and (b_n)$ are two sequences of M such that $\lim_{n\to\infty}\|a_n\| \le m, \lim_{n\to\infty}\|b_n\| \le m$ and $\lim_{n\to\infty}\|t_n a_n + (1-t_n)b_n\| = m$ hold for some $m \ge 0$. Then $\lim_{n\to\infty}\|a_n - b_n\| = 0$.

3. Two Lemmas

Lemma (9): Let B be a non-empty closed convex subset of a normed space M, $T:: \to B$ be a firmly noninexpensivep and satisfying Lipschitz $S: B \to B$ be satisfying condition (C_λ).Let

1-(a_n) be as in (1) where $(\alpha_n) \in (0,1), n \in N$.

2-(z_n) be as in (2) where $(\alpha_n), (\beta_n)$ and $(\gamma_n) \in [0,1]$.

If $F(S,T) \neq \emptyset$, then $\lim_{n\to\infty}\|a_n - a^*\|$ and $\lim_{n\to\infty}\|z_n - a^*\|$ exist $\forall\, a^* \in F(S,T)$.

Proof: Let $a^* \in F(T,S)$.

By using condition(C_λ), we have

$$\lambda\|a^* - Sa^*\| = 0 \le \|b_n - a^*\| \xrightarrow{yields} \|Sb_n - a^*\| \le \|b_n - a^*\|.$$

Then

1-$\|a_{n+1} - a^*\| = \|Sb_n - a^*\|$

$$\begin{aligned}
&\le \|b_n - a^*\| \\
&\le (1-\alpha_n)\|a_n - a^*\| + \alpha_n\|Ta_n - a^*\| \\
&\le (1-\alpha_n)\|a_n - a^*\| + \alpha_n(1-t)\|Ta_n - a^*\| + \alpha_n t\|a_n - a^*\| \\
&\le (1-\alpha_n)\|a_n - a^*\| + \alpha_n K(1-t)\|a_n - a^*\| + \alpha_n t\|a_n - a^*\| \\
&\le (1-\alpha_n)\|a_n - a^*\| + \alpha_n[(1-t)K + t]\|a_n - a^*\| \\
&\le (1-\alpha_n)\|a_n - a^*\| + \alpha_n\|a_n - a^*\| \\
&\le \|a_n - a^*\|
\end{aligned}$$

Then $\lim_{n\to\infty}\|a_n - a^*\|$ exists $\forall a^* \in F(T,S)$.

2-$\|v_n - a^*\| = \|(1-\gamma_n)z_n + \gamma_n Tz_n - a^*\|$

$$\begin{aligned}
&\le (1-\gamma_n)\|z_n - a^*\| + \gamma_n\|Tz_n - a^*\| \\
&\le (1-\gamma_n)\|z_n - a^*\| + \gamma_n(1-t)\|Tz_n - a^*\| + \gamma_n t\|z_n - a^*\| \\
&\le (1-\gamma_n)\|z_n - a^*\| + \gamma_n K(1-t)\|z_n - a^*\| + \gamma_n t\|z_n - a^*\| \\
&\le (1-\gamma_n)\|z_n - a^*\| + \gamma_n[(1-t)K + t]\|z_n - a^*\| \\
&\le \|z_n - a^*\|
\end{aligned}$$

$$\begin{aligned}
\|u_n - a^*\| &\le (1-\beta_n)\|z_n - a^*\| + \beta_n\|Tv_n - a^*\| \\
&\le (1-\beta_n)\|z_n - a^*\| + \beta_n[(1-t)K + t]\|v_n - a^*\| \\
&\le \|z_n - a^*\|
\end{aligned}$$

Now

$$\begin{aligned}
\|z_{n+1} - a^*\| &\le (1-\alpha_n)\|z_n - a^*\| + \alpha_n\|Su_n - a^*\| \\
&\le (1-\alpha_n)\|z_n - a^*\| + \alpha_n\|u_n - a^*\|
\end{aligned}$$

$\leq \|z_n - a^*\|$

Then $\lim_{n\to\infty}\|z_n - a^*\|$ exists $\forall a^* \in F(T,S)$.

Lemma (10): Let M be a uniformly convex Banach space and B be a nonempty closed convex subset of M. Let:

1-$T: B \to B$ be firmly nonexpansive map and satisfying Lipschitz, $S: B \to B$ be affine and satisfying condition$(C_\lambda) and\ (a_n)$ be as in (1) .

2-T:B→B be firmly nonexpansive map and satisfying Lipschitz, S:B→B be satisfying condition$(C_\lambda) and\ (z_n)$ be as in (2). Suppose that condition (I) holds. If $F(S,T) \neq \emptyset$, then $\lim_{n\to\infty}\|Ta_n - a^*\| = 0 = \lim_{n\to\infty}\|Sa_n - a^*\|$ & $\lim_{n\to\infty}\|Tz_n - a^*\| = 0 = \lim_{n\to\infty}\|Sz_n - a^*\|$.

Proof: Let $a^* \in F(T,S)$.

1- As proved by lemma (9), $\lim_{n\to\infty}\|a_n - a^*\|$ exists. Suppose that $\lim_{n\to\infty}\|a_n - a^*\| = c$, $\forall\ c \geq 0$.

If $c = 0$, there is nothing to prove.

Now, suppose $c > 0$,

Since, $\|a_{n+1} - a^*\| = \|Sb_n - a^*\|$

$\leq \|b_n - a^*\|$

$\|b_n - a^*\| \leq (1-\alpha_n)\|a_n - a^*\| + \alpha_n\|Ta_n - a^*\|$

$\leq \|a_n - a^*\|$

Then $\lim_{n\to\infty}\|b_n - a^*\| = c$.

Next consider

$c = \|b_n - a^*\| \leq (1-\alpha_n)\|a_n - a^*\| + \alpha_n\|Ta_n - a^*\|$

By applying lemma (9),we obtain

$$\lim_{n\to\infty}\|Ta_n - a_n\| = 0.$$

Now

$c = \lim_{n\to\infty}\|a_{n+1} - a^*\| = \lim_{n\to\infty}\|Sb_n - a^*\|$

$\|Sb_n - a^*\| = \|S[(1-\alpha_n)a_n + \alpha_n Ta_n - a^*\|$

$\leq (1-\alpha_n)\|Sa_n - a^*\| + \alpha_n\|STa_n - a^*\|$

By applying Lemma (8) , we have

$$\lim_{n\to\infty}\|Sa_n - STa_n\| = 0.$$

Next, by using condition (I), we obtain

$$\|Sa_n - a^*\| \leq \|Sa_n - STa_n\| + \|STa_n - a^*\|$$
$$\leq 2\|Sa_n - STa_n\| \to 0\ as\ n \to \infty$$

Thus $\lim_{n\to\infty}\|Sa_n - a_n\| = 0$

2- As proved by lemma (9), $\lim_{n\to\infty}\|z_n - a^*\|$ exists. Suppose that $\lim_{n\to\infty}\|z_n - a^*\| = c, \forall\ c \geq 0$.

If $c = 0$, there is nothing to prove.

Now, suppose $c > 0$,

Since $\|Tz_n - a^*\| \leq \|z_n - a^*\|$, and as proved by lemma (3.1)

$$\|Su_n - a^*\| \leq \|u_n - a^*\|\ and\ \|Tv_n - a^*\| \leq \|v_n - a^*\|.$$

Then,

$$\lim_{n\to\infty}\|Tz_n - a^*\| \leq c, \lim_{n\to\infty}\|Su_n - a^*\| \leq c\ and\ \lim_{n\to\infty}\|Tv_n - a^*\| \leq c.$$

Moreover

$\lim_{n\to\infty}\|z_{n+1} - a^*\| = c$

$c = \|z_{n+1} - a^*\| \leq (1-\alpha_n)\|z_n - a^*\| + \alpha_n\|Su_n - a^*\|$

By applying lemma (9), we get
$\lim_{n\to\infty} \|z_n - Su_n\| = 0$

Now
$\|u_n - z_n\| \le (1-\beta_n)\|z_n - z_n\| + \beta_n\|Tv_n - z_n\| \ = 0$
Then, $\lim_{n\to\infty}\|u_n - z_n\| = 0$.

Since, $\lim_{n\to\infty}\|u_n - a^*\| \le c \ and \ \|z_n - a^*\| \le \|z_n - Su_n\| + \|Su_n - a^*\|$,
which implies to
$c \le \lim_{n\to\infty} inf\|u_n - a^*\|$

That gives $\lim_{n\to\infty}\|u_n - a^*\| = c$, so
$c = \|u_n - a^*\| \le (1-\beta_n)\|z_n - a^*\| + \beta_n\|Tv_n - a^*\|$
$\quad \le (1-\alpha_n\beta_n)\|z_n - a^*\| + \alpha_n\beta_n\|Tz_n - a^*\|$

By lemma (9), we obtain:
$\lim_{n\to\infty}\|z_n - Tz_n\| = 0.$

Next
$\|z_n - Sz_n\| \le \|z_n - Su_n\| + \|Su_n - z_n\| + \|z_n - Sz_n\|$
Letting n→∞, we have:
$\|z_n - Sz_n\| \le \|z_n - Sz_n\|$
That means $\lim_{n\to\infty}\|z_n - Sz_n\| = 0$.

4. Convergence and Equivalence Results

Theorem (11): Let M be a uniformly convex Banach space. Let $B, S, T, (a_n)$ $and\ (z_n)$ be as in lemma (10) and T, S satisfying condition (A). If $F(T,S) \ne \emptyset$, then $(a_n)\ and\ (z_n)$ converge strongly to a common fixed point of T and S.

Proof: Now, we will show that (a_n) is strong convergence. By lemma (10), $\lim_{n\to\infty}\|a_n - a^*\|$ exists. Suppose that $\lim_{n\to\infty}\|a_n - a^*\| = c, c \ge 0$.
From lemma (9), we have $\|a_{n+1} - a^*\| \le \|a_n - a^*\|$
That gives
$inf_{a^*\in F}\|a_{n+1} - a^*\| \le inf_{a^*\in F}\|a_n - a^*\|$
Which means, $d(a_{n+1}, F) \le d(a_n, F) \xrightarrow{yields} \lim_{n\to\infty} d(a_n, F)$ exists.
By using condition (A), we have
$\lim_{n\to\infty} g(d(a_n, F) \le \lim_{n\to\infty}\|a_n - Ta_n\| = 0.$
Or
$\lim_{n\to\infty} g(d(a_n, F) \le \lim_{n\to\infty}\|a_n - Sa_n\| = 0.$
In both situations, we obtain

$$\lim_{n\to\infty} g(d(a_n, F) = 0$$

Since g is a non-decreasing function and $g(0) = 0$. It follows that $\lim_{n\to\infty} d(a_n, F) = 0$.
Now to show that (a_n) Is a Cauchy sequence in B. Let $\epsilon > 0, \lim_{n\to\infty} d(a_n, F) = 0, \exists$ a positive integer n_0, such that:
$d(a_n, F) < \frac{\epsilon}{4}, \qquad \forall n \ge n_0$

In particular.

$\inf\{\|a_n - a^*\|, a^* \in F\} < \dfrac{\epsilon}{2}$

Thus,it must exist $a^{**} \in F(T,S)$ such that $\|a_n - a^{**}\| < \frac{\epsilon}{2}$.

Now, $\forall\, n, w \geq n_0$, we obtain:

$\|a_{n+w} - a_n\| \leq \|a_{n+w} - a^{**}\| + \|a_n - a^{**}\| < \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} = \epsilon$

Hence, (a_n) Is Cauchy sequence in the B of M. Then (a_n) converges to a point $p \in B$.

$\lim_{n\to\infty} d(a_n, F) = 0 \xrightarrow{yields} d(p, F) = 0.$

Since F is closed, hence $p \in F(T,S)$.

By utilizing the same procedure, we can prove (z_n) convergence strongly.

Theorem (12): Let $T: B \to B$ be a firmly nonexpansive and satisfying lipschitz, $S: B \to B$ satisfying condition (C_λ), with $F(S,T) \neq \emptyset$ and,

1-(a_n) be as in (1) and $(\alpha_n) \in (0,1)$ satisfying $\sum_{k=0}^{\infty} \alpha_k = \infty$.

2-(z_n) be as in (2) and $(\alpha_n), (\beta_n)\ and\ (\gamma_n) \in [0,1]$ satisfying $\sum_{k=0}^{\infty} \alpha_k = \infty$.

Then (a_n) & (z_n) converge to a unique common fixed point $a^* \in F(S,T)$.

Proof:

1-$\|b_n - a^*\| \leq (1-\alpha_n)\|a_n - a^*\| + \alpha_n\|Ta_n - a^*\|$

$\leq (1-\alpha_n)\|a_n - a^*\| + \alpha_n[(1-t)K + t]\|a_n - a^*\|$

Suppose $\xi = (1-t)K + t$

$\leq (1-(1-\xi)\alpha_n)\|a_n - a^*\|$

$\|a_{n+1} - a^*\| = \|Sb_n - a^*\|$

$\leq \|b_n - a^*\|$

$\leq (1-(1-\xi)\alpha_n)\|a_n - a^*\|$

By induction

$\|a_{n+1} - a^*\| \leq \prod_{i=0}^{n} \leq (1-(1-\xi)\alpha_i)\|a_0 - a^*\|$

$\leq \|a_0 - a^*\| e^{-(1-\xi)\sum_{i=0}^{\infty}\alpha_i}$

Since $\sum_{i=0}^{\infty} \alpha_i = \infty, e^{-(1-\xi)\sum_{i=0}^{\infty}\alpha_i} \to 0\ as\ n \to \infty$.

Thus $\lim_{n\to\infty}\|a_n - a^*\| = 0$.

2-$\|v_n - a^*\| \leq (1-\gamma_n)\|z_n - a^*\| + \gamma_n\|Tz_n - a^*\|$

$\leq (1-\gamma_n)\|z_n - a^*\| + \gamma_n[(1-t)K + t]\|z_n - a^*\|$

Setting $\xi = (1-t)K + t$

$\leq (1-\gamma_n + \gamma_n\xi)\|z_n - a^*\|$

$\|u_n - a^*\| \leq (1-\beta_n)\|z_n - a^*\| + \beta_n\|Tv_n - a^*\|$

$\leq (1-\beta_n)\|z_n - a^*\| + \beta_n\xi\|v_n - a^*\|$

$\leq (1-\beta_n)\|z_n - a^*\| + \beta_n\xi(1-\gamma_n + \gamma_n\xi)\|z_n - a^*\|$

$\leq (1-\beta_n)\|z_n - a^*\| + \beta_n(1-\gamma_n + \gamma_n\xi)\|z_n - a^*\|$

Now

$\|z_{n+1} - a^*\| \leq (1-\alpha_n)\|z_n - a^*\| + \alpha_n\|Su_n - a^*\|$

$\leq (1-\alpha_n)\|z_n - a^*\| + \alpha_n\|u_n - a^*\|$

$\leq (1-\alpha_n)\|z_n - a^*\| + \alpha_n(1-\beta_n)\|z_n - a^*\|$

$+\alpha_n\beta_n(1-\gamma_n+\gamma_n\xi)\|z_n-a^*\|$
$\leq [1-\alpha_n\beta_n\gamma_n+\alpha_n\beta_n\gamma_n\xi]\|z_n-a^*\|$
$\leq [1-\alpha_n\beta_n\gamma_n]\|z_n-a^*\|$

By induction

$\|z_{n+1}-a^*\| \leq \prod_{i=0}^{n}[1-\alpha_i\beta_i\gamma_i]\|z_0-a^*\|$
$\leq \|z_0-a^*\|e^{-\sum_{i=0}^{n}\alpha_i\beta_i\gamma_i}$

Since $\sum_{i=0}^{\infty}\alpha_i\,\beta_i\gamma_i=\infty, e^{-\sum_{i=0}^{n}\alpha_i\beta_i\gamma_i} \to 0 \; as \; n\to\infty$.
Thus, $\lim_{n\to\infty}\|z_n-a^*\|=0$.

Theorem (13): Let $T: B\to B$ be a firmly nonexpansive mapping and satisfying lipachitz, $S: B\to B$ satisfying condition(C_λ) and $a^*\in B$ be a common fixed point of S and T. Let (a_n) and (z_n) be the Picard-Mann and Noor iterations defined in (1) and (2).

Suppose $(\alpha_n), (\beta_n)\; and\; (\gamma_n)$ satisfied the following conditions:
1-$(\alpha_n)\; and\; (\beta_n) \in (0,1), \forall\, n\geq 0$.
2-$\sum\alpha_n=\infty$.
3-$\sum\alpha_n\beta_n<\infty$.

If $z_0=a_0\; and\; R(T), R(S)$ are bounded, then the Picard-Mann iteration sequence (a_n) converges strongly to a^* $(a_n\to a^*)$ and the Noor iteration sequence (z_n) converges strongly to $a^*(z_n\to a^*)$.

Proof: Since the range of T and S is bounded, let:

$M=sup_{a\in B}\{\|Ta\|\}+\|a_0\|<\infty$

and

$M=sup_{a\in B}\{\|Tz\|\}+\|z\|<\infty$

Then

$\|a_n\|\leq M, \|b_n\|\leq M, \|z_n\|\leq M, \|u_n\|\leq M, \|v_n\|\leq M$

Therefore

$\|Ta_n\|\leq M, \|Tz_n\|\leq M$
$\|a_{n+1}-z_{n+1}\|=\|Sb_n-(1-\alpha_n)z_n-\alpha_nSu_n\|$
$\leq \|Sb_n-z_n\|+\alpha_n\|Su_n-z_n\|$
$\leq \|b_n-a^*\|+\alpha_n\|u_n-a^*\|+(1+\alpha_n)\|z_n-a^*\|$

$\|b_n-a^*\|\leq(1-\alpha_n)\|a_n-a^*\|+\alpha_n(M+\|a^*\|)$
$\|v_n-a^*\|\leq(1-\gamma_n)\|z_n-a^*\|+\gamma_n\|Tz_n-a^*\|$

Since T is Lipschitzain and firmly nonexpansive, setting $\xi=k-kt+t$
$\leq(1-\gamma_n)\|z_n-a^*\|+\gamma_n\xi\|z_n-a^*\|$
$\leq\|z_n-a^*\|$
$\|u_n-a^*\|\leq(1-\beta_n)\|z_n-a^*\|+\beta_n\|Tv_n-a^*\|$
$\leq(1-\beta_n)\|z_n-a^*\|+\beta_n\xi\|v_n-a^*\|$
$\leq\|z_n-a^*\|$
$\leq M+\|a^*\|$

Then

$\|a_{n+1}-z_{n+1}\|\leq\|b_n-a^*\|+\alpha_n\|u_n-a^*\|+(1+\alpha_n)\|z_n-a^*\|$

$$\leq (1-\alpha_n)\|a_n - a^*\| + \alpha_n(M + \|a^*\|) +$$
$$\alpha_n(M + \|a^*\|) + (1+\alpha_n)(M + \|a^*\|)$$
$$\leq (1-\alpha_n)\|a_n - z_n\| + (1-\alpha_n)(M + \|a^*\|)$$
$$+2\alpha_n(M + \|a^*\|) + (1+\alpha_n)(M + \|a^*\|)$$
$$\leq (1-\alpha_n)\|a_n - z_n\| + 2\,(1+\alpha_n)(M + \|a^*\|)$$

Let
$\mu_n = \|a_n - z_n\|$, $\omega_n = (2{+}2\alpha_n)(M + \|a^*\|)$
and $\frac{\omega_n}{\delta_n} \to 0 \; as \; n \to \infty$. By applying lemma (7), we get:
$\lim_{n\to\infty} \|a_n - w_n\| = 0$
If $a_n \to a^* \in F(T,S)$, then
$\|z_n - a^*\| \leq \|z_n - a_n\| + \|a_n - a^*\| \to 0 \; as \; n \to \infty.$
If $z_n \to a^* \in F(T,S)$, then
$\|a_n - a^*\| \leq \|a_n - z_n\| + \|z_n - a^*\| \to 0 \; as \; n \to \infty.$

Theorem (14): Let $T: B \to B$ be a firmly nonexpansive mapping and satisfying Lipschitz with $Kt < 1$ and $S: B \to B$ satisfying condition (C_λ). Suppose that the Picard-Mann and Noor iteration converge to the same common fixed point a*. Then picard-Mann iteration converges faster than Noor iteration.

Proof: Let $a^* \in F(T,S)$. Then, for Picard-Mann iteration.
$\|b_n - a^*\| \leq (1-\alpha_n)\|a_n - a^*\| + \alpha_n\|Ta_n - a^*\|$
Setting $\xi = (1-t)K + t$, then we have
$$\leq (1-(1-\xi)\alpha_n)\|a_n - a^*\|$$
Next
$$\|a_{n+1} - a^*\| = \|Sb_n - a^*\|$$
$$\leq \|b_n - a^*\|$$
$$\leq (1-(1-\xi)\alpha_n)\|a_n - a^*\|$$
$$\leq .$$
$$\leq .$$
$$\leq (1-(1-\xi)\alpha)^n\|a_1 - a^*\|$$
Let $f_n = (1-(1-\xi)\alpha)^n\|a_1 - a^*\|$

Now, Noor iteration.
$$\|v_n - a^*\| \leq (1-\gamma_n)\|z_n - a^*\| + \gamma_n\|Tz_n - a^*\|$$
$$\leq (1-\gamma_n)\|z_n - a^*\| + \gamma_n\xi\|z_n - a^*\|$$
$$= \|z_n - a^*\|$$
$$\|u_n - a^*\| \leq (1-\beta_n)\|z_n - a^*\| + \beta_n\|Tv_n - a^*\|$$
$$\leq .$$
$$\leq (1-(1-\xi)\beta_n)\|z_n - a^*\|$$
Then
$$\|z_{n+1} - a^*\| \leq (1-\alpha_n)\|z_n - a^*\| + \alpha_n\|Su_n - a^*\|$$
$$\leq (1-\alpha_n + \alpha_n(1-(1-\xi)\beta_n)\|z_n - a^*\|$$
Assume that
$$\alpha_n \leq (1-\alpha_n + \alpha_n(1-(1-\xi)\beta_n)$$
$$\leq \alpha_n\|z_n - a^*\|$$
$$\leq .$$
$$\leq .$$
$$\leq \alpha^n\|z_1 - a^*\|$$

Let $g_n = \alpha^n \|z_1 - a^*\|$

Now,
$$\frac{f_n}{g_n} = \frac{(1-(1-\xi)\alpha)^n \|a_1 - a^*\|}{\alpha^n \|z_1 - a^*\|} \leq \left(1-(1-\xi)\right)^n \frac{\|a_1 - a^*\|}{\|z_1 - a^*\|} \to 0 \quad as\ n \to \infty.$$
Then, (a_n) converges faster than $(z_n)\ to\ a^*$.

Example (15): Let $B = [0,\infty)\ and\ T, S: B \to B$ be an mappings defined by $Ta = \frac{3-a}{2}$ and $Sa = \frac{1+4a}{5}\ \forall a \in B$. Choose $\alpha_n = \beta_n = \gamma_n = \frac{1}{2}$, $\forall n$ with initial value $a_1 = 20$. The Picard-Mann iteration converges faster than Noor iteration, as shown in **Table 1.** and **Figure 1**.

Table 1. Numerical results corresponding to $a_1 = 20$ for 30 steps.

n	Picard-Mann	Noor	n	Picard-Mann	Noor
0	20	20	**16**	1.0000	1.0353
1	4.8000	13.8250	**17**	1.0000	1.0238
2	1.7600	9.6569	**18**	1.0000	1.0161
3	1.1520	6.8434	**19**	1.0000	1.0109
4	1.0304	4.9443	**20**	1.0000	1.0073
5	1.0061	3.6624	**21**	1.0000	1.0049
6	1.0012	2.7971	**22**	1.0000	1.0033
7	1.0002	2.2131	**23**	1.0000	1.0023
8	1.0000	1.8188	**24**	1.0000	1.0015
9	1.0000	1.5527	**25**	1.0000	1.0010
10	1.0000	1.3731	**26**	1.0000	1.0007
11	1.0000	1.2518	**27**	1.0000	1.0005
12	1.0000	1.1700	**28**	1.0000	1.0003
13	1.0000	1.1147	**29**	1.0000	1.0002
14	1.0000	1.0774	**30**	1.0000	1.0001
15	1.0000	1.523			

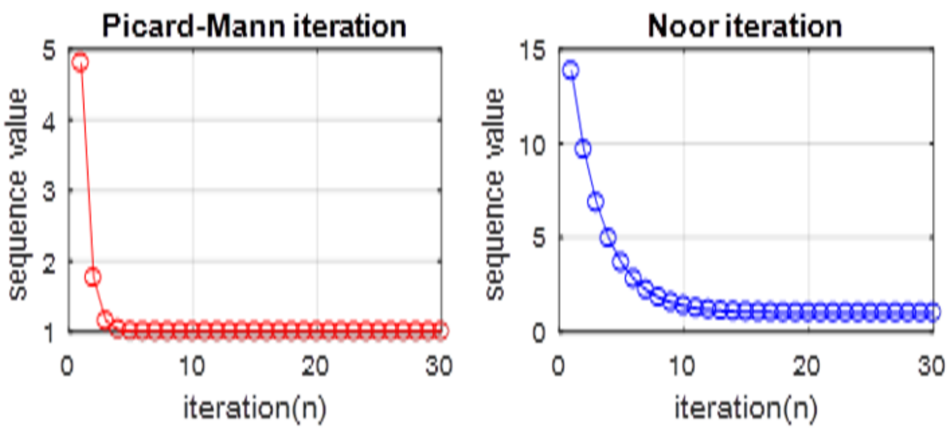

Figure 1: Convergence behavior corresponding to $a_1 = 20$ for 30 steps.

5. Application

The following mixed type of Volterra-Fredholm functional nonlinear integral equation that is appeared in [14]. We use theorem (14) to solve it:

$$a(t) = G(t, a(t), \int_{x_1}^{t_1} \dots \int_{x_n}^{t_n} Ꝁ(t, r, a(r))dr, \int_{x_1}^{y_1} \dots \int_{x_n}^{y_n} Ꟊ(t, r, a(r))dr) \qquad (3)$$

Where:

$[x_1, y_1] \times \dots \times [x_n, y_n]$ be an interval in R^n, Ꝁ, Ꟊ: $[x_1, y_1] \times \dots \times [x_n, y_n] \times [x_1, y_1] \times \dots \times [x_n, y_n] \times R \to R$ continuous functions and G: $[x_1, y_1] \times \dots \times [x_n, y_n] \times R^3 \to R$.

Assume that the following conditions are accomplished:

i- Ꝁ, Ꟊ $\in C([x_1, y_1] \times \dots \times [x_n, y_n] \times [x_1, y_1] \times \dots \times [x_n, y_n] \times R)$.

ii- $G \in C([x_1, y_1] \times \dots \times [x_n, y_n] \times R^3)$.

iii- $\exists\, positive\ constants$ ç, ϱ, è such that $|G(t, a_1, b_1, c_1) - G(t, a_2, b_2, c_2)| \le$ ç$|a_1 - a_2| + \varrho|b_1 - b_2| +$ è$|c_1 - c_2|\ \forall t \in [x_1, y_1] \times \dots \times [x_n, y_n], a_1, a_2, b_1, b_2, c_1, c_2 \in R$.

iv- $\exists\, positive\ constants\ S_{Ꝁ}\ and\ S_{Ꟊ}$ such that $|Ꝁ(t, r, a) - Ꝁ(t, r, b)| \le E_{Ꝁ}|a - b|$&$|Ꟊ(t, r, a) - Ꟊ(t, r, b)| \le E_{Ꟊ}|a - b|\ \forall t \in [x_1, y_1] \times \dots \times [x_n, y_n]\ and\ a, b \in R$.

v- ç $+ (\varrho E_{Ꝁ} +$ è$E_{Ꟊ})(y_1 - x_1) \dots (y_n - x_n) < 1,\ a^* \in C([x_1, y_1] \times \dots \times [x_n, y_n])$.

Theorem (16)[14]. Suppose that conditions (i-v) are satisfied. Then, the equation (3) has a unique solution $a^* \in C([x_1, y_1] \times \dots \times [x_n, y_n])$.

Theorem (17): We deem Banach space $M = C([x_1, y_1] \times \dots \times [x_n, y_n], \|.\|)$, such that satisfying $\sum_{k=0}^{\infty} \alpha_k = \infty$. Let (a_n) be as shown in step (1) and a map $T: M \to M$ is defined by

$$Ta(t) = G(t, a(t), \int_{x_1}^{t_1} \dots \int_{x_n}^{t_n} Ꝁ(t, r, a(r))dr, \int_{x_1}^{y_1} \dots \int_{x_n}^{y_n} Ꟊ(t, r, a(r))dr))$$

Suppose that the conditions (i-v) are accomplished. Then, the equation (3) has a unique solution $a^* in\ C([x_1, y_1] \times \dots \times [x_n, y_n])$ and the Picard-Mann iteration converges to a^*.

Proof: To prove $a_n \to a^*\ as\ n \to \infty$. Let

$$\|a_{n+1} - a^*\| = \|Sb_n - a^*\|$$
$$= |Sb_n(t) - Sa^*(t)|$$
$$= G(t, b_n(t), \int_{x_1}^{t_1} \dots \int_{x_n}^{t_n} Ꝁ(t, r, b_n(r))dr, \int_{x_1}^{y_1} \dots \int_{x_n}^{y_n} Ꟊ(t, r,\ b_n(r))d$$
$$- G(t, a^*(t), \int_{x_1}^{t_1} \dots \int_{x_n}^{t_n} Ꝁ(t, r, a^*(r))dr, \int_{x_1}^{y_1} \dots \int_{x_n}^{y_n} Ꟊ(t, r, a^*(r))dr)$$
$$\le ç|b_n(t) - a^*(t)| + \varrho \left| \int_{x_1}^{t_1} \dots \int_{x_n}^{t_n} Ꝁ(t, r, b_n(r))dr, \int_{x_1}^{t_1} \dots \int_{x_n}^{t_n} Ꝁ(t, r,\ a^*(r))d \right|$$
$$+ è \left| \int_{x_1}^{t_1} \dots \int_{x_n}^{t_n} Ꟊ(t, r, b_n(r))dr, \int_{x_1}^{y_1} \dots \int_{x_n}^{y_n} Ꟊ(t, r, a^*(r))dr) \right|$$
$$\le [ç + (\varrho E_{Ꝁ} + èE_{Ꟊ})(y_1 - x_1) \dots (y_n - x_n)]\|b_n - a^*\|$$

Since,

$$\|b_n - a^*\| \le (1 - \alpha_n)\|a_n - a^*\| + \alpha_n\|Ta_n(t) - Ta^*(t)\|$$

$$\leq (1-\alpha_n)\|a_n - a^*\|$$
$$+\alpha_n \left| G(t.b_n(t)). \int_{x_1}^{t_1} \dots \int_{x_n}^{t_n} Ҁ(t.r.b_n(r))dr. \int_{x_1}^{y_1} \dots \int_{x_n}^{y_n} Ӷ(t.r.\ b_n(r)) - G(t.a^*(t)). \int_{x_1}^{t_1} \dots \int_{x_n}^{t_n} Ҁ(t.r.a^*(r))dr. \int_{x_1}^{y_1} \dots \int_{x_n}^{y_n} Ӷ(t.r.a^*(r))dr) \right|$$
$$\leq (1-\alpha_n)\|a_n - a^*\| + \alpha_n[ç + (\varrho E_{Ҁ} + E_{Ӷ})(y_1 - x_1)\dots(y_n - x_n)]$$
$$\|a_n - a^*\|$$
$$\leq \{1-(1-\alpha_n[ç + (\varrho E_{Ҁ} + èE_{Ӷ})(y_1 - x_1)\dots(y_n - x_n)]\}\|a_n - a^*\|$$
$$\leq \|a_0 - a^*\| \prod_{k=0}^{n} \{1-(1-\alpha_n[ç + (\varrho E_{Ҁ} + èE_{Ӷ})(y_1 - x_1)\dots(y_n - x_n)]\}$$

By condition (v), $1-\alpha_n[ç + (\varrho E_{Ҁ} + èE_{Ӷ})(y_1 - x_1)\dots(y_n - x_n)] < 1$
Now, under using theorem (12), we obtain that equation (3) has a unique solution $a^* \in C([x_1, y_1] \times \dots \times [x_n, y_n])$ and Picard-Mann iteration converges to a^*.

In the same scope you can see the results in [15]. and [16]. where Hasan and Abed established weak convergence theorems by using appropriate conditions for approximating common fixed points and equivalence between the convergence of the Picard-Mann iteration scheme and Liu el.at iteration scheme in Banach spaces.

6.Conclusion

In the setting of 2-normed spaces [16]. we define firmly nonexpansive and generalized nonexpansive maps. Then, we study the convergence of Picard-Mann iteration and Noor iteration.

References

1. Diaz, J.B.; Metcalf, F.T. On the structure of the set of subsequential limit points of successive approximation. *Bull. Am. Math. Sco*. **1967**, *73*, 516-519.
2. Suzuki, T. Fixed point theorems and convergence theorems for some generalized-nonexpansive mappings. *J. Math. Anal. Appl*. **2008**, *340*, 1088–1095.
3. Dhompongsa, S.; Inthakon, W.; Kaewkhao, A.; Edelstein, S. Method and Fixed point theorems for some generalized nonexpansive mappings. *J.Math. Anal. Appl*.**2009**, *350*,12-17.
4. Garcial-Falset, J.; Llorens-Fuster, E.; Suzuki, T. Fixed point theory for a class of generalized nonexpansive mappings. *J. Math. Anal. Appl*. **2011**, *375*, 185-195.
5. Bruck, RE. Nonexpansive projections on subsets of Banach spaces. *Pac. J. Math*. **1973**, *47*, 341-355.
6. Khan, S.H.; Kim, J.K. Common fixed points of two nonexpansive mappings by a modified faster iteration scheme. *Bull. Korean. Math. Soc*. **2010**, *47*, *5*, 973-985.
7. Khan, S.H. A Picard-Mann hybrid iteration process. *Fixed Point Theory App*l. **2013**, *2013*, 69
8. Noor, M.A. New approximation schemes for general variational inequalities. *J. Math. Anal. Appl*. **2000**, *251*, 217-229.
9. Rajshree, D.; Balwant, T.S. A hybrid iteration for asymptotically strictly pesudocontractive mappings. *Journal of Inequalities and Applications*. **2014**, *2014*, *374*, 1-11, doi: 10.1186/1029-242X-2014-374.
10. Anupam, S.; Mohammad, I. Approximating fixed points of generalize nonexpansive mappings Via faster iteration schemes. *Fixed point theory*. **2014**, *4*, *4*, 605-623.

11. Khan, S.H.; Abbas, M.; Khan, A.R. Common Fixed Points of two Nonexpansive Mappings by a new One-step iteration Process. *Iranian Journal of Science & Technology, Transuction A*. **2009**, *1*, *33*, *A3*.
12. Deli, W.; Yisheng, S.; Xinwen, M. Common Fixed Points for Non-commuting Generalized (f,g)-Nonexpansive Maps. *Applied Mathematical Sciences*. **2008**, *2*, *52*, 2597-2602.
13. Weng, X. Fixed point iteration for local strictly pseudo-contractive mapping. Proc. Amer. Math. Soc. **1991**, *113*, *3*.
14. Craciun, C.; Serban, M.A. A nonlinear integral equation via Picard operators. *Fixed point theory*. **2011**, *12*, *1*, 57-70.
15. Abed, S.S.; Hasan, Z.M., Convergence theorems of a finite-step iteration algorithm under two finite families of total asymptotically quasi-nonexpansive maps .*IEEE Explore Digital Library* .International Conference on Advanced Science and Engineering (ICOASE). **2018**, dol:10.1109/ICOASE.2018.8548873.
16. Abed, S.S.; Abed Ali, R.S. Fixed Point for Asymptotically Non-Expansive Mappings in 2-Banach Space. *Ibn Al-Haitham Jour. for Pure & Appl. Sci*.**2014**,*27*,*1*,343-350.