Ibn Al-Haitham Jour.for Pure & Appl .Sci. 32 (2) 2019

Ibn Al Haitham Journal for Pure and Applied Science

Journal homepage: http://jih.uobaghdad.edu.iq/index.php/j/index

Approximaitly Prime Submodules and Some Related Concepts

Ali Sh. Ajeel **Haibat K. Mohammadali**

Department of Mathematics, College of Computer Science and Mathematics, University of Tikrit, Tikrit, Iraq

Ali.shebl@st.tu.edu.iq H.mohammadali@tu.edu.iq

Article history: Received 27 December 2018, Accepted 20 January 2019, Publish May 2019

Doi: 10.30526/32.2.2148

Abstract

In this research note approximately prime submodules is defined as a new generalization of prime submodules of unitary modules over a commutative ring with identity. A proper submodule K of an R-module Y is called an approximaitly prime submodule of Y (for short app-prime submodule), if when ever $ry \in K$, where $r \in R$, $y \in Y$, implies that either $y \in K + soc(Y)$ or $rY \subseteq K + soc(Y)$. So, an ideal I of a ring R is called app-prime ideal of R if I is an app-prime submodule of R-module R. Several basic properties, characterizations and examples of approximaitly prime submodules were given. Furthermore, the definition of approximaitly prime radical of submodules of modules were introduced, and some of it is properties were established.

Keywords: Prime submodules, Approximaitly prime submodules, Approximaitly prime radical, Socle of submodules.

1. Introduction

Throughout this article, we consider all rings as commutative rings with identity, and all modules as unital R-modules. A proper submodule K of an R- module Y is prime, if whenever $ry \in K$, for $r \in R, y \in Y$, then either $y \in K$ or $r \in [K:_R Y]$where $[K:Y] = \{r \in R: rY \subseteq K\}$. The class of prime submodules was introduced and systematically studied in 1978 by Dauns [1]. as a generalization of the class of prime ideals of rings and recently has received a good of attention from several authors see [2-8]. In this paper, we will recall some basic definitions. The socle of a module Y denoted by $soc(Y)$ is the intersection of all essential submodules of Y [9]. where a non-zero submodule K of an R-module Y is called essential if $K \cap E \neq (0)$ for each non-zero submodule E of Y [9]. An element y in a module Y over integral domain R is torsion element if $rx \neq 0$ for all $r \in R$ [9]. The set of all torsion elements of Y denoted by $\tau(Y)$ is a submodule of Y. If $\tau(Y) = 0$ then Y is said to be torsion free [9]. An R- module Y is multiplication if each submodule E is the form IY for some ideal I of R or $E = [E:Y]Y$ [10]. A subset S of a ring R is called multiplicatively closed subset of R if $1 \in S$ and $ab \in S$ for every $a, b \in S$ [11]. If E is a submodule of an R-module Y, and S is a multiplicatively closed subset of R, then $E(S) = \{y \in Y: \exists t \in S \text{ such that } ty \in E\}$ is a submodule of Y and

$E \subseteq E(S)$ [11]. A non-zero R-module Y is compressible, if it is passable to embed Y in every non-zero submodule of Y [12].

2. Approximaitly Prime Submodules

In this section, we introduce the definition of approximaitly prime submodule as a generalization of a prime submodule, and give some basic properties, examples and characterizations of this concept.

Definition (1)

A proper submodule K of an R-module Y is called an approximaitly prime submodule of Y (for short app-prime submodule), if whenever $ry \in K$, where $r \in R$, $y \in Y$, implies that either $y \in K + soc(Y)$ or $rY \subseteq K + soc(Y)$. So, an ideal I of a ring R is called app-prime ideal of R if I is an app-prime submodule of R-module R.

The following results are characterizations of app-prime submodules.

Proposition (2)

Let Y be an R-module, and E be a submodule of Y. Then E is an app-prime submodule of Y if and only if for every submodule D of Y and every ideal I of R such that $ID \subseteq E$, implies that either $D \subseteq E + soc(Y)$ or $I \subseteq [E + soc(Y): Y]$.

Proof

($\Longrightarrow$) Assume that $ID \subseteq E$, where I is an ideal of R, and D is a submodule of Y, and suppose that $D \nsubseteq E + soc(Y)$, then there exists $d \in D$ such that $d \notin E + soc(Y)$. Since $ID \subseteq E$, then for any $x \in I$, $xd \in E$. But E is an app-prime submodule of Y, and $d \notin E + soc(Y)$, hence $x \in [E + soc(Y): Y]$. Thus $I \subseteq [E + soc(Y): Y]$.

($\Longleftarrow$) Assume that $ay \in E$, where $a \in R$, $y \in Y$, then $(a)(y) \subseteq E$, so by hypothesis either $(a) \subseteq [E + soc(Y): Y]$ or $(y) \subseteq E + soc(Y)$. That is either $y \in E + soc(Y)$ or $a \in [E + soc(Y): Y]$.

The following corollary is a consequence immediately of a Proposition (2).

Corollary (3)

Let Y be an R-module, and E be a submodule of Y. Then E is an app-prime submodule of Y if and only if for every submodule D of Y and any $a \in R$ with $aD \subseteq E$, implies that either $D \subseteq E + soc(Y)$ or $a \in [E + soc(Y): Y]$.

Remark (4)

It is clear that every prime submodule of an R-module Y is an app-prime submodule of Y, while the converse is not true as the following example shows that:

Example (5)

Consider the Z-module Z_{12} , and $E = \langle\bar{4}\rangle$, $soc(Z_{12}) = \langle\bar{2}\rangle$.Therefore each $a \in Z$, $y \in Z_{12}$, if $ay \in E$, then either $y \in \langle\bar{4}\rangle + \langle\bar{2}\rangle = \langle\bar{2}\rangle$ or $a \in [\langle\bar{4}\rangle + \langle\bar{2}\rangle: Z_{12}] = 2Z$. Thus E is an app-prime submodule of Z_{12} , but E is not prime submodule of Z_{12} , because $2.\bar{2} \in E$, but neither $\bar{2} \in E$ nor $2 \in [E: Z_{12}] = 4Z$.

Proposition (6)

Let Y be an R-module, and E be a submodule of Y, with $soc(Y) \subseteq E$. Then E is a prime submodule of Y if and only if E is an app-prime submodule of Y.

Proof

It is clear

The following corollaries are direct consequence of proposition (2.6).

Corollary (7)

Let Y be an R-module, and E be a submodule of Y, with $soc(Y)=0$. Then E is a prime submodule of Y if and only if E is an app-prime submodule of Y.

It is well-known that a torsion free Z-module Y has zero socle [13]. so we set the following result.

Corollary (8)

Let Y be a torsion free Z-module, and E be a submodule of Y. Then E is a prime submodule of Y if and only if E is an app-prime submodule of Y.

Proposition (9)

Let E be an app-prime submodule of an R-module Y, with $soc(Y)\subseteq E$. Then $[E:Y]$ is an app-prime ideal of R.

proof

It is followed by proposition (6) and by [14, Prop. 2.8].

The convers of proposition (9) is not true in general, as the following example explain that.

Example (10)

Let $Y=Z\bigoplus Z$, $R=Z$, and $E=\langle(2,0)\rangle$, then $[E:Y]=(0)$ is an app-prime ideal in a ring Z. But E is not an app-prime submodule of Y.

Recall that an R-module Y is called singular module provided $Z(Y)=Y$. At the other extreme, we say that Y is non-singular module provided $Z(Y)=0$, where $Z(Y)=\{y\in Y : yI=0\ for\ some\ I\in\tau(R)\}$ where $\tau(R)$ the set of all essential right ideals of the ring R, [9].

The following proposition shows that the converse of proposition (9) is true under certain conditions.

Proposition (11)

Let Y be a multiplication non-singular R-module, and E be a proper submodule of Y, with $soc(Y)\subseteq E$. Then E is an app-prime submodule of Y if and only if $[E:Y]$ is an app-prime ideal of R.

Proof

$(\Longrightarrow)$ Follows by proposition (9).

$(\Longleftarrow)$ Let $ry\in E$, where $r\in R$, $y\in Y$, then $r(y)\subseteq E$. But Y is multiplication, then $(y)=IY$ for some ideal I of R. Thus $rIY\subseteq E$, it follows that $rI\subseteq[E:Y]$. Since $[E:Y]$ is an app-prime ideal of R, then either $I\subseteq[E:Y]+soc(R)$ or $r\in\big[[E:Y]+soc(R):R\big]=[E:Y]+soc(R)$. Hence either $IY\subseteq[E:Y]Y+soc(R)Y$ or $rY\subseteq[E:Y]Y+soc(R)Y$. But Y is a non-singular, then by [9]. we have $soc(R)Y=soc(Y)$. Thus, either $IY\subseteq E+soc(Y)$ or $rY\subseteq E+soc(Y)$. Hence either $y\in E+soc(Y)$ or $r\in[E+soc(Y):Y]$. Hence E is an app-prime submodule of Y.

Proposition (12)

Let Y be a faithful multiplication R-module, and E be a proper submodule of Y, with $soc(Y)\subseteq E$. Then E is an app-prime submodule of Y if and only if $[E:Y]$ is an app-prime ideal of R.

Proof

($\Longrightarrow$) Follows by proposition (9).

($\Longleftarrow$) Let $ry \in E$, where $r \in R$, $y \in Y$, then $r(y) \subseteq E$. But Y is multiplication, then $(y) = IY$ for some ideal I of R. Thus $rIY \subseteq E$, it follows that $rI \subseteq [E\colon Y]$. Since $[E\colon Y]$ is an app-prime ideal of R, then by corollary (3) either $I \subseteq [E\colon Y] + soc(R)$ or $r \in \big[[E\colon Y] + soc(R)\colon R\big] = [E\colon Y] + soc(R)$. Thus either $IY \subseteq [E\colon Y]Y + soc(R)Y$ or $rY \subseteq [E\colon Y]Y + soc(R)Y$. Since Y is a faithful multiplication, then by [10,cor.2.14]. we have $soc(R)Y = soc(Y)$. It follows that either $IY \subseteq E + soc(Y)$ or $rY \subseteq E + soc(Y)$. Hence either $y \in E + soc(Y)$ or $r \in [E + soc(Y)\colon Y]$. Hence E is an app-prime submodule of Y.

Proposition (13)

Let Y be an R-module, and E be a submodule of Y such that $[E + soc(Y)\colon Y]$ is a maximal ideal of R. Then E is an app-prime submodule of Y.

Proof

Let $sy \in E$, where $s \in R$, $y \in Y$, with $s \notin [E + soc(Y)\colon Y]$. Since $[E + soc(Y)\colon Y]$ is a maximal ideal of R, then $R = \langle s \rangle + [E + soc(Y)\colon Y]$, where $\langle s \rangle$ is an ideal of R generated by s. Thus, there exists $r \in R$ and $i \in [E + soc(Y)\colon Y]$ such that $1 = rs + i$, it follows that $y = rsy + iy \in E + soc(Y)$. Hence E is an app-prime submodule of Y.

Proposition (14)

Let Y be an R-module, and E be a proper submodule of Y, with $[D\colon Y] \nsubseteq [E + soc(Y)\colon Y]$, and $E + soc(Y)$ proper submodule of D for each submodule D of Y such that $[E + soc(Y)\colon Y]$ is a prime ideal of R. Then E is an app-prime submodule of Y.

Proof

Assume that $sy \in E$, where $s \in R$, $y \in Y$, with $y \notin E + soc(Y)$. Then $E + soc(Y) \subsetneq E + soc(Y) + \langle y \rangle = D$ and so $[D\colon Y] \nsubseteq [E + soc(Y)\colon Y]$, then there exists $r \in [D\colon Y]$ and $r \notin [E + soc(Y)\colon Y]$. That is $rY \subseteq D$ and $rY \nsubseteq E + soc(Y)$. That is $rY \subseteq D$ implies that $srY \subseteq s(E + soc(Y) + \langle y \rangle) \subseteq E + soc(Y)$. It follows that $sr \in [E + soc(Y)\colon Y]$. But $[E + soc(Y)\colon Y]$ is a prime ideal of R, then $s \in [E + soc(Y)\colon Y]$. Hence E is an app-prime submodule of Y.

Note

It is well-know that if Y is a non-zero multiplication module, then $[D\colon Y] \nsubseteq [E\colon Y]$ for each submodule D of Y with $E \subsetneq D$, where E is a proper submodule of Y [14, Rem.2.15].

Now, we get the following corollary as a direct consequence of proposition (14).

Corollary (15)

Let Y be a multiplication R-module, and E be a proper submodule of Y, with$[E + soc(Y)\colon Y]$ is prime ideal of R, and $E + soc(Y) \subsetneq D$ for each submodule D of Y. Then E is an app-prime submodule of Y.

Proposition (16)

Let Y be an R-module, and E, D are submodules of Y, with $E \subsetneq D$. If E is an app-prime submodule of Y and $soc(Y) \subseteq soc(D)$, then E is an app-prime submodule of D.

Proof

Suppose that $ay \in E$, where $a \in R$, $y \in D$. Since E is an app-prime submodule of Y, then either $y \in E + soc(Y)$ or $aY \subseteq E + soc(Y)$. But $soc(Y) \subseteq soc(D)$, implies that either $y \in E + soc(D)$ or $aY \subseteq E + soc(D)$. Thus E is an app-prime submodule of D.

Proposition (17)

Let Y be an R-module, and E be a submodule of Y, with $E + soc(Y)$ is an app-prime submodule of Y. Then E is an app-prime submodule of Y.

Proof

Suppose that $ay \in E$, where $a \in R$, $y \in Y$. Hence $ay \in E \subseteq E + soc(Y)$, and so $ay \in E + soc(Y)$. But $E + soc(Y)$ is an app-prime submodule of Y, then either $y \in E + soc(Y) + soc(Y) = E + soc(Y)$ or $aY \subseteq E + soc(Y) + soc(Y) = E + soc(Y)$. Thus E is an app-prime submodule of Y.

Proposition (18)

Let Y be an R-module, and E be a submodule of Y, with $[E + soc(Y): Y]$ is a prime ideal of R. Then $E(S) \subseteq E + soc(Y)$ for each multiplicatively closed subset S of R with $S \cap [E + soc(Y): Y] = \phi$ if and only if E is an app-prime submodule of Y.

Proof

($\Longrightarrow$) Assume that $ay \in E$, where $a \in R$, $y \in Y$, and suppose that $y \notin E + soc(Y)$ and $a \notin [E + soc(Y): Y]$. Since S is a multiplicatively closed subset of R, then $S = \{1, a, a^2, a^3, \dots\}$, and since $[E + soc(Y): Y]$ is a prime ideal of R, then it is clear that $S \cap [E + soc(Y): Y] = \phi$. But $y \notin E + soc(Y)$, implies that $y \notin E(S)$ and hence $ay \notin E$ which contradiction. Thus, either $y \in E + soc(Y)$ or $a \in [E + soc(Y): Y]$, therefore E is an app-prime submodule of Y.

($\Longleftarrow$) Let $y \in E(S)$, then there exists $s \in S$ such that $sy \in E$. But E is an app-prime submodule of Y, so either $y \in E + soc(Y)$ or $s \in [E + soc(Y): Y]$. But $s \in [E + soc(Y): Y]$, implies that $s \in S \cap [E + soc(Y): Y] = \phi$, which is a contradiction. Thus $y \in E + soc(Y)$ and hence $E(S) \subseteq E + soc(Y)$.

Proposition (19)

Let Y be an R-module, and I be a maximal ideal of R, with $IY + soc(Y) \neq Y$. Then IY is an app-prime submodule of Y.

Proof

Clearly, $I \subseteq [IY + soc(Y): Y]$.That is there exists $a \in [IY + soc(Y): Y]$ and $a \notin I$, then $R = I + \langle a \rangle$, where $\langle a \rangle$ is an ideal of R generated by a, thus there exist $r \in R$ and $i \in I$ such that $1 = i + ar$. Hence $y = iy + ary$ for each $y \in Y$. It follows that $y \in IY + soc(Y)$ for each $y \in Y$, hence $Y \subseteq IY + soc(Y)$, it follows that $IY + soc(Y) = Y$ which is a contradiction. Then $a \in I$ and hence $[IY + soc(Y): Y] \subseteq I$, it follows that $[IY + soc(Y): Y] = I$ is a maximal ideal of R, hence by proposition (13) IY is an app-prime submodule of Y.

Proposition (20)

Let Y be a faithful multiplication R-module, and I is an app-prime ideal of R. Then IY is an app-prime submodule of Y.

Proof

Let $ay \in IY$, where $a \in R$, $y \in Y$, then $a(y) \subseteq IY$. But Y is multiplication, then $(y) = JY$ for some ideal J of R. It follows that $aJY \subseteq IY$, and so $aJ \subseteq I + ann(Y) = I$. But I is an app-prime ideal of R, then by corollary (3) either $J \subseteq I + soc(R)$ or $a \in [I + soc(R): R] = I + soc(R)$. Thus either $JY \subseteq IY + soc(R)Y$ or $aY \subseteq IY + soc(R)Y$. But Y be a faithful multiplication module then by [10, cor. 2.14]. we have $soc(R)Y = soc(Y)$. Thus either

$aY \subseteq IY + soc(Y)$ or $JY \subseteq IY + soc(Y)$. That is either $a \in [IY + soc(Y): Y]$ or $y \in IY + soc(Y)$. Hence IY is an app-prime submodule of Y.

Proposition (21)

Let Y be a finitely generated multiplication non-singular R-module, and I is an app-prime ideal of R, with $ann(Y) \subseteq I$. Then IY is an app-prime submodule of Y.

Proof

Let $ay \in IY$, where $a \in R$, $y \in Y$, then $a(y) \subseteq IY$. But Y is multiplication, then $(y) = JY$ for some ideal J of R. It follows that $aJY \subseteq IY$, and so $aJ \subseteq I + ann(Y) = I$. But I is an app-prime ideal of R, then by corollary (3) either $J \subseteq I + soc(R)$ or $a \in [I + soc(R): R] = I + soc(R)$. Thus either $JY \subseteq IY + soc(R)Y$ or $aY \subseteq IY + soc(R)Y$. But Y is non-singular, then by [9]. we have $soc(R)Y = soc(Y)$. Thus either $aY \subseteq IY + soc(Y)$ or $JY \subseteq IY + soc(Y)$. That is either $a \in [IY + soc(Y): Y]$ or $y \in IY + soc(Y)$. Hence IY is an app-prime submodule of Y.

Proposition (22)

Let Y be an R-module, and E be a proper submodule of Y with $[D: Y] \nsubseteq [E: Y]$ for each submodule D of Y such that $E \subsetneq D$ and $soc(Y) \subseteq E$. then E is an app-prime submodule of Y if and only if $\frac{Y}{E}$ is a compressible R-module.

Proof

($\Longrightarrow$) Assume that E is an app-prime submodule of Y and D be a submodule of Y with $E \subsetneq D$, therefore $\frac{D}{E}$ is a non-zero submodule of $\frac{Y}{E}$, we are going to emmbed $\frac{Y}{E}$ inside $\frac{D}{E}$. Since $[D: Y] \nsubseteq [E: Y]$, then there exists $r \in [D: Y]$ and $r \notin [E: Y]$. That is $rY \nsubseteq E$. Define $f: \frac{Y}{E} \longrightarrow \frac{D}{E}$ by $f(y + E) = ry + E$ for each $y \in Y$. It is clear that f is an R-homomorphism. To prove that f is one to one. Suppose that $f(y_1 + E) = f(y_2 + E)$, then $ry_1 + E = ry_2 + E$, so $ry_1 - ry_2 \in E$, that is $r(y_1 - y_2) \in E$. but E is an app-prime submodule of Y, so either $y_1 - y_2 \in E + soc(Y)$ or $rY \subseteq E + soc(Y)$. Since $soc(Y) \subseteq E$, it follows that either $y_1 - y_2 \in E$ or $rY \subseteq E$. But $rY \nsubseteq E$, hence $y_1 - y_2 \in E$, so $y_1 + E = y_2 + E$. Thus f is monomorphism and $\frac{Y}{E}$ is a compressible.

($\Longleftarrow$) Suppose that $\frac{Y}{E}$ is a compressible R-module, and $ry \in E$, where $r \in R$, $y \in Y$, with $y \notin E + soc(Y)$ that is $y \notin E$ since $soc(Y) \subseteq E$. Then $\frac{\langle y \rangle + E}{E}$ is a submodule of $\frac{Y}{E}$, hence there is a monomorphism $: \frac{Y}{E} \longrightarrow \frac{\langle y \rangle + E}{E}$, that is $Ker\ g = E$. Let $y_1 \in Y$, then $g(y_1 + E) \in \frac{\langle y \rangle + E}{E}$, then $\exists\ a \in R$ such that $g(y_1 + E) = ay_1 + E$ that is $g(ry_1 + E) = ary_1 + E = E$, implies that $ry_1 + E \in Ker\ g$ that is $ry_1 + E = E$, it follows that $ry_1 \in E$ for each $y_1 \in Y$, hence $rY \subseteq E \subseteq E + soc(Y)$, that is $rY \subseteq E + soc(Y)$. Thus E is an app-prime submodule of Y.

So, we get the following corollary as a direct consequence of proposition (22).

Corollary (23)

Let Y be a multiplication R-module, and E be a proper submodule of Y with $soc(Y) \subseteq E$. Then E is an app-prime submodule of Y if and only if $\frac{Y}{E}$ is a compressible R-module.

Proposition (24)

Let Y be an R-module, and E be a proper submodule of Y such that $[E+soc(Y):Y]=[E+soc(Y):D]$ for each submodule D of Y with $E\subsetneq D$ and $E+soc(Y)\subsetneq E$.Then E is an app-prime submodule of Y.

Proof

Suppose that $ay\in E$, where $a\in R$, $y\in Y$, with $y\notin E+soc(Y)$. Let $D=E+soc(Y)+\langle y\rangle$, then $E+soc(Y)\subsetneq E+soc(Y)+\langle y\rangle$, it follows that $y\in D$. And so $a\in[E:E+soc(Y)+\langle y\rangle]\subseteq[E+soc(Y):E+soc(Y)+\langle y\rangle]=[E+soc(Y):Y]$. Hence $a\in[E+soc(Y):Y]$. It follows that E is an app-prime submodule of Y.

Remark (25)

The intersection of two app-prime submodules of an R-module Y need not be an app-prime submodule of Y, as the following example shows that:

Example (26)

Let $Y=Z$, $R=Z$, $E=2Z$ and $D=3Z$. E and D are app-prime submodules of Y, but $E\cap D=6Z$ is not app-prime submodule of Y since $2.3\in 6Z,\ 2\in R,\ 3\in Z,$ but $3\notin(E\cap D)+soc(Y)$ and $2\notin[(E\cap D)+soc(Y):Y]=6Z$

Proposition (27)

Let Y be an R-module, E and D are two app-prime submodules of Y with $E\subseteq soc(Y)$ and $D\subseteq soc(Y)$. Then $E\cap D$ is an app-prime submodule of Y.

Proof

Suppose that $ay\in E\cap D$, where $a\in R$, $y\in Y$, then $ay\in E$ and $ay\in D$. Since E and D are app-prime submodules of Y, so either $y\in E+soc(Y)$ or $aY\subseteq E+soc(Y)$ and either $y\in D+soc(Y)$ or $aY\subseteq D+soc(Y)$. But $E\subseteq soc(Y)$ and $D\subseteq soc(Y)$, then either $y\in soc(Y)$ or $aY\subseteq soc(Y)$ and either $y\in soc(Y)$ or $aY\subseteq soc(Y)$, it follows that either $y\in(E\cap D)+soc(Y)$ or $aY\subseteq(E\cap D)+soc(Y)$. Hence $E\cap D$ is an app-prime submodule of Y.

Proposition (28)

Let Y be an R-module, E and D are two submodules of Y with D is not contained in E and $soc(Y)\subseteq D$. If E is an app-prime submodule of Y then $E\cap D$ is an app-prime submodule of D.

Proof

Since D is not contained in E, then $E\cap D$ is a proper submodule of D. Now, let $ay\in E\cap D$, where $a\in R$, $y\in D$, then $ay\in E$ and $ay\in D$. But E is app-prime submodule of Y, then either $y\in E+soc(Y)$ or $aY\subseteq E+soc(Y)$. But $y\in D$ and $ay\in D$, then we have either $y\in[E+soc(Y)]\cap D$ or $aD\subseteq[E+soc(Y)]\cap D$. But $soc(Y)\subseteq D$, so by modular law we have either $y\in(E\cap D)+(soc(Y)\cap D)$ or $aD\subseteq(E\cap D)+s(oc(Y)\cap D)$. But by [13]. Coro. 9.9] we have $soc(D)=soc(Y)\cap D$, hence either $y\in(E\cap D)+soc(D)$ or $aD\subseteq(E\cap D)+soc(D)$. Thus $E\cap D$ is an app-prime submodule of D.

Proposition (29)

Let Y be an R-module, and E be a submodule of Y such that $=\cap D_i$, where D_i is a prime submodule of Y for each $i\in I$. Then E is an app-prime submodule of Y.

Proof

Let $ay\in E$, where $a\in R$, $y\in Y$, then $ay\in D_i$ for each $i\in I$. Since D_i is a prime submodule of Y for each $i\in I$, so either $y\in D_i$ or $a\in[D_i:Y]$. That is either $y\in\cap D_i\subseteq\cap$

$D_i + soc(Y)$ or $aY \subseteq D_i$, it follows that $aY \subseteq \cap D_i \subseteq \cap D_i + soc(Y)$. Hence either $y \in \cap D_i + soc(Y)$ or $a \in [\cap D_i + soc(Y): Y]$. Therefore E is an app-prime submodule of Y.

The following proposition shows that the invers image of app-prime submodule is app-prime.

Proposition (30)

Let $f \in Hom\,(Y, Y')$ be an R-epimorphism, and E be an app-prime submodule of Y'. Then $f^{-1}(E)$ is an app-prime submodule of Y.

Proof

It is clear that $f^{-1}(E)$ is a proper submodule of Y. Now, suppose that $ay \in f^{-1}(E)$, where $a \in R$, $y \in Y$, then $f(ay) \in E$. But E is an app-prime submodule of Y', implies that either $f(y) \in E + soc(Y')$ or $aY' \subseteq E + soc(Y')$. If $f(y) \in E + soc(Y')$, then $y \in f^{-1}(E) + f^{-1}\big(soc(Y')\big) \subseteq f^{-1}(E) + soc(Y)$ [15,Theo.(1.4)a]. That is $y \in f^{-1}(E) + soc(Y)$. If $aY' \subseteq E + soc(Y')$ and $f(Y) \subseteq Y'$, then $af(Y) \subseteq aY' \subseteq E + soc(Y')$. That is $aY \subseteq f^{-1}(E) + f^{-1}\big(soc(Y')\big) \subseteq f^{-1}(E) + soc(Y)$ [15,Theo.(1.4)a]. Hence $aY \subseteq f^{-1}(E) + soc(Y)$. Thus $f^{-1}(E)$ is an app-prime submodule of Y.

Proposition (31)

Let $f \in Hom\,(Y, Y')$ be an R-epimorphism, and E be an app-prime submodule of Y with $Ker\, f \subseteq E$. Then $f(E)$ is an app-prime submodule of Y'.

Proof

$f(E)$ is a proper submodule of Y'. If not, that is $f(E) = Y'$, let $y \in Y$, then $f(y) \in Y' = f(E)$, so there exists $x \in E$ such that $f(y) = f(x)$, that is $f(y - x) = 0$, implies that $y - x \in Ker\, f \subseteq E$, it follows that $y \in Y$, hence $E = Y$ contradiction. Now suppose that $ay' \in f(E)$, where $a \in R$, $y' \in Y'$. Since f is an epimorphism, and $y' \in Y'$, then there exists $y \in Y$ such that $f(y) = y'$,that is $ay' = af(y) = f(ay) \in f(E)$, so there exists $x \in E$ such that $f(x) = f(ay)$, it follows that $f(x - ay) = 0$, so $x - ay \in Ker\, f \subseteq E$, then $ay \in E$. but E be an app-prime submodule of Y, then either $y \in E + soc(Y)$ or $aY \subseteq E + soc(Y)$, and hence either $f(y) \in f(E) + f(soc(Y))$ or $af(Y) \subseteq f(E) + f(soc(Y))$. But by [15,Theo.(1.4)a]. we have $f(soc(Y) \subseteq soc(Y')$. So, we have either $f(y) = y' \in f(E) + soc(Y')$ or $aY' \subseteq f(E) + soc(Y')$. Hence $f(E)$ is an app-prime submodule of Y'.

As a direct consequence of proposition (30) and proposition (31), we set the following result.

Corollary (32)

Let Y be an R-module, E and D are two submodules of Y with $D \subseteq E$. Then $\frac{E}{D}$ is an app-prime submodule of $\frac{Y}{D}$ if and only if E is an app-prime submodule of Y.

Proposition (33)

Let Y be an R-module, E and D are two submodules of Y and F is an app-prime submodule of Y with $E \cap D \subseteq F$ and $[E: Y] \nsubseteq [F + soc(y): Y]$ then $D \subseteq F + soc(Y)$.

Proof

Since $[E: Y] \nsubseteq [F + soc(y): Y]$, then there exists $a \in [E: Y]$ but $a \notin [F + soc(Y): Y]$. Let $x \in D$, so $ax \in D$ and $aY \subseteq E$, so $ax \in E$, implies that $ax \in E \cap D \subseteq F$. But F ia an app-prime submodule of Y and $a \notin [F + soc(Y): Y]$ then $x \in F + soc(Y)$. Thus $D \subseteq F + soc(Y)$.

3. Approximaitly Prime Radical of Submodules

In this section we introduce the notion of approximaitly prime radical of a submodule, and we establish several properties of this notion that are similarly to those of radical of submodules.

Definition (34)

Let Y b an R-module, and E is a submodule of Y.An app-prime radical of a submodule E denoted by $App-rad_Y(E)$ is defined as the intersection of all approximaitly prime submodules of Y which contain E, if there exists no approximaitly prime submodule containing E, we put $App-rad_Y(E)=Y$.

In the following proposition we introduce some basic properties of approximaitly prime radical.

Proposition (35)

Let $f\in Hom\,(Y,Y')$ be an R-epimorphism, and E is a submodule of Y wihe $Ker\,f\subseteq E$. Then $f\big(App-rad_Y(E)\big)=App-rad_{Y'}(f(E))$.

Proof

Since $App-rad_Y(E)=\cap K$ where the intersection runs over all app-prime submodules K of Y with $E\subseteq K$, so $f\big(App-rad_Y(E)\big)=f(\cap K)$. Since $Ker\,f\subseteq E\subseteq K$, then by [13, Lemm.(3.1.10) c]. $f\big(App-rad_Y(E)\big)=\cap f(K)$ where the intersection runs over all app-prime submodules $f(K)$ of Y' with $f(E)\subseteq f(K)$. Thus $f\big(App-rad_Y(E)\big)=App-rad_{Y'}(f(E))$.

Proposition (36)

Let $f\in Hom\,(Y,Y')$ be an R-epimorphism, and D' is a submodule of Y'. Then $f^{-1}\big(App-rad_{Y'}(D')\big)=App-rad_Y(f^{-1}(D'))$.

Proof

Since $App-rad_{Y'}(D')=\cap L'$ where the intersection runs over all app-prime submodules L' of $Y'$with $D'\subseteq L'$. Hence by [15,Lemm.(3.1.10)a]. $f^{-1}\big(App-rad_{Y'}(D')\big)=f^{-1}(\cap L')=\cap f^{-1}(L')$ where the intersection runs over all app-prime submodules $f^{-1}(L')$ of Y with $f^{-1}(D')\subseteq f^{-1}(L')$. It follows that $f^{-1}\big(App-rad_{Y'}(E)\big)=App-rad_Y(f^{-1}(E))$.

Proposition (37)

Let Y be an R-module, E and D are two submodules of Y. Then:

(1) $E\subseteq App-rad_Y(E)$.

(2) If $E\subseteq D$, then $App-rad_Y(E)\subseteq App-rad_Y(D)$.

(3) $App-rad_Y\big(App-rad_Y(E)\big)=App-rad_Y(E)$.

(4) $App-rad_Y(E\cap D)\subseteq App-rad_Y(E)\cap App-rad_Y(D)$.

(5) $App-rad_Y(E+D)=App-rad_Y\big(App-rad_Y(E)+App-rad_Y(D)\big)$.

Proof

(1) Since $App-rad_Y(E)=\cap K$ where the intersection runs over all app-prime submodules K of Y with $E\subseteq K$, so $E\subseteq App-rad_Y(E)$.

(2) Suppose that $E\subseteq D$, and let K be an app-prime submodule of Y with $D\subseteq K$, then $E\subseteq D\subseteq K$, implies that $E\subseteq K$. Thus $App-rad_Y(E)\subseteq App-rad_Y(D)$.

(3) By part (1) we have $App-rad_Y(E)\subseteq App-rad_Y\big(App-rad_Y(E)\big)$. But $App-rad_Y\big(App-rad_Y(E)\big)=\cap N$, where the intersection runs over all app-prime submodules N of Y with $App-rad_Y(E)\subseteq N$, again by (1) $E\subseteq App-rad_Y(E)$. Thus $App-$

$rad_Y(App - rad_Y(E)) \subseteq App - rad_Y(E)$. Hence $App - rad_Y(App - rad_Y(E)) = App - rad_Y(E)$.

(4) Let K be an app-prime submodule of Y contining E and D. Since $E \cap D \subseteq D \subseteq K$, so : $App - rad_Y(E \cap D) \subseteq K$. Thus $App - rad_Y(E \cap D) \subseteq App - rad_Y(D)$. By same way $App - rad_Y(E \cap D) \subseteq App - rad_Y(E)$.

Hence $App - rad_Y(E \cap D) \subseteq App - rad_Y(E) \cap App - rad_Y(D)$.

(5) Since $E + D \subseteq App - rad_Y(E) + App - rad_Y(D)$, then by (2) we have $App - rad_Y(E + D) \subseteq App - rad_Y(App - rad_Y(E) + App - rad_Y(D))$. Now, let K be an app-prime submodule of Y with $E + D \subseteq K$, we prove that $App - rad_Y(E) + App - rad_Y(D) \subseteq K$. Since $E + D \subseteq K$ and $E \subseteq E + D$, $D \subseteq E + D$, then $E \subseteq K$ and $D \subseteq K$. Hence $App - rad_Y(E) + App - rad_Y(D) \subseteq K$. Therefore $App - rad_Y(App - rad_Y(E) + App - rad_Y(D)) \subseteq App - rad_Y(E + D)$ and we have $App - rad_Y(App - rad_Y(E) + App - rad_Y(D)) = App - rad_Y(E + D)$.

Recall that a submodule E of an R-module Y is called completely irreducible, if for any submodules L_1, L_2 of Y, $L_1 \cap L_2 \subseteq E$, implies that either $L_1 \subseteq E$ or $L_2 \subseteq E$ [11].

Proposition (38)

Let E and D are two submodules of an R-module Y. If every app-prime submodule of Y which contains $E \cap D$ is a completely irreducible submodule, then $App - rad_Y(E \cap D) = App - rad_Y(E) \cap App - rad_Y(D)$.

Proof

$App - rad_Y(E \cap D) \subseteq App - rad_Y(E) \cap App - rad_Y(D)$ holds by proposition (37)(4). Now, if $App - rad_Y(E \cap D) = Y$, then $App - rad_Y(E) + App - rad_Y(D) = Y$. If $App - rad_Y(E \cap D) \neq Y$, then there exists an app-prime submodule K of Ysuch that $E \cap D \subseteq K$ or $D \subseteq K$, so that either $App - rad_Y(E) \subseteq K$ or $App - rad_Y(D) \subseteq K$ because every app-prime submodule containing $E \cap D$ is completely irreducible, then we have either $App - rad_Y(E) \subseteq App - rad_Y(E \cap D)$ or $App - rad_Y(D) \subseteq App - rad_Y(E \cap D)$. Therefore $App - rad_Y(E \cap D) \subseteq App - rad_Y(E) \cap App - rad_Y(D)$, and hence $App - rad_Y(E \cap D) = App - rad_Y(E) \cap App - rad_Y(D)$.

4. Conclusion

In this paper an approximaitly prime submodules are introduced and studied as a new generalization of prime submodules, also we introduced and studied the approximaitly prime radical of modules.

The main results of this study are the following.

1) A proper submodule E of an R-module Y is an app-prime submodule if and only $ID \subseteq E$, with I is an ideal of R and D is a submodule of Y, implies that either $D \subseteq E + soc(Y)$ or $I \subseteq [E + soc(Y): Y]$.

2) Every prime submodule is an app-prime submodule, while the converse is not true see Remark (4) and Example (5).

3) In multiplication non-singular R-module Y a proper submodule E of Y with $soc(Y) \subseteq E$ is an app-prime submodule of Y if and only if $[E: Y]$ is an app-prime ideal of R.

Also, this result is satisfied if Y is faithful multiplication with $soc(Y) \subseteq E$ see proposition (12).

4) If E is a proper submodule of Y, with $[E + soc(Y): Y]$ is a prime ideal of R. Then $E(S) \subseteq E + soc(Y)$ with $S \cap [E + soc(Y): Y] = \phi$ if and only if E is an app-prime submodule of Y.

5) If I is a maximal ideal of R, with $IY + soc(Y) \neq Y$. Then IY is an app-prime submodule This result is true if Y faithful multiplication (finitely generated multiplication non-singular) see proposition (20), proposition (21).
6) If $[E + soc(Y): Y] = [E + soc(Y): D]$ for each submodule D of Y with $E \subsetneq D$ and $E + soc(Y) \subsetneq E$.Then E is an app-prime.
7) The invers image and homomorphic image of an app-prime submodule is an app-prime submodule see proposition (30), proposition (31).
8) We introduced and studied $App - rad_Y(E)$ and state several basic properties of this notion for example see proposition (36), proposition (37) and proposition (38).

References

1. Dauns, J. Prime Modules. *J. reine Angew, Math*.**1978**, *2*, 156-181.
2. Lu, C.P. Prime Submodules of Modules. *comm. Math. Univ. Sancti Pauli*.**1984**, *33*, 61-69.
3. McCasland, R.L.; Smith, P.F. Prime Submodules of Noetherian Modules, *Rocky Mountain. J. of Math*. **1993**, *23, 3*, 1041-1062.
4. McCasland, R.L.; Moore, M.C.; Smith, P.F. On spectrum of Modules over Commutative Rings. *Comm. Algebra*.**1997**, *25*, *1*, 79-103.
5. Moore, M.E.; Smith, S.J. Prime and Radical Submodules over Commutative Rings. *Comm. Algebra*. **2002**, *30*, *10*, 5037-5064.
6. Ebrahimi, S.A.; Esmaeili, K.S. On McCasland Submodules. *International Math. Forum*. **2007**, *2*, *46*, 2255-2260.
7. Hussin, S.A.WE-Prime submodules and WE-semi-prime submodules. *Ibn-Al-Haitham Journal for Pure and Applied Science*. **2018**, *3*, *3*, 109-117.
8. Hussin, W.A. WN-2-Absorbing and WES- 2-Absorbing submoduled. *Ibn-Al-Haitham Jornal for Pure and Applied Science*.**2018**, *31*, *3*, 118-125.
9. Gooderal, K.R. *Ring Theory, Nonsingular Ring and Modules*. Marcel. Dekker. New York. **1976**.
10. Abd El-Bast, Z.; Smith, P.F. Multiplication Modules. *Comm. Algebra*.**1988**, *16*, *4*, 755-779.
11. Larsen, M.D.; McCarthy, P.J. *Multiplicative Theory of Ideals*. Academic press. New York and London. **1971**
12. Zalmanowiz, J. *Dense Rings of Liner transformations.* Ring Theory II. Marcel Dekker New York. **1997**
13. Anderson, F.W.; Fuller, K.R. *Rings and Categories of Modules*. Springer-Verlag. New York. **1992**
14. Athab, E.A. Prime and Semi prime Submodules. M.Sc. Thesis. College of Science, University of Baghdad. **1996**
15. Kasch, F. *Modules and Rings*. London Math. Soc. Monographs. New York. **1982**