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Integral Transforms of New Subclass of Meromorphic Univalent Functions Defined by Linear Operator I

Aqeel Ketab AL-khafaji

Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Babylon, Iraq.

aqeelketab@gmail.com

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Abstract

New class $A^*(a,c,k,\beta,\alpha,\gamma,\mu)$ is introduced of meromorphic univalent functions with positive coefficient $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n, (a_n\geq 0, z\in U^*, \forall n\in \mathbb{N}=\{1,2,3,\dots\})$ defined by the integral operator in the punctured unit disc $U^*=\{z\in\mathbb{C}:0<|z|<1\}$, satisfying $\left|\frac{z^2\left(I^k(L^*(a,c)f(z))\right)''+2z\left(I^k(L^*(a,c)f(z))\right)'}{\beta z\left(I^k(L^*(a,c)f(z))\right)''-\alpha(1+\gamma)z\left(I^k(L^*(a,c)f(z))\right)'}\right|<\mu, (0<\mu\leq 1, 0\leq\alpha,\gamma<1, 0<\beta\leq\frac{1}{2}, k=1,2,3,\dots)$. Several properties were studied like coefficient estimates, convex set and weighted mean.

Keywords: Meromorphic univalent function; coefficient estimates; convex set; weighted mean.

1. Introduction

Let W^* denote the class of functions of the form:

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n, (z\in U^*, n\in \mathbb{N}=\{1,2,3,\dots\}), \quad (1)$$

which are analytic and meromorphic univalent in the punctured unit disc

$U^*=\{z\in\mathbb{C}:0<|z|<1\}=U-\{0\}$.

The Hadamard product [1]. (convolution) of function $f(z)$ in (1) and a function $g(z)$:

$$g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}b_nz^n, (z\in U^*, \mathbb{N}=\{1,2,3,\dots\}), \quad (2)$$

is defined in the class W^* as

$$(f * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n, (z \in U^*, \forall\, n \in \mathbb{N} = \{1,2,3, \dots\}). \quad (3)$$

Let A^* be a subclass of the class W^* of functions of the form:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, (a_n \geq 0, z \in U^*, n \in \mathbb{N} = \{1,2,3, \dots\}). \quad (4)$$

A function f in the class A^* is said to be meromorphic starlike and meromorphic convex of order δ [2]. $(0 \leq \delta < 1, z \in, U^* f'(z) \neq 0)$, respectively if $-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \delta$ and $-Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \delta$.

In 2013, Juma and Zirar [3]. defined the function $\widetilde{\emptyset}\,(a, c; z)$ as follows:

$$\widetilde{\emptyset}\,(a, c; z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| z^n,\ z \in U^*$$

for $(c \in \mathbb{C}, c \neq 0, -1, -2, \dots\ and\ a \in \mathbb{C} - \{0\}), where\ (a)_n = a(a+1)_{n-1}$ is the Pochhammer symbol.

Gaussian hypergeometric function $\left({}_2F_1(b, a, c; z) = \sum_{n=0}^{\infty} \frac{(b)_n (a)_n}{(c)_n n!} z^n \right)$ was used, where $\widetilde{\emptyset}\,(a, c; z) = \frac{1}{z}\ {}_2F_1(1, a, c; z)$ and the Hadamard product for $f \in A^*$ corresponding to the function $\widetilde{\emptyset}\,(a, c; z)$, the linear operator $L^*(a, c)$ [3] defined on A^* by

$$L^*(a, c)f(z) = \widetilde{\emptyset}\,(a, c; z) * f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| a_n z^n, \quad (5)$$

and

$I^0\big(L^*(a, c)f(z)\big) = L^*(a, c)f(z)$, and for $k = 1,2,3, \dots$

$$I^k\big(L^*(a, c)f(z)\big) = z\left(I^{k-1}\big(L^*(a, c)f(z)\big)\right)' + \frac{2}{z}$$

$$= \frac{1}{z} + \sum_{n=0}^{\infty} n^k \left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| z^n. \quad (6)$$

In [4]. Darus and Frasin studied the operator $I^k\big(L^*(a, c)f(z)\big)$.

Now, the condition for the function f which is defined in (4) belongs to a class $A^*(a, c, k, \beta, \alpha, \gamma, \mu)$, where $A^*(a, c, k, \beta, \alpha, \gamma, \mu) \subset A^*$ according to Equation (6).

Definition 1: A function $f \in A^*$ of the form (1) is said to be in the class $A^*(a, c, k, \beta, \alpha, \gamma, \mu)$, if satisfies the following condition:

$$\left|\frac{z^2\left(I^k\left(L^*(a,c)f(z)\right)\right)''+2z\left(I^k\left(L^*(a,c)f(z)\right)\right)'}{\beta z\left(I^k\left(L^*(a,c)f(z)\right)\right)''-\alpha(1+\gamma)z\left(I^k\left(L^*(a,c)f(z)\right)\right)'}\right|<\mu, \tag{7}$$

where $0<\mu\leq 1, 0\leq\alpha,\gamma<1, 0<\beta\leq\frac{1}{2}$, $k=1,2,3,\ldots$.

In this paper, A new class $A^*(a,c,k,\beta,\alpha,\gamma,\mu)$ of meromorphic univalent functions is studied and discussed the positive coefficient defined by integral operator in the punctured unit disc U^*. Several properties are resulted such as, coefficient estimates, convex set, extreme point and obtain some interested results. See also References [5-9].

2. Results

In this section we introduce the results of the study in the two subsections:

2.1. Coefficient Estimates

In the first theorem, the necessary and sufficient condition is given to be the function f in the class $A^*(a,c,k,\beta,\alpha,\gamma,\mu)$.

Theorem 1: A function f(z) defined by (4) is in the class $A^*(a,c,k,\beta,\alpha,\gamma,\mu)$ if and only if:

$$\sum_{n=1}^{\infty}\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right|n^{k+1}\left[3n+\mu\left(\alpha(1+\gamma)-\beta(n-1)\right)-1\right]a_n\leq\mu[2\beta+\alpha(1+\gamma)], \tag{8}$$

where $0<\mu\leq 1, 0\leq\alpha,\gamma<1, 0<\beta\leq\frac{1}{2}$, $k=1,2,3,\ldots$.

Proof

Assume that (8) holds true. It is enough to show that:

$$S=\left|z^2\left(I^k\left(L^*(a,c)f(z)\right)\right)''+2z\left(I^k\left(L^*(a,c)f(z)\right)\right)'\right| \\ -\mu\left|\beta z\left(I^k\left(L^*(a,c)f(z)\right)\right)''-\alpha(1+\gamma)z\left(I^k\left(L^*(a,c)f(z)\right)\right)'\right|<0,$$

for $|z|=r<1$, from (8), that resulted:

$$S=\left|z^2\left(2z^{-3}+\sum_{n=1}^{\infty}\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right|n^{k+1}(n-1)a_nz^{n-2}\right)\right. \\ \left.+2z\left(-z^{-2}+\sum_{n=1}^{\infty}\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right|n^{k+1}a_nz^{n-1}\right)\right| \\ -\mu\left|\beta z^2\left(2z^{-3}+\sum_{n=1}^{\infty}\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right|n^{k+1}(n-1)a_nz^{n-2}\right)\right. \\ \left.-\alpha(1+\gamma)z\left(-z^{-2}+\sum_{n=1}^{\infty}\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right|n^{k+1}a_nz^{n-1}\right)\right|$$

$$= \left| \sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| n^{k+1}(3n-1)a_n z^n \right| - \mu \left| \big(2\beta + \alpha(1+\gamma)\big) z^{-2} + \sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| n^{k+1}[\beta(n-1) - \alpha(1+\gamma)] a_n z^n \right|$$

$$\leq \sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| n^{k+1}(3n-1)a_n z^n - \mu\big(2\beta + \alpha(1+\gamma)\big) r^{-2} - \mu \sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| n^{k+1}[\beta(n-1) - \alpha(1+\gamma)] a_n r^n$$

$$< \sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| n^{k+1}\big[3n + \mu\big(\alpha(1+\gamma) - \beta(n-1)\big) - 1\big] a_n r^n - \mu[2\beta + \alpha(1+\gamma)] < 0.$$

Hence, $f \in A^*(a, c, k, \beta, \alpha, \gamma, \mu)$.

Conversely, let $f(z) \in A^*(a, c, k, \beta, \alpha, \gamma, \mu)$, then (7) holds true, so: we have:

$$\left| \frac{z^2 \left(I^k\big(L^*(a,c)f(z)\big)\right)'' + 2z\left(I^k\big(L^*(a,c)f(z)\big)\right)'}{\beta z\left(I^k\big(L^*(a,c)f(z)\big)\right)'' - \alpha(1+\gamma)z\left(I^k\big(L^*(a,c)f(z)\big)\right)'} \right|$$

$$= \left| \frac{\sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| n^{k+1}(3n-1)a_n z^n}{\big(2\beta + \alpha(1+\gamma)\big) z^{-2} + \sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| n^{k+1}[\beta(n-1) - \alpha(1+\gamma)] a_n z^n} \right| < \mu .$$

Since $Re(z) \leq |z|$ for all z, it follows that:

$$Re \left\{ \frac{\sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| n^{k+1}(3n-1)a_n z^n}{\big(2\beta + \alpha(1+\gamma)\big) z^{-2} + \sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| n^{k+1}[\beta(n-1) - \alpha(1+\gamma)] a_n z^n} \right\} \leq \mu .$$

Now, we choose the value of z on the real axis so that $I^k\big(L^*(a,c)f(z)\big)$ is real.

Letting $z \to 1^-$ through real values, we obtain:

$$\sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| n^{k+1}\big[3n + \mu\big(\alpha(1+\gamma) - \beta(n-1)\big) - 1\big] a_n \leq \mu[2\beta + \alpha(1+\gamma)] .$$

Hence, the result follows ■

Finally, sharpness follows if we take

$$f(z) = \frac{1}{z} + \frac{\mu[2\beta + \alpha(1+\gamma)]}{\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{k+1}\left[3n + \mu\left(\alpha(1+\gamma) - \beta(n-1)\right) - 1\right]} z^n . \qquad (9)$$

$$(n = 1,2, \dots).$$

Corollary 1: If f(z) defined by (4) is in the class $A^*(a, c, k, \beta, \alpha, \gamma, \mu)$, then:

$$a_n \le \frac{\mu[2\beta + \alpha(1+\gamma)]}{\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{k+1}\left[3n + \mu\left(\alpha(1+\gamma) - \beta(n-1)\right) - 1\right]} \ , \qquad (10)$$

where $0 < \mu \le 1, 0 \le \alpha, \gamma < 1, 0 < \beta \le \frac{1}{2}, n \in \mathbb{N}, \ k = 1,2,3, \dots$.

Now, the function was defined $f_i(z)$ $(i = 1,2,3, \dots)$, as follows:

$$f_i(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,i} z^n \ (a_{n,i} \ge 0 , n \in \ \mathbb{N}). \qquad (11)$$

2.2. Convex Set

Here, the class $A^*(a, c, k, \beta, \alpha, \gamma, \mu)$ will prove as a convex set and give some result about it.

Theorem 2: The class $A^*(a, c, k, \beta, \alpha, \gamma, \mu)$ is convex set.

Proof

Let the functions $f_i(z)$ $(i = 1,2)$, defined by (11), be in the class $A^*(a, c, k, \beta, \alpha, \gamma, \mu)$, then for every e $(0 \le e \le 1)$, that showed must:

$[(1-e)f_1(z) + ef_2(z)] \in \ A^*(a, c, k, \beta, \alpha, \gamma, \mu)$.

Thus, we obtain:

$$(1-e)f_1(z) + ef_2(z) = z^{-1} + \sum_{n=1}^{\infty}\left[(1-e)a_{n,1} + ea_{n,2}\right] z^n ,$$

and

$$\sum_{n=1}^{\infty} \frac{\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{k+1}\left[3n + \mu\left(\alpha(1+\gamma) - \beta(n-1)\right) - 1\right]}{\mu[2\beta + \alpha(1+\gamma)]} \left[(1-e)a_{n,1} + ea_{n,2}\right]$$

$$= (1-e)\sum_{n=1}^{\infty}\frac{\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{k+1}\left[3n+\mu\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\right]}{\mu[2\beta+\alpha(1+\gamma)]}a_{n,1} + e\sum_{n=1}^{\infty}\frac{\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{k+1}\left[3n+\mu\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\right]}{\mu[2\beta+\alpha(1+\gamma)]}a_{n,2} \le 1\,.$$

Therefore, by Theorem (1), the result followed ■

Theorem 3: Let the functions $f_i(z)$ $(i=1,2)$, defined by (11) be in the class $A^*(a,c,k,\beta,\alpha,\gamma,\mu)$, then

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty}\left(a_{n,1}^2 + a_{n,2}^2\right)z^n\,, \tag{12}$$

In the class $A^*(a,c,k,\beta,\alpha,\gamma,\delta)$, where:

$$\delta \le \frac{\mu^2[2\beta+\alpha(1+\gamma)][\alpha(1+\gamma)-\beta(n-1)]}{\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^2\left[3n+\mu\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\right]^2 - \mu^2[2\beta+\alpha(1+\gamma)]}\,.$$

Proof

Since $f_i(z) \in A^*(a,c,k,\beta,\alpha,\gamma,\mu)$, then by Theorem (1), we have:

$$\sum_{n=1}^{\infty}\left(\frac{\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{k+1}\left[3n+\mu\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\right]}{\mu[2\beta+\alpha(1+\gamma)]}\right)^2 a_{n,1}^2$$

$$\le \sum_{n=1}^{\infty}\left(\frac{\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{k+1}\left[3n+\mu\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\right]}{\mu[2\beta+\alpha(1+\gamma)]}\, a_{n,1}\right)^2 \le 1, \tag{13}$$

and

$$\sum_{n=1}^{\infty}\left(\frac{\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{k+1}\left[3n+\mu\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\right]}{\mu[2\beta+\alpha(1+\gamma)]}\right)^2 a_{n,2}^2$$

$$\le \sum_{n=1}^{\infty}\left(\frac{\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{k+1}\left[3n+\mu\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\right]}{\mu[2\beta+\alpha(1+\gamma)]}\, a_{n,1}\right)^2 \le 1. \tag{14}$$

It follows from (13) and (14), that:

$$\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{k+1}\left[3n+\mu\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\right]}{\mu[2\beta+\alpha(1+\gamma)]}\right)^{2}\left(a_{n,1}^{2}+a_{n,2}^{2}\right)\leq 1\,.$$

But $g(z)\in A^{*}(a,c,k,\beta,\alpha,\gamma,\delta)$ if and only if:

$$\sum_{n=1}^{\infty}\frac{\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{k+1}\left[3n+\delta\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\right]}{\delta[2\beta+\alpha(1+\gamma)]}\left(a_{n,1}^{2}+a_{n,2}^{2}\right)\leq 1. \qquad (15)$$

The inequality (15) is satisfied if:

$$\frac{\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{k+1}\left[3n+\delta\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\right]}{\delta[2\beta+\alpha(1+\gamma)]} \leq \frac{\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{2(k+1)}\left[3n+\mu\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\right]^{2}}{\mu^{2}[2\beta+\alpha(1+\gamma)]^{2}}\,.$$

Hence:

$$\delta\leq\frac{\mu^{2}[2\beta+\alpha(1+\gamma)][\alpha(1+\gamma)-\beta(n-1)]}{\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{2}\left[3n+\mu\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\right]^{2}-\mu^{2}[2\beta+\alpha(1+\gamma)]}\,. \qquad (16)$$

$$=L(n)\,.$$

Since $L(n)$ is an increasing function of n $(n\geq 1)$, letting $n=2$ in (16), we get:

$$\delta\leq\frac{\mu^{2}[2\beta+\alpha(1+\gamma)][\alpha(1+\gamma)-\beta]}{\left|\frac{(a)_{3}}{(c)_{3}}\right| 4[5+\mu(\alpha(1+\gamma)-\beta)]^{2}-\mu^{2}[2\beta+\alpha(1+\gamma)]}\,.$$

This completes the proof ■

Theorem 4: Let the functions $f_i(z)$ $(i=1,2,\dots,m)$, defined by (12) be in the class $A^{*}(a,c,k,\beta,\alpha,\gamma,\mu)$, then:

$$q_{1}(z)=\frac{1}{z}+\sum_{n=1}^{\infty}e_{n}z^{n}\,,\quad (e_{n}\geq 0, n\in\mathbb{N})$$

In the class $A^{*}(a,c,k,\beta,\alpha,\gamma,\mu)$, where:

$$e_{n}=\frac{1}{m}\sum_{i=1}^{m}a_{n,i}\quad ,(n=1,2,\dots).$$

Proof

Since $f_i(z)\in A^{*}(a,c,k,\beta,\alpha,\gamma,\mu)$, for all ($i=1,2,3,\dots$), it follows from theorem (1) that:

$$\sum_{n=1}^{\infty}\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{k+1}\big[3n+\mu\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\big]a_{n,i} \leq \mu[2\beta+\alpha(1+\gamma)]\,.$$

Hence:

$$\sum_{n=1}^{\infty}\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{k+1}\big[3n+\mu\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\big]e_n\,.$$

$$=\sum_{n=1}^{\infty}\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{k+1}\big[3n+\mu\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\big]\left(\frac{1}{m}\sum_{i=1}^{m}a_{n,i}\right).$$

$$=\frac{1}{m}\sum_{i=1}^{m}\left(\sum_{n=1}^{\infty}\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{k+1}\big[3n+\mu\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\big]a_{n,i}\right) \leq \mu[2\beta+\alpha(1+\gamma)],$$

Therefore by theorem (1), we get $q_1(z) \in A^*(a,c,k,\beta,\alpha,\gamma,\delta)$ ■

Theorem 5: Let the functions f_i defined by (11), be in the class $A^*(a,c,k,\beta,\alpha,\gamma,\mu)$, (for all $i=1,2,\ldots,m$), Then the function:

$$q_2(z)=\sum_{i=1}^{m}c_i f_i(z)\,,\quad (c_i\geq 0)$$

Belongs to the class $A^*(a,c,k,\beta,\alpha,\gamma,\mu)$, where:

$$\sum_{i=1}^{m}c_i=1,\quad (c_i\geq 0).$$

Proof

For every $i=1,2,3,\ldots$, it follows from theorem (1) that:

$$\sum_{n=1}^{\infty}\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{k+1}\big[3n+\mu\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\big]a_{n,i} \leq \mu[2\beta+\alpha(1+\gamma)]\,.$$

But

$$q_2(z)=\sum_{i=1}^{m}c_i f_i(z)=\sum_{i=1}^{m}c_i\left(z^{-1}+\sum_{n=1}^{\infty}a_{n,i}z^n\right)=z^{-1}+\sum_{n=1}^{\infty}\left(\sum_{i=1}^{m}c_i a_{n,i}\right)z^n.$$

Therefore

$$\sum_{n=1}^{\infty}\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right| n^{k+1}\big[3n+\mu\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\big]\left(\sum_{i=1}^{m}c_i a_{n,i}\right)$$

$$= \sum_{i=1}^{m} c_i \left(\sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| n^{k+1} \left[3n + \mu\big(\alpha(1+\gamma) - \beta(n-1)\big) - 1 \right] a_{n,i} \right)$$

$$\leq \sum_{i=1}^{m} c_i \, \mu[2\beta + \alpha(1+\gamma)] = \mu[2\beta + \alpha(1+\gamma)],$$

This end of the proof

Definition 2 [2]**:** The weighted mean $w_j(z)$ of functions f and g, defined by: $w_j = \frac{1}{2}[(1-j)f(z) + (1+j)g(z)], \quad 0 < j < 1.$

Theorem 6. Let the functions $f_i(z)$ $(i = 1,2)$, defined by (11), be in the class $A^*(a, c, k, \beta, \alpha, \gamma, \mu)$, then the function, then the weighted men of $f_i(z)$ $(i = 1,2)$, is also in the class $A^*(a, c, k, \beta, \alpha, \gamma, \delta)$.

Proof

By Definition (2), we have

$$w_j = \frac{1}{2}[(1-j)f_1(z) + (1+j)f_2(z)]$$

$$= \tfrac{1}{2}\left[(1-j)\left(z^{-1} + \textstyle\sum_{n=1}^{\infty} a_{n,1} z^n\right) + (1+j)\left(z^{-1} + \textstyle\sum_{n=1}^{\infty} a_{n,2} z^n\right)\right]$$

$$= z^{-1} + \sum_{n=1}^{\infty} \frac{1}{2}\left[(1-j)a_{n,1} + (1+j)a_{n,2}\right] z^n ,$$

Since $f_i(z)$ $(i = 1,2)$, in the class $A^*(a, c, k, \beta, \alpha, \gamma, \mu)$, then by Theorem (1), we have:

$$\sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| n^{k+1} \left[3n + \mu\big(\alpha(1+\gamma) - \beta(n-1)\big) - 1 \right] a_{n,1} \leq \mu[2\beta + \alpha(1+\gamma)],$$

and

$$\sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| n^{k+1} \left[3n + \mu\big(\alpha(1+\gamma) - \beta(n-1)\big) - 1 \right] a_{n,2} \leq \mu[2\beta + \alpha(1+\gamma)] .$$

Hence

$$\sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| n^{k+1} \left[3n + \mu\big(\alpha(1+\gamma) - \beta(n-1)\big) - 1 \right] \frac{1}{2}\left[(1-j)a_{n,1} + (1+j)a_{n,2}\right]$$

$$= \frac{1}{2}(1-j) \sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| n^{k+1} \left[3n + \mu\big(\alpha(1+\gamma) - \beta(n-1)\big) - 1 \right] a_{n,1}$$

$$+\frac{1}{2}(1+j)\sum_{n=1}^{\infty}\left|\frac{(a)_{n+1}}{(c)_{n+1}}\right|n^{k+1}\left[3n+\mu\big(\alpha(1+\gamma)-\beta(n-1)\big)-1\right]a_{n,2}$$

$$\leq \frac{1}{2}\mu[2\beta+\alpha(1+\gamma)]+\frac{1}{2}\mu[2\beta+\alpha(1+\gamma)]$$

$$=\mu[2\beta+\alpha(1+\gamma)].$$

So $w_j \in A^*(a,c,k,\beta,\alpha,\gamma,\delta)$

3. Conclusions

From above and [10] we can use this class to generate another using the definition of meromorphic multivalent function. Also by suitable operator with meromorphic multivalent function can getting on a good class studies.

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