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Pseudo Quasi-2-Absorbing Submodules and Some Related Concepts

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Abstract

Let R be a ring and let A be a unitary left R-module. A proper submodule H of an R-module A is called 2-absorbing , if $rsa \in H$, where $r, s \in R, a \in A$, implies that either $ra \in H$ or $sa \in H$ or $rs \in [H:A]$, and a proper submodule H of an R-module A is called quasi-prime , if $rsa \in H$, where $r, s \in R, a \in A$, implies that either $ra \in H$ or $sa \in H$. This led us to introduce the concept pseudo quasi-2-absorbing submodule, as a generalization of both concepts above, where a proper submodule H of an R-module A is called a pseudo quasi-2-absorbing submodule of A, if whenever $rsta \in H$,where $r, s, t \in R, a \in A$, implies that either $rsa \in H + soc(A)$ or $sta \in H + soc(A)$ or $rta \in H + soc(A)$, where $soc(A)$ is socal of an R-module A. Several basic properties, examples and characterizations of this concept are given. Moreover, we investigate relationships between pseudo quasi-2-absorbing submodule and other classes of submodules.

Keywords: Prime submodules, quasi-prime submodules, 2-absorbing submodules, quasi-2-absorbing submodules, pseudo quasi-2-absorbing submodules.

1. Introduction and Preliminaries

Throughout this dissertation all ring is commutative with identity and all R-modules are left unitary. A proper submodule H of an R-module A is called a prime submodule if whenever $ra \in H$, with $r \in R, a \in A$, implies that either $a \in H$ or $r \in [H:A]$ [1]. Prime submodules play an important role in the module theory over a commutative ring. There are several generalizations of the notion of prime submodules such as, quasi prime submodule, where a proper submodule H of an R-module A is called a quasi-prime, if whenever $rsa \in H$, with $r, s \in R, a \in A$, implies that either $ra \in H$ or $sa \in H$ [2]. WE-prime submodules and WE-semi prime submodules which appear in [3]. The concept of prime submodule was generalized by Darani and Soheilnia to 2-absorbing submodule, where a proper submodule H of an R-module A is called 2-absorbing, if whenever $rsa \in H$, with $r, s \in R, a \in A$, implies that either $ra \in H$ or $sa \in H$ or $rs \in [H:A]$[4]. There are several generalizations of 2-absorbing submodules such as WN-2-absorbing submodules and WNS- 2-absorbing submodules which appear in [5]. The concept of quasi-2-absorbing submodule, was introduced in 2018 as a generalization of 2-absorbing submodule, where a proper submodule H of an R-module A is called a quasi-2-absorbing, if whenever $rsta \in H$, with $r,s,t \in R, a \in A$, implies that either $rsa \in H$ or $sta \in H$ or $rta \in H$[6]. In this paper we establish new concept called pseudo quasi-2-absorbing submodule as generalization of (prime, quasi-prime, 2-absorbing and quasi-2-absorbing) submodules. Several basic properties examples, and relationships of pseudo quasi-2-absorbing submodules, with other classes of submodules are studied. Socle of a module A denoted by $soc(A)$ defined to be the intersection of all essential submodules of A [7]. Where a submodule H of an R-module A is called essential, if H has non-zero intersection with every non-zero submodule of A [7]. Recall that a non-zero proper ideal I of R is called 2-absorbing ideal of R, if whenever $a,b,c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$[8]. Recall that an R-module A is multiplication if every submodule H of A is of the form $H = IA$ for some ideal I of R [9].

2. Pseudo quasi-2-Absorbing Submodules

In this section, we introduced the definition of a pseudo quasi-2-absorbing submodule

Definition (1)

A proper submodule H of an R-module A is called a pseudo quasi-2-absorbing submodule, if whenever $rsta \in H$, with $r,s,t \in R, a \in A$, implies that either $rsa \in H + soc(A)$ or $rta \in H + soc(A)$ or $sta \in H + soc(A)$. And a proper ideal I of a ring R is called a pseudo quasi-2-absorbing, if I is a pseudo quasi-2-absorbing submodule of an R-module R.

The following proposition gives characterization of a pseudo quasi-2-absorbing submodules.

Proposition (2)

Let A be an R-module, and K is a submodule of A. Then K is a pseudo quasi-2-absorbing submodule of A if and only if for every ideals J_1, J_2, J_3 of R and submodule L of A with $J_1J_2J_3L \subseteq K$, implies that either $J_1J_2L \subseteq K + soc(A)$ or $J_1J_3L \subseteq K + soc(A)$ or $J_2J_3L \subseteq K + soc(A)$.

Proof

($\Longrightarrow$) Suppose that $J_1J_2J_3L \subseteq K$, where J_1, J_2, J_3 are ideals of R, and L is a submodule of A with $J_1J_2L \nsubseteq K + soc(A)$ and $J_1J_3L \nsubseteq K + soc(A)$ and $J_2J_3L \nsubseteq K + soc(A)$. Thus, there exists $x_1, x_2, x_3 \in L$ and $r_1 \in J_1, r_2 \in J_2$ and $r_3 \in J_3$ such that $r_1r_2x_1 \notin K + soc(A)$ and $r_1r_3x_2 \notin K + soc(A)$ and $r_2r_3x_3 \notin K + soc(A)$. But $r_1r_2r_3x_1 \in K$, and K is a pseudo quasi-2-absorbing submodule of A, with $r_1r_2x_1 \notin K + soc(A)$, then we have $r_1r_3x_1 \in K + soc(A)$ or $r_2r_3x_1 \in K + soc(A)$. Again $r_1r_2r_3x_2 \in K$ and $r_1r_3x_2 \notin K + soc(A)$, implies that either $r_1r_2x_2 \in K + soc(A)$ or $r_2r_3x_2 \in K + soc(A)$. Also, $r_1r_2r_3x_3 \in K$ and $r_2r_3x_3 \notin K + soc(A)$, implies that either $r_1r_2x_3 \in K + soc(A)$ or $r_1r_3x_3 \in K + soc(A)$. Thus either $J_1J_2L \subseteq K + soc(A)$ or $J_1J_3L \subseteq K + soc(A)$ or $J_2J_3L \subseteq K + soc(A)$.

($\Longleftarrow$) Suppose that $rsta \in K$, where $r,s,t \in R, a \in A$ then $(r)(s)(t)(a) \subseteq K$, so by hypothesis, either $(r)(s)(a) \subseteq K + soc(A)$ or $(r)(t)(a) \subseteq K + soc(A)$ or $(s)(t)(a) \subseteq K + soc(A)$. Thus either $rsa \in K + soc(A)$ or $rta \in K + soc(A)$ or $sta \in K + soc(A)$. Hence K is a pseudo quasi-2-absorbing submodule of A.

As a direct consequence of proposition (2) we get the following result.

Corollary (3)

Let A be an R-module, and K is a submodule of A. Then K is a pseudo quasi-2-absorbing submodule of A if and only if for each $r,s,t \in R$ and for each submodule L of A with $rstL \subseteq K$, implies that either $rsL \subseteq K + soc(A)$ or $rtL \subseteq K + soc(A)$ or $stL \subseteq K + soc(A)$.

Remarks and Examples (4)

1- It is clear that every quasi-prime submodule of an R-module A is a pseudo quasi-2-absorbing submodule of A, while the converse is not true in general. For the converse consider the following example:

In the Z-module Z_4 , the submodule $H = \langle\bar{0}\rangle$ is pseudo quasi-2-absorbing, but not quasi-prime since $2.2.\bar{1} \in \langle\bar{0}\rangle$, but $2.\bar{1} \notin \langle\bar{0}\rangle$. Since $soc(Z_4) = \langle\bar{2}\rangle$, it is clear that for each $r,s \in Z$ and $a \in Z_4$, if $rsa \in \langle\bar{0}\rangle$, implies that either $ra \in \langle\bar{0}\rangle + soc(Z_4)$ or $sa \in \langle\bar{0}\rangle + soc(Z_4)$ or $rs \in [\langle\bar{0}\rangle + soc(Z_4): Z_4]$.

2- It is clear that every prime submodule of an R-module A is a pseudo quasi-2-absorbing submodule of A, while the converse is not true in general. For the converse see the following example:

In the Z-module Z_4 , the submodule $H = \langle\bar{0}\rangle$ is pseudo quasi-2-absorbing, but not prime, since $2.\bar{2} \in H$, but $\bar{2} \notin H$ and $2 \notin [H: Z_4]$.

3- It is clear that every 2-absorbing submodule of an R-module A is a pseudo quasi-2-absorbing submodule of A, while the converse is not true in general. For the converse see the following example:

In the Z-module Z_{12} , the submodule $H = \langle\bar{0}\rangle$ is pseudo quasi-2-absorbing, but not 2-absorbing, since $2.3.\bar{2} \in H$, but $2.\bar{2} = 4 \notin H$ and $3.\bar{2} \notin H$ and $2.3 \notin [H: Z_{12}] = 12Z$. Since $soc(Z_{12}) = \langle\bar{2}\rangle$, it is clear that for all $r,s,t \in Z$ and $a \in Z_{12}$, if $rsta \in \langle\bar{0}\rangle$, implies that either $rsa \in \langle\bar{0}\rangle + soc(Z_{12})$ or $sta \in \langle\bar{0}\rangle + soc(Z_{12})$ or $rta \in \langle\bar{0}\rangle + soc(Z_{12})$.

4- It is clear that every quasi-2-absorbing submodule of an R-module A is a pseudo quasi-2-absorbing submodule of A, while the converse is not true in general. For the converse see the following example:

In the Z-module Z_{24} , the submodule $H = \langle\bar{0}\rangle$ is pseudo quasi-2-absorbing, but not quasi-2-absorbing, since $4.3.2.\bar{1} \in H$, but $4.3.\bar{1} \notin H$ and $4.2.\bar{1} \notin H$ and $3.2.\bar{1} \notin H$. Since $soc(Z_{24}) = \langle\bar{4}\rangle$, it is clear that H is a pseudo quasi-2-absorbing submodule of Z_{24}.

Proposition (5)

Let A be an R-module, and H is a proper submodule of A, with $[H + soc(A): a]$ is 2-absorbing ideal of R for each $a \in A$.Then H is a pseudo quasi-2-absorbing submodule of A.

Proof

Assume that $rsta \in H$, where $r,s,t \in R, a \in A$. Since $rsta \in H \subseteq H + soc(A)$, implies that $rst \in [H + soc(A): a]$. But $[H + soc(A): a]$ is a 2-absorbing ideal of R, then either $rs \in [H + soc(A): a]$ or $rt \in [H + soc(A): a]$ or $st \in [H + soc(A): a]$. That is either $rsa \in H + soc(A)$ or $rta \in H + soc(A)$ or $sta \in H + soc(A)$. Hence H is a pseudo quasi-2-absorbing submodule of A.

Proposition (6)

Let A be an R-module and H is a pseudo quasi-2-absorbing submodule of A, with $soc(A) \subseteq H$. Then $[H: A]$ is 2-absorbing (hence a pseudo quasi-2-absorbing) ideal of R.

Proof

Let $rst \in [H: A]$, where $r, s, t \in R$, then $rstA \subseteq H$, it follows that $rsta \in H$ for all $a \in A$. But H is a pseudo quasi-2-absorbing submodule of A, implies that either $rsa \in H + soc(A)$ or $rta \in H + soc(A)$ or $sta \in H + soc(A)$. But $soc(A) \subseteq H$, implies that $H + soc(A) = H$. That is either $rsa \in H$ or $rta \in H$ or $sta \in H$ for all $a \in A$. Hence either $rsA \subseteq H$ or $rtA \subseteq H$ or $stA \subseteq H$. Thus either $rs \in [H: A]$ or $rt \in [H: A]$ or $st \in [H: A]$. That is $[H: A]$ is 2-absorbing ideal of R, hence is a pseudo quasi-2-absorbing ideal of R.

Recall that an R-module A is called faithful if $ann(A) = 0$ [7].

Before we give the converse of proposition (2.6), we recalled the following lemmas.

Lemma (7) [9, coro 2.14]**.**

Let M be faithful multiplication R-module, then $soc(R)M = soc(M)$.

Proposition (8)

Let A be a faithful multiplication R-module and H is a proper submodule of A. If $[H: A]$ is a pseudo quasi-2-absorbing ideal of R, then H is a pseudo quasi-2-absorbing submodule of A

Proof

Let $rsta \in H$, with $r, s, t \in R, a \in A$, then $rst(a) \subseteq H$. But A is a multiplication, so $(a) = IA$ for some ideal I of R. Thus $rstIA \subseteq H$, it follows that $rstI \subseteq [H: A]$. But $[H: A]$ is a pseudo quasi-2-absorbing ideal of R, then by corollary (3) either $rsI \subseteq [H: A] + soc(R)$ or $stI \subseteq [H: A] + soc(R)$ or $rtI \subseteq [H: A] + soc(R)$. Hence either $rsIA \subseteq [H: A]A + soc(R)A$ or $stIA \subseteq [H: A]A + soc(R)A$ or $rtIA \subseteq [H: A]A + soc(R)A$. But $[H: A]A = H$ and by lemma (7) $soc(R)A = soc(A)$. Thus, either $rs(a) \subseteq H + soc(A)$ or $st(a) \subseteq H + soc(A)$ or $rt(a) \subseteq H + soc(A)$. Therefore H is a pseudo quasi-2-absorbing submodule of A.

Recall that an R-module A is called singular module provided $Z(A) = A$. At the other extreme, we say that A is non-singular module provided $Z(A) = 0$, where $Z(A) = \{x \in A : xI = 0 \ for\ some\ I \in \mathcal{T}(R)\}$, where $\mathcal{T}(R)$ is the set of all essential right ideals of the ring R [7].

Lemma (9) [7, coro 1.2.6].

If M is a non-singular R-module, then $soc(R)M = soc(M)$.

Proposition (10)

Let A be a non-singular multiplication R-module and H is a proper submodule of A. If $[H: A]$ is pseudo quasi-2-absorbing ideal of R, then H is a pseudo quasi-2-absorbing submodule of A.

Proof

Similarly, as in proposition (6), by using lemma (9).

Lemma (11)[10, coro of Theo. 9].

Let I and J be ideals of a ring R and M is a finitely generated multiplication R-module. Then $IM \subseteq JM$ if and only if $I \subseteq J + annM$.

Proposition (12)

Let A be a faithful finitely generated multiplication R-module. If I is a pseudo quasi-2-absorbing ideal of R, then IA is a pseudo quasi-2-absorbing submodule of A.

Proof

Let $rsta \in IA$, where $r, s, t \in R, a \in A$, that is $rst(a) \subseteq IA$. But A is a multiplication, then $(a) = JA$ for some ideal J of R. Thus $rstJA \subseteq IA$, and so by lemma (11), $rstJ \subseteq I + annA$, but A is faithful, then $annA = (0)$, hence $rstJ \subseteq I$. But I is a pseudo quasi-2-

absorbing ideal of R, then by corollary (3) either $rsJ \subseteq I + soc(R)$ or $rtJ \subseteq I + soc(R)$ or $stJ \subseteq I + soc(R)$. Thus either $rsJA \subseteq IA + soc(R)A$ or $rtJA \subseteq IA + soc(R)A$ or $stJA \subseteq IA + soc(R)A$. But by lemma (7) $soc(R)A = soc(A)$. Hence either $rsJA \subseteq IA + soc(A)$ or $rtJA \subseteq IA + soc(A)$ or $stJA \subseteq IA + soc(A)$, it follows that $rs(a) \in IA + soc(A)$ or $rt(a) \in IA + soc(A)$ or $st(a) \in IA + soc(A)$. Therefore IA is a pseudo quasi-2-absorbing submodule of A.

By using lemma (9) and lemma (11) we get the following result.

Proposition (13)

Let A be a finitely generated multiplication non-singular R-module. If I is a pseudo quasi-2-absorbing ideal of R with $annA \subseteq I$, then IA is a pseudo quasi-2-absorbing submodule of A.

Proof

Similar steps of proposition (12).

Proposition (14)

Let A be an R-module and H is a proper submodule of A, with $soc(A) \subseteq H$.Then H is a pseudo quasi-2-absorbing submodule of A if and only if $[H:_A rst] = [H:_A rs] \cup [H:_A rt] \cup [H:_A st]$ for all $r, s, t \in R$.

Proof

($\Longrightarrow$) Let $a \in [H:_A rst]$, implies that $rsta \in H$. But H is a pseudo quasi-2-absorbing submodule of A, then either $rsa \in H + soc(A)$ or $rta \in H + soc(A)$ or $sta \in H + soc(A)$. But $soc(A) \subseteq H$, then $H + soc(A) = H$, it follows that either $rsa \in H$ or $rta \in H$ or $sta \in H$, implies that either $a \in [H:_A rs]$ or $a \in [H:_A rt]$ or $a \in [H:_A st]$. That is $a \in [H:_A rs] \cup [H:_A rt] \cup [H:_A st]$. Clearly that $[H:_A rs] \cup [H:_A rt] \cup [H:_A st] \subseteq [H:_A rst]$. Thus $[H:_A rst] = [H:_A rs] \cup [H:_A rt] \cup [H:_A st]$.

($\Longleftarrow$) Suppose that $rsta \in H$ with $r, s, t \in R, a \in A$, implies that $a \in [H:_A rst] = [H:_A rs] \cup [H:_A rt] \cup [H:_A st]$, implies that either $a \in [H:_A rs]$ or $a \in [H:_A rt]$ or $a \in [H:_A st]$. That is either $rsa \in H \subseteq H + soc(A)$ or $rta \in H \subseteq H + soc(A)$ or $sta \in H \subseteq H + soc(A)$. Hence H is a pseudo quasi-2-absorbing submodule of A.

Proposition (15)

Let A be an R-module and H is a proper submodule of A, with $soc(A) \subseteq H$.Then H is a pseudo quasi-2-absorbing submodule of A if and only if $[H:_R rsa] = [H:_R ra] \cup [H:_R sa]$ for all $r, s \in R, a \in A$.

Proof

($\Longrightarrow$) Let $t \in [H:_R rsa]$, implies that $rsta \in H$. But H is a pseudo quasi-2-absorbing submodule of A, and let $rsa \notin H + soc(A)$, then either $rta \in H + soc(A)$ or $sta \in H + soc(A)$. But $soc(A) \subseteq H$, then $H + soc(A) = H$. Hence either $rta \in H$ or $sta \in H$. Thus either $t \in [H:_R ra]$ or $t \in [H:_R sa]$, it follows that $t \in [H:_R ra] \cup [H:_R sa]$, hence $[H:_R rsa] \subseteq [H:_R ra] \cup [H:_R sa]$. Consequently $[H:_R rsa] = [H:_R ra] \cup [H:_R sa]$.

($\Longleftarrow$) Suppose that $rsta \in H$ with $r, s, t \in R, a \in A$, with $rsa \notin H + soc(A)$, then $t \in [H:_R rsa] = [H:_R ra] \cup [H:_R sa]$, implies that either $t \in [H:_R ra]$ or $t \in [H:_R sa]$, it follows that $rta \in H \subseteq H + soc(A)$ or $sta \in H \subseteq H + soc(A)$, hence either $rta \in H + soc(A)$ or $sta \in H + soc(A)$. Therefore H is a pseudo quasi-2-absorbing submodule of A.

Proposition (16)

Let A be an R-module and H is a pseudo quasi-2-absorbing submodule of A, with $soc(A) \subseteq H$, then $[H:_R rsta] = [H:_R rsa] \cup [H:_R rta] \cup [H:_R sta]$ for all $r, s, t \in R, a \in A$.

Proof

Let $c \in [H:_R rsta]$, implies that $rst(ca) \in H$. But H is a pseudo quasi-2-absorbing submodule of A, then either $rs(ca) \in H + soc(A)$ or $rt(ca) \in H + soc(A)$ or $st(ca) \in H + soc(A)$. But $soc(A) \subseteq H$, then $H + soc(A) = H$, it follows that either $rsca \in H$ or $rtca \in H$ or $stca \in H$. Hence $c \in [H:_R rsa]$ or $c \in [H:_R rta]$ or $c \in [H:_R sta]$. Therefore $c \in [H:_R rsa] \cup [H:_R rta] \cup [H:_R sta]$, hence$[H:_R rsta] \subseteq [H:_R rsa] \cup [H:_R rta] \cup [H:_R sta]$. Consequently, the equality holds.

Proposition (17)

Let A be an R-module, H and K are submodules of A such that $K \subsetneq H$ and H is an essential submodule of A. If K is a pseudo quasi-2-absorbing submodule of A, then K is a pseudo quasi-2-absorbing submodule of H.

Proof

Let $rsta \in K$, with $r, s, t \in R, a \in H$. Since K is a pseudo quasi-2-absorbing submodule of A, then either $rsa \in K + soc(A)$ or $rta \in K + soc(A)$ or $sta \in K + soc(A)$. But H is essential submodule of A, then by $[5, \text{Exs}. 10]$. we have $soc(A) = soc(H)$. Hence either $rsa \in K + soc(H)$ or $rta \in K + soc(H)$ or $sta \in K + soc(H)$. Therefore K is a pseudo quasi-2-absorbing submodule in H.

Proposition (18)

Let A be an R-module, H and K are submodules of A such that $K \subsetneq H$ and $soc(A) \subseteq soc(H)$. If K is a pseudo quasi-2-absorbing submodule of A, then K is a pseudo quasi-2-absorbing submodule of H.

Proof: Similar steps as proposition (17).

Remark (19)

The intersection of two pseudo quasi-2-absorbing submodules of an R-module A need not to be pseudo quasi-2-absorbing submodule of A, as the following example explains that:
In the Z-module Z, the submodules $5Z$ and $4Z$ are pseudo quasi-2-absorbing submodules of Z, but $5Z \cap 4Z = 20Z$ is not a pseudo quasi-2-absorbing of Z, since $2.2.5.1 \in 20Z$, but $2.2.1 \notin 20Z + soc(Z)$ and $2.5.1 \notin 20Z + soc(Z)$.

Proposition (20)

Let A be an R-module, H_1 and H_2 are pseudo quasi-2-absorbing submodules of A, with $H_1 \subseteq soc(A)$ and $H_2 \subseteq soc(A)$. Then $H_1 \cap H_2$ is a pseudo quasi-2-absorbing submodule of A.

Proof

Let $rsta \in H_1 \cap H_2$, where $r, s, t \in R, a \in A$, then $rsta \in H_1$ and $rsta \in H_2$. But H_1 and H_2 are pseudo quasi-2-absorbing submodules of A, so either $rsa \in H_1 + soc(A)$ or $rta \in H_1 + soc(A)$ or $sta \in H_1 + soc(A)$ and either $rsa \in H_2 + soc(A)$ or $rta \in H_2 + soc(A)$ or $sta \in H_2 + soc(A)$. But $H_1 \subseteq soc(A)$ and $H_2 \subseteq soc(A)$, implies that $H_1 \cap H_2 \subseteq soc(A)$, and hence $H_1 + soc(A) = soc(A)$, $H_2 + soc(A) = soc(A)$ and $H_1 \cap H_2 + soc(A) = soc(A)$. Thus either $rsa \in H_1 \cap H_2 + soc(A)$ or $rta \in H_1 \cap H_2 + soc(A)$ or $sta \in H_1 \cap H_2 + soc(A)$. Hence $H_1 \cap H_2$ is a pseudo quasi-2-absorbing submodule of A.

Proposition (21)

Let A be an R-module and H_1 is a pseudo quasi-2-absorbing submodule of A and H_2 is a submodule of A, with $soc(A) \subseteq H_2$ and H_2 is not contained in H_1. Then $H_1 \cap H_2$ is a pseudo quasi-2-absorbing submodule of A.

Proof

Since H_2 is not contained in H_1, then $H_1 \cap H_2$ is a proper submodule of H_2. Let $rsta \in H_1 \cap H_2$, where $r,s,t \in R, a \in H_2$, then $rsta \in H_1$ and $rsta \in H_2$. But H_1 is a pseudo quasi-2-absorbing submodule of A, then either $rsa \in H_1 + soc(A)$ or $rta \in H_1 + soc(A)$ or $sta \in H_1 + soc(A)$. But $a \in H_2$, it follows that either $rsa \in (H_1 + soc(A)) \cap H_2$ or $rta \in (H_1 + soc(A)) \cap H_2$ or $sta \in (H_1 + soc(A)) \cap H_2$. By hypothesis $soc(A) \subseteq H_2$, then by $[11, lemma\ 2.3.15]$. we have either $rsa \in (H_1 \cap H_2) + (soc(A) \cap H_2)$ or $rta \in (H_1 \cap H_2) + (soc(A) \cap H_2)$ or $sta \in (H_1 \cap H_2) + (soc(A) \cap H_2)$. But by $[12, coro. 9.9]$. $soc(H_2) = soc(A) \cap H_2$. Hence either $rsa \in (H_1 \cap H_2) + soc(H_2)$ or $rta \in (H_1 \cap H_2) + soc(H_2)$ or $sta \in (H_1 \cap H_2) + soc(H_2)$. Therefore $H_1 \cap H_2$ is a pseudo quasi-2-absorbing submodule of A.

Proposition (22)

Let $h \in Hom(A, \acute{A})$ be an R-epimorphism and K is a pseudo quasi-2-absorbing submodule of $\acute{A}$. Then $h^{-1}(K)$ is a pseudo quasi-2-absorbing submodule of A.

Proof

It is clear that $h^{-1}(K)$ is a proper submodule of A. Let $rsta \in h^{-1}(K)$, where $r,s,t \in R, a \in A$, then $rsth(a) \in K$, as K is a pseudo quasi-2-absorbing submodule of $\acute{A}$, implies that either $rsh(a) \in K + soc(\acute{A})$ or $rth(a) \in K + soc(\acute{A})$ or $sth(a) \in K + soc(\acute{A})$. That is either$rsa \in h^{-1}(K) + h^{-1}\left(soc(\acute{A})\right) \subseteq h^{-1}(K) + soc(A)$ or $rta \in h^{-1}(K) + h^{-1}\left(soc(\acute{A})\right) \subseteq h^{-1}(K) + soc(A)$ or $sta \in h^{-1}(K) + h^{-1}\left(soc(\acute{A})\right) \subseteq h^{-1}(K) + soc(A)$, it follows that either $rsa \in h^{-1}(K) + soc(A)$ or $rta \in h^{-1}(K) + soc(A)$ or $sta \in h^{-1}(K) + soc(A)$. Therefore $h^{-1}(K)$ is a pseudo quasi-2-absorbing submodule of A.

Proposition (23)

Let $h \in Hom(A, \acute{A})$ be an R-epimorphism and L is a pseudo quasi-2-absorbing submodule of A, with $kerh \subseteq L$. Then $h(L)$ is a pseudo quasi-2-absorbing submodule of $\acute{A}$.

Proof

$h(L)$ is a proper submodule of $\acute{A}$, if not that is $h(L) = \acute{A}$. Let $a \in A$ then $h(a) \in \acute{A} = h(L)$, then $h(l) = h(a)$ for some $l \in L$, hence $h(l - a) = 0$, implies that $l - a \in kerh \subseteq L$, it follows that $a \in L$, hence $L = A$ contradiction. Let $rst\acute{a} \in h(L)$, where $r,s,t \in R$ and $\acute{a} \in \acute{A}$, since h is on to, then $\acute{a} = h(a)$ for some $a \in A$, hence $rsth(a) \in h(L)$, it follows that $rsth(a) = h(l)$ for some $l \in L$, that is $h(rsta - l) = 0$, implies that $rsta - l \in kerh \subseteq L$, it follows that $rsta \in L$. But L is a pseudo quasi-2-absorbing submodule of A, then either $rsa \in L + soc(A)$ or $rta \in L + soc(A)$ or $sta \in L + soc(A)$. It follows that either $rsh(a) \in h\big(L + soc(A)\big) \subseteq h(L) + soc(\acute{A})$ or $rth(a) \in h\big(L + soc(A)\big) \subseteq h(L) + soc(\acute{A})$ or $sth(a) \in h\big(L + soc(A)\big) \subseteq h(L) + soc(\acute{A})$. That is either $rs\acute{a} \in h(L) + soc(\acute{A})$ or $rt\acute{a} \in h(L) + soc(\acute{A})$ or $st\acute{a} \in h(L) + soc(\acute{A})$. Therefore $h(L)$ is a pseudo quasi-2-absorbing submodule of $\acute{A}$.

Proposition (24)

Let $A = A_1 \oplus A_2$ be an R-module, where A_1, A_2 are R-modules and $H = H_1 \oplus H_2$ be a submodule of A, where H_1, H_2 are submodules of A_1, A_2 respectively, with $H \subseteq soc(A)$. If H is a pseudo quasi-2-absorbing submodule of A, then H_1 and H_2 are pseudo quasi-2-absorbing submodules of A_1, A_2 respectively.

Proof

Let $rsta_1 \in H_1$, where $r, s, t \in R, a_1 \in A_1$, then $rst(a_1, 0) \in H$. But H is a pseudo quasi-2-absorbing submodule of A, then either $rs(a_1, 0) \in H + soc(A)$ or $rt(a_1, 0) \in H + soc(A)$ or $st(a_1, 0) \in H + soc(A)$. But $H \subseteq soc(A)$, implies that $H + soc(A) = soc(A)$ and $soc(A) = soc(A_1) \oplus soc(A_2)$. If $rs(a_1, 0) \in soc(A_1) \oplus soc(A_2)$, implies that $rsa_1 \in soc(A_1) \subseteq H_1 + soc(A_1)$. If $rt(a_1, 0) \in soc(A_1) \oplus soc(A_2)$, implies that $rta_1 \in soc(A_1) \subseteq H_1 + soc(A_1)$, also in similar way we get $sta_1 \in H_1 + soc(A_1)$. Therefore H_1 is a pseudo quasi-2-absorbing submodule of A_1.

Similarly, H_2 is a pseudo quasi-2-absorbing submodule of A_2.

Proposition (25)

Let A_1, A_2 be two R-modules and $A = A_1 \oplus A_2$. Then

a) H is a pseudo quasi-2-absorbing submodule of A_1, with $H \subseteq soc(A_1)$ and A_2 is a semi simple if and only if $H \oplus A_2$ is a pseudo quasi-2-absorbing submodule of A.

b) K is a pseudo quasi-2-absorbing submodule of A_2, with $K \subseteq soc(A_2)$ and A_1 is a semi simple if and only if $K \oplus A_1$ is a pseudo quasi-2-absorbing submodule of A.

Proof

a) $(\Longrightarrow)$ Let $rst(a_1, a_2) \in H \oplus A_2$, where $r, s, t \in R$ and $(a_1, a_2) \in A$, implies that $rsta_1 \in H$ and $rsta_2 \in A_2$. Since H is a pseudo quasi-2-absorbing submodule of A_1, and $H \subseteq soc(A_1)$, then either $rsa_1 \in H + soc(A_1) = soc(A_1)$ or $rta_1 \in H + soc(A_1) = soc(A_1)$ or $sta_1 \in H + soc(A_1) = soc(A_1)$. Since A_2 is a semi simple, then by $[10, P\ 121]$. $soc(A_2) = A_2$, then $rs(a_1, a_2) \in soc(A_1) \oplus A_2 = soc(A_1) \oplus soc(A_2) = soc(A_1 \oplus A_2) \subseteq H \oplus A_2 + soc(A_1 \oplus A_2)$. If

$rt(a_1, a_2) \in soc(A_1) \oplus A_2 = soc(A_1) \oplus soc(A_2) = soc(A_1 \oplus A_2) \subseteq H \oplus A_2 + soc(A_1 \oplus A_2)$.

If $st(a_1, a_2) \in soc(A_1) \oplus A_2 = soc(A_1) \oplus soc(A_2) = soc(A_1 \oplus A_2) \subseteq H \oplus A_2 + soc(A_1 \oplus A_2)$.

Hence $H \oplus A_2$ is a pseudo quasi-2-absorbing submodule of A.

$(\Longleftarrow)$ Let $rsta_1 \in H$, where $r, s, t \in R, a_1 \in A_1$, then for each $a_2 \in A_2$, $rst(a_1, a_2) \in H \oplus A_2$. But $H \oplus A_2$ is a pseudo quasi-2-absorbing submodule of A, so either $rs(a_1, a_2) \in H \oplus A_2 + soc(A_1 \oplus A_2)$ or $rt(a_1, a_2) \in H \oplus A_2 + soc(A_1 \oplus A_2)$ or $st(a_1, a_2) \in H \oplus A_2 + soc(A_1 \oplus A_2)$. If

$rs(a_1, a_2) \in H \oplus A_2 + soc(A_1) \oplus soc(A_2) = H \oplus A_2 + soc(A_1) \oplus A_2 = H \oplus A_2 + \big(H + soc(A_1)\big) \oplus A_2 = (H + soc(A_1)) \oplus A_2$. It follows that $rta_1 \in H + soc(A_1)$.

Similarly, if $rt(a_1, a_2) \in H \oplus A_2 + soc(A_1) \oplus soc(A_2)$, implies that:

$rt(a_1, a_2) \in (H + soc(A_1)) \oplus A_2$, it follows that $rta_1 \in H + soc(A_1)$.

Similarly, we get $sta_1 \in H + soc(A_1)$. Therefore H is a pseudo quasi-2-absorbing submodule of A_1.

The proof of (b) is similarly.

3. Conclusion

In this research we introduce and study the concept pseudo quasi-2-absorbing submodules as a generalization of quasi-prime and 2-absorbing submodules. The main results of this study are the following:

1- A proper submodule K of an R-module A is a pseudo quasi-2-absorbing submodule of A if and only if for every ideals J_1, J_2, J_3 of R and submodule L of A with $J_1J_2J_3L \subseteq K$, implies that either $J_1J_2L \subseteq K + soc(A)$ or $J_1J_3L \subseteq K + soc(A)$ or $J_2J_3L \subseteq K + soc(A)$.

2- Let A be a faithful finitely generated multiplication R-module and I is a pseudo quasi-2-absorbing ideal of R, then IA is a pseudo quasi-2-absorbing submodule of A.

3- A proper submodule H of an R-module A, with $soc(A) \subseteq H$ is a pseudo quasi-2-absorbing submodule of A if and only if $[H:_A rst] = [H:_A rs] \cup [H:_A rt] \cup [H:_A st]$ for all $r, s, t \in R$.

4- A proper submodule H of an R-module A, with $soc(A) \subseteq H$ is a pseudo quasi-2-absorbing submodule of A if and only if $[H:_R rsa] = [H:_R ra] \cup [H:_R sa]$ for all $r, s \in R, a \in A$.

5- The intersection of two pseudo quasi-2-absorbing submodules of an R-module A need not to be pseudo quasi-2-absorbing. This explain by example see Remark (19). But under certain conditions the intersection are satisfies see Proposition (20).

6- The inverse image and homomorphism image of pseudo quasi-2-absorbing submodule is pseudo quasi-2-absorbing see Proposition (22), (23).

7- The direct summand of pseudo quasi2-absorbing submodule is a pseudo quasi-2-absorbing submodule see Proposition (24).

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