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New Technique for Solving Autonomous Equations

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Abstract

This paper demonstrates a new technique based on a combined form of the new transform method with homotopy perturbation method to find the suitable accurate solution of autonomous Equations with initial condition. This technique is called the transform homotopy perturbation method (THPM). It can be used to solve the problems without resorting to the frequency domain. The implementation of the suggested method demonstrates the usefulness in finding exact solution for linear and nonlinear problems. The practical results show the efficiency and reliability of technique and easier implemented than HPM in finding exact solutions. Finally, all algorithms in this paper implemented in MATLAB version 7.12.

Keywords: Initial value problem, Autonomous Equation, Homotopy perturbation method, Transformation.

1. Introduction

The most of the phenomena that arise in mathematical physics and engineering fields can be described by partial differential Equations (PDEs), for example, in chemical diffusion, the heat flow, thermo elasticity and the wave propagation phenomena are well described by PDEs [1–4]. So, it's important tool for describing natural phenomena of science and engineering models. Most of these problems are difficult to solve them analytically. So, the researchers due to the get approximate solution. The importance of obtaining the exact or approximate solution of nonlinear PDEs is still a significant problem that needs new methods to get exact or approximate solutions. In the last two decades, various powerful methods have been proposed for obtaining exact and approximate analytic solutions. Among these are the Adomian decomposition method (ADM) [5,6]. the homotopy perturbation method (HPM) [7, 8]. the homotopy analysis method (HAM), the differential transform method, Laplace decomposition method [9]. and the variational iteration method (VIM) [10,11].

In this research we suggest new technique to solve one of the most important of amplitude Equations is the autonomous Equation which describes the appearance of the stripe pattern in two dimensional systems. Moreover, this Equation was applied to a number of problems in variety systems, e.g., Rayleigh-Benard convection, Faraday instability, nonlinear optics, chemical reactions and biological systems. The approximate solutions of the autonomous

Equation were presented by differential transformation method [12]. reduce differential transformation [13]. In this paper a reliable transform homotopy perturbation method is proposed and applied for solving autonomous Equation. The technique can be employed to linear and nonlinear problems. Moreover, some examples are illustrative for demonstrating the advantage of the technique.

2. Partial Differential Equations

Partial differential Equations are a type of differential Equation, i.e, a relation involving an unknown function (or functions say dependent variables) of several independent variables and their partial derivatives with respect to those variables. Partial differential Equations appear frequently in all areas of physics and engineering. In recent years, we have seen a dramatic increase in the use of these Equations in areas such biology, chemistry, chemical engineering, computer science (partially in relation to image processing and graphics) and economics. A PDE is called linear if the power of the dependent variable and each partial derivative contained in the Equation is one and the coefficients of the dependent variable and the coefficients of each partial derivative are constants or independent variables. However, if any of these conditions is not satisfied, the Equation is called nonlinear [4].

3. New Transform

New transform was introduced by Luma and Alaa [14]. to solve differential Equations and engineering problems. Apart from other advantages of new transform (NT) over other integral transforms such as accuracy and simplicity, in [15]. consist a very interesting fact about this transform.

The new transform of a function $f(t)$, defined by:

$$\bar{f}(u) = \mathbb{T}\{f(t)\} = \int_0^\infty e^{-t} f\left(\frac{t}{u}\right) dt, \tag{1}$$

Here some basic properties of the NT are introduced:

If a, b are constants, f(t) and g(t) are functions having NT, then:

1. Linearity Property: $\mathbb{T}\{af(t) + bg(t)\} = a\mathbb{T}\{f(t)\} + b\mathbb{T}\{g(t)\}$; a, b ϵ R
2. Convolution Property: $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$ (2)
3. $\mathbb{T}\{t^n\} = \frac{n!}{v^n}$, $v \neq 0$, $n = 0,1,2,3,...$
4. Differentiation Property: $\mathbb{T}\{f'\} = v(\mathbb{T}\{f\} - f(0))$

For more details see [14,15].

4. Basic Idea of Suggested Method

To illustrate the ideas of new transform homotopy perturbation method (NTHPM) firstly rewrite the initial value problem in autonomous Equation in the form:

$$u_t(x,t) = cu_{xx}(x,t) + c_1 u(x,t) - c_2 u^n(x,t) \tag{3}$$

With the initial condition: $u\,(x,t) = g(x)$ (4)

Where c_1 and c_2 are real numbers and c and n are positive integers.

Taking new transformation on both sides of the Equation (3) and using the linearity property of the new transform gives:

$$T\{u_t(x,t)\} = cT\{u_{xx}(x,t)\} + c_1 T\{u(x,t)\} - c_2 T\{u^n(x,t)\} \tag{5}$$

By applying the differentiation property of new transform, we have:

$$vT\{u(x,t)\} - vu(x,0) = cT\{u_{xx}(x,t)\} + c_1T\{u(x,t)\} - c_2T\{u^n(x,t)\} \tag{6}$$

Thus, we get:

$$(v - c_1)T\{u(x,t)\} = vg(x) + cT\{u_{xx}(x,t)\} - c_2T\{u^n(x,t)\} \tag{7}$$

$$T\{u(x,t)\} = \frac{vg(x)}{v-c_1} + \frac{c}{v-c_1}T\{u_{xx}(x,t)\} - \frac{c_2}{v-c_1}T\{u^n(x,t)\} \tag{8}$$

Taking the inverse of new transform on Equation (6), we obtain:

$$u(x,t) = T^{-1}\left\{\frac{vg(x)}{v-c_1}\right\} + T^{-1}\left(\frac{c}{v-c_1}T\{u_{xx}(x,t)\} - \frac{c_2}{v-c_1}T\{u^n(x,t)\}\right) \tag{9}$$

In the HPM, the basic assumption is that the solutions can be written as a power series in *p* as:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{10}$$

and the nonlinear term *N(u)* = u^n, $n > 1$ can be presented by an infinite series as:

$$N(u) = \sum_{n=0}^{\infty} p^n H_n(x,t) \tag{11}$$

Where $p \in [0,1]$ is an embedding parameter. Hn (u) is He polynomials [16,17]. that can be generated by formula given below:

$$H_n(u_0, u_1, \dots\dots\dots, u_n) = \frac{1}{n!}\frac{d^n}{dp^n}\left[N\left(\left(\sum_{i=0}^{N} p^i u_i(x,t)\right)\right)\right], \; n = 0,1,2, \dots\dots \tag{12}$$

Substituting (10) and (11) in (9), we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = T^{-1}\left\{\frac{vg(x)}{v-c_1}\right\} + pT^{-1}\left(\frac{c}{v-c_1}T\{\sum_{n=0}^{\infty} p^n\, u_{xx}(x,t)\} - \frac{c_2}{v-c_1}T\{\sum_{n=0}^{\infty} p^n H_n\}\right) \tag{13}$$

which is the coupling of the new transform and the HPM using He's polynomials. Comparing the coefficient of like powers of *p*, the following approximations are obtained:

$$p^0: u_0(x,t) = T^{-1}\left\{\frac{vg(x)}{v - c_1}\right\}$$

$$p^1: u_1(x,t) = T^{-1}\left(\frac{c}{v-c_1}T\left\{u_{0_{xx}}(x,t)\right\} - \frac{c_2}{v-c_1}T\{H_0\}\right)$$

$$p^2: u_2(x,t) = T^{-1}\left(\frac{c}{v-c_1}T\left\{u_{1_{xx}}(x,t)\right\} - \frac{c_2}{v-c_1}T\{H_1\}\right)$$

$$p^3: u_3(x,t) = T^{-1}\left(\frac{c}{v-c_1}T\left\{u_{2_{xx}}(x,t)\right\} - \frac{c_2}{v-c_1}T\{H_2\}\right) \tag{14}$$

Proceeding in this same manner, the rest of the components $u_n(x, t)$ can be completely obtained and the series solution is thus entirely determined. Finally, we approximate the analytical solution $u(x,t)$ by truncated series:

$$u(x,t) = \lim_{N\to\infty}\left(\sum_{n=0}^{N} u_n(x,t)\right) \tag{15}$$

The above series solutions generally converge very rapidly.

5. Illustrative Examples

In this section, some initial value problems are presented to show the advantages of the proposed method which can be applied to non-linear problem.

Example 1

Consider linear 2nd order autonomous Equation:

$$u_t(x,t) = u_{xx}(x,t) - 3u(x,t) \tag{16}$$

With the initial condition:

$$u\,(x,t) = e^{2x} \tag{17}$$

Taking the new transform on both sides of Equation (16) subject to the initial condition (17), we have:

$$T\{u_t(x,t)\} = T\{u_{xx}(x,t)\} - 3T\{u(x,t)\} \tag{18}$$

By applying the differentiation property of new transform, we get:

$$vT\{u(x,t)\} - vu(x,0) = T\{u_{xx}(x,t)\} - 3T\{u(x,t)\} \tag{19}$$

Thus, we have

$$(v+3)T\{u(x,t)\} = ve^{2x} + T\{u_{xx}(x,t)\} \tag{20}$$

$$T\{u(x,t)\} = \frac{ve^{2x}}{v+3} + \frac{1}{v+3}T\{u_{xx}(x,t)\} \tag{21}$$

Taking the inverse new transform on Equation (21), we get:

$$u(x,t) = T^{-1}\left\{\frac{ve^{2x}}{v+3}\right\} + T^{-1}\left(\frac{1}{v+3}T\{u_{xx}(x,t)\}\right) \tag{22}$$

Now, applying the HPM, we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = e^{2x-3t} + pT^{-1}\left(\frac{1}{v+3}T\{\sum_{n=0}^{\infty} p^n\, u_{xx}(x,t)\}\right) \tag{23}$$

Comparing the coefficients of like powers of *p*, we have:

$$p^0: u_0(x,t) = e^{2x-3t}$$

$$p^1: u_1(x,t) = T^{-1}\left(\frac{1}{v+3}T\{u_{0_{xx}}(x,t)\}\right) = T^{-1}\left(\frac{1}{v+3}T\{4e^{2x}e^{-3t}\}\right) = 4e^{2x}T^{-1}\left(\frac{v}{(v+3)^2}\right) = 4te^{2x-3t}$$

$$p^2: u_2(x,t) = T^{-1}\left(\frac{1}{v+3}T\{u_{1_{xx}}(x,t)\}\right) = T^{-1}\left(\frac{1}{v+3}T\{16te^{2x}e^{-3t}\}\right) = T^{-1}\left(\frac{1}{v+3} * \frac{16e^{2x}v}{(v+3)^2}\right) \tag{24}$$

$$= 8t^2e^{2x-3t} = T^{-1}\left(8e^{2x} * \frac{2!v}{(v+3)^{2+1}}\right)$$

$$p^3: u_3(x,t) = T^{-1}\left(\frac{1}{v+3}T\{u_{2_{xx}}(x,t)\}\right) = T^{-1}\left(\frac{1}{v+3}T\{32e^{2x}t^2e^{-3t}\}\right) = 32e^{2x}T^{-1}\left(\frac{1}{v+3} * \frac{2!\,v}{(v+3)^3}\right)$$

$$= 32e^{2x}T^{-1}\left(\frac{2!v}{(v+3)^4}\right) = \frac{32}{3}e^{2x}T^{-1}\left(\frac{3!v}{(v+3)^{3+1}}\right) = \frac{32}{3}t^3e^{2x-3t}$$

And so on, therefore the solution $u(x, t\,)$ is given by:

$$u(x,t) = e^{2x-3t}\left(1 + 4t + 8t^2 + \frac{32t^3}{3} + \cdots\right) = e^{2x-3t}\left(1 + (4t) + \frac{(4t)^2}{2!} + \frac{(4t)^3}{3!} + \cdots\right) = e^{2x+t} \qquad (25)$$

Example 2

Consider nonlinear 2nd order autonomous Equation:

$$u_t(x,t) = 5u_{xx}(x,t) + 2u(x,t) + u^2(x,t) \qquad (26)$$

With the initial condition:

$$u\,(x,t) = \beta \qquad (27)$$

where β is arbitrary constant. Taking the new transform on both sides of Equation (23) subject to the initial condition (24), we have:

$$T\{u_t(x,t)\} = 5T\{u_{xx}(x,t)\} + 2T\{u(x,t)\} + T\{u^2(x,t)\} \qquad (28)$$

$$vT\{u(x,t)\} - vu(x,0) = 5T\{u_{xx}(x,t)\} + 2T\{u(x,t)\} + T\{u^2(x,t)\} \qquad (29)$$

$$(v-2)T\{u(x,t)\} = v\beta + 5T\{u_{xx}(x,t)\} + T\{u^2(x,t)\} \qquad (30)$$

$$T\{u(x,t)\} = \frac{v\beta}{v-2} + \frac{5}{v-2}T\{u_{xx}(x,t)\} + \frac{1}{v-2}T\{u^2(x,t)\} \qquad (31)$$

Taking the inverse new transform on Equation (28), we obtain:

$$u(x,t) = T^{-1}\left\{\frac{v\beta}{v-2}\right\} + T^{-1}\left(\frac{5}{v-2}T\{u_{xx}(x,t)\} + \frac{1}{v-2}T\{u^2(x,t)\}\right) \qquad (32)$$

Now, applying the HPM, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \beta\, e^{2t} + pT^{-1}\left(\frac{5}{v-2}T\{\sum_{n=0}^{\infty} p^n\, u_{xx}(x,t)\} + \frac{1}{v-2}T\{\sum_{n=0}^{\infty} p^n H_n\}\right) \qquad (33)$$

Comparing the coefficients of like powers of *p*, we have:

$$p^0: u_0(x,t) = \beta e^{2t}$$

$$p^1: u_1(x,t) = T^{-1}\left(\frac{5}{v-2}T\{u_{0_{xx}}(x,t)\} + \frac{1}{v-2}T\{H_0(u)\}\right)$$

$$p^2: u_2(x,t) = T^{-1}\left(\frac{5}{v-2}T\{u_{1_{xx}}(x,t)\} + \frac{1}{v-2}T\{H_1(u)\}\right) \qquad (34)$$

$$p^3: u_3(x,t) = T^{-1}\left(\frac{5}{v-2}T\{u_{2_{xx}}(x,t)\} + \frac{1}{v-2}T\{H_2(u)\}\right)$$

Therefore the solution $u(x, t)$ is given by:

$$u(x,t) = e^{2t}\,(\beta + \frac{\beta^2}{2}e^{2t}(e^{2t}-1) + \frac{\beta^3}{4}e^{2t}(e^{2t}-1)^2 + \frac{\beta^4}{8}e^{2t}(e^{2t}-1)^3 + \cdots$$
$$= \frac{2\beta e^{2t}}{2 + \beta(1-e^{2t})}$$

There is many research study the convergence of methods such [18-25]. but here we introduce other manner illustrated in the following section.

6. The Convergence of the Solution

Now, we need to show the convergence of series form to the exact form as the following.

Lemma 1 If f be continues function then:

$$\frac{\partial}{\partial t}\int_0^t f(t-\tau)d\tau = f(t)$$

Proof

Suppose that:

$$\int f(x)dx = F(x) + c$$

Assume that $x = t - \tau$ then $dx = -d\tau$ then:

$$\frac{\partial}{\partial t}\int_0^t f(t-\tau)d\tau = -\frac{\partial}{\partial t}\int_t^0 f(x)dx = \frac{\partial}{\partial t}\int_0^t f(x)dx = \frac{\partial}{\partial t}[F(x)|_0^t] = \frac{\partial}{\partial t}[F(t) - F(0)]$$

$$= \frac{\partial}{\partial t}F(t) - \frac{\partial}{\partial t}F(0) = f(t)$$

$$so, \qquad \frac{\partial}{\partial t}\int_0^t f(t-\tau)d\tau = f(t)$$

Lemma 2 Let $\mathbb{T}$ is new transform. Then:

$$\frac{\partial}{\partial t}\left(\mathbb{T}^{-1}\left\{\frac{1}{u}\ \mathbb{T}\left\{f(x,t)\right\}\right\}\right) = f(x,t)\ ,$$

Proof

Using property 2 and 3 of NT, and lemma (1), we have

$$\frac{\partial}{\partial t}\left(\mathbb{T}^{-1}\left\{\frac{1}{u}\ \mathbb{T}\left\{f(x,t)\right\}\right\}\right) = \frac{\partial}{\partial t}\left(\mathbb{T}^{-1}\left\{\frac{1}{u}\ \mathbb{T}\left\{1\right\}\mathbb{T}\left\{f(x,t)\right\}\right\}\right) = \frac{\partial}{\partial t}\left(\mathbb{T}^{-1}\left\{\mathbb{T}\left\{1 * f(x,t)\right\}\right\}\right)$$

$$= \frac{\partial}{\partial t}(1 * f(x,t)) = \frac{\partial}{\partial t}\left(\int_0^t f(x,t-\tau)d\tau\right) = f(x,t)$$

Theorem 1 (Convergence Theorem)

If the series form given in Equation (10) with $p = 1$, i.e.,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{35}$$

is convergent. Then the limit point converges to the exact solution of Equation (3), where u_n ($n = 0, 1, \dots$) are calculated by NTHPM, i.e.,

$$\left.\begin{aligned} u_0(x,t) + u_1(x,t) &= \mathbb{T}^{-1}\left\{f + \frac{1}{v\alpha}\ \mathbb{T}\left\{-R[u_0]\right\}\right\} \\ u_n(x,t) &= -\mathbb{T}^{-1}\left\{\frac{1}{v\alpha}\mathbb{T}\{R[u_{n-1}]\}\right\}, n > 1 \end{aligned}\right\} \tag{36}$$

Proof

Suppose that Equation (32) converge to the limit point say as:

$$w(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{37}$$

Now, from right hand side of Equation (3) we have:

$$u_t(x,t) = cu_{xx}(x,t) + c_1 u(x,t) - c_2 u^n(x,t)$$

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\sum_{n=0}^{\infty} u_n(x,t) \ = \frac{\partial}{\partial t}\left[u_0 + u_1 + \sum_{n=2}^{\infty} u_n(x,t)\right]$$

$= \frac{\partial}{\partial t}\left[\mathbb{T}^{-1}\left\{f + \frac{1}{v\alpha}\,\mathbb{T}\left\{-R[u_0]\right\}\right\} - \sum_{n=2}^{\infty}\mathbb{T}^{-1}\left\{\frac{1}{v\alpha}\left[\mathbb{T}\{R[u_{n-1}]\}\right]\right\}\right] = \frac{\partial f}{\partial t} - R[u_0] -$
$\frac{\partial}{\partial t}\left(\sum_{n=2}^{\infty}\mathbb{T}^{-1}\left\{\frac{1}{v}\left[\mathbb{T}\{R[u_n]\}\right]\right\}\right) = 0 - R[u_0] - \sum_{n=0}^{\infty}\frac{\partial}{\partial t}\left(\mathbb{T}^{-1}\left\{\frac{1}{v}\left[\mathbb{T}\{R[u_n]\}\right]\right\}\right)$ (38)

By lemma (2), Equation (35) becomes

$$\frac{\partial w}{\partial t} = -\sum_{n=0}^{\infty} R[u_n] = -R\left[\sum_{n=0}^{\infty} u_n\right] = -Rw = c\,w_{xx} + c_1 w - c_2 w^n$$

Then $w(x, t)$ is satisfying Equation (3). So, its exact solution.

7. Conclusion

We employed the combination of new transform suggested by Luma and Alaa with HPM method to get a closed form solution of the 2nd order autonomous Equation PDE linear and nonlinear. The new method is free of unnecessary mathematical complexities. Although the problem considered has no exact solution, the accuracy, efficiency and reliability of the new method are guaranteed. The convergence of obtained solution to the exact solution by using NTHPM is proved.

This method provides an effective and efficient way of solving a wide range of non-linear PDEs. The advantage of this method is its capability of combining two powerful methods for obtaining exact solutions for non-linear PDEs. This research considers the effectiveness of the suggested method in solving non-linear 2nd order autonomous Equation

References

1. Salih, H.; Tawfiq, L.N.M.; Yahya, Z.R.; Zin, S.M. Solving Modified Regularized Long Wave Equation Using Collocation Method. *Journal of Physics: Conference Series*.**2018**,*1003*,*012062*,1–10, doi:10.1088/1742-6596/1003/1/012062
2. Tawfiq, L.N.M.; Jabber, A.K. Steady State Radial Flow in Anisotropic and Homogenous in Confined Aquifers. *Journal of Physics: Conference Series*.**2018**, *1003*,*012056*,1-12, doi:10.1088/1742-6596/1003/1/012056
3. Salih, H.; Tawfiq, L.N.M.; Yahya, Z.R.; Zin, S.M. Numerical solution of the Equation Modified Equal Width Equation by using Cubic Trigonometric B-Spline Method. *International Journal of Engineering & Technology*.**2018**, *7*, *37*, 340-344.
4. Wazwaz, A.M. Partial Differential Equations and Solitary Waves Theory, Springer, Beijing and Berlin.**2009**, ISBN 978-3-642-00250-2
5. Adomian, G. *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Boston, MA.**1994**.
6. Tawfiq, L.N.M.; Jasim, K.A.; Abdulhmeed, E.O. Numerical Model for Estimation the Concentration of Heavy Metals in Soil and its Application in Iraq. *Global Journal of Engineering science and Researches*.**2016**, *3*, *3*, 75-81.
7. Tawfiq, L.N.M.; Jabber, A.K. Mathematical Modeling of Groundwater Flow. *Global Journal of Engineering science and Researches*.**2016**,*3*,*10*,15-22.
8. He, J.H. Homotopy perturbation technique. *Computer Methods in Applied Mechanics and Engineering*.**1999**, *178*, *3-4*, 257-262, doi:10.1016/s0045-7825(99)00018-3.

9. Sweilam, N.H.; Khader, M.M. Exact solutions of some coupled nonlinear partial differential Equations using the homotopy perturbation method. *Computers & Mathematics with Applications*.**2009**, *58*, 2134–2141.
10. He, J.H.; Wu, G.C.; Austin, F. The variational iteration method which should be followed. *Nonlinear Science Letters A1*.**2010**, 1–30.
11. He, J.H.; Wu, X.H. Variational iteration method: new development and applications. *Computers & Mathematics with Applications*.**2007**, *54*, 881–894.
12. Aasaraai, A. Analytic solution for Newell-Whitehead-Segel Equation by differential transform method. *Middle East J. Sci. Res*.**2011**, *10, 2*, 270–273.
13. Saravanan, A.; Magesh, N. A comparison between the reduced differential transform method and the Adomian decomposition method for the Newell-Whitehead-Segel. *Egyptian Math. Soc*.**2013**, *21*, 259–265.
14. Tawfiq, L.N.M.; Jabber, A.K. New Transform Fundamental Properties and its Applications. *Ibn Alhaitham Journal for Pure and Applied Science*.**2018**, *31, 1*,151-163, doi: http://dx.doi.org/10.30526/31.2.1954
15. Jabber, A.K. Design Mathematical Model for Groundwater and its Application in Iraq, Ph.D. Thesis, Department of Mathematics, College of Education for Pure Science Ibn Al-Haitham, University of Bagdad, Bagdad, Iraq, **2018**.
16. Yusufoglu, E. Numerical solution of Diffusing Equation by the Laplace decomposition algorithm. *Applied Mathematics and Computation*.**2006**, *177*, 572–580.
17. Khan, Y. An effective modification of the Laplace decomposition method for nonlinear Equations. *Journal of Nonlinear Sciences and Numer. Simulation*. **2009**, *10*, 1373–1376.
18. Tawfiq, L.N.M.; Rasheed, H.W. On Solution of Non Linear Singular Boundary Value Problem. *Ibn Al-Haitham Journal for Pure & Applied Sciences*. **2013**, *26, 3*, 320- 328.
19. Tawfiq, L.N.M.; Hussein, R.W. Efficient Semi-Analytic Technique for Solving Nonlinear Singular Initial Value Problems. *Ibn Al-Haitham Journal for Pure & Applied Sciences*.**2014**, *27*, *3*, 513-522.
20. Tawfiq, L.N.M.; Hilal, M.M. Solution of 2nd Order Nonlinear Three-Point Boundary Value Problems By Semi-Analytic Technique. *Ibn Al-Haitham Journal for Pure & Applied Sciences*. **2014**, *27*, *3*, 501-512.
21. Tawfiq, L.N.M.; Rasheed, H.W. On Solution of Regular Singular Ordinary Boundary Value Problem. *Ibn Al-Haitham Journal for Pure & Applied Sciences*.**2013**, *26*, *2*, 249-256.
22. Tawfiq, L.N.M.; Yassien, S.M. Solution of High Order Ordinary Boundary Value Problems Using Semi-Analytic Technique. *Ibn Al-Haitham Journal for Pure & Applied Sciences*.**2013**, *26*, *1*, 281-291.
23. Tawfiq, L.N.M.; Hussein, R.W. On Solution of Regular Singular Initial Value Problems. Ibn Al-Haitham Journal for Pure & Applied Sciences. **2013**, *26*, *1*, 257- 264.
24. Tawfiq, L.N.M.; Hussein, A.A.T. Design feed forward neural network to solve singular boundary value problems. *ISRN Applied Mathematics*. **2013**, *2013*, 1-8, http://dx.doi.org/10.1155/2013/650467
25. Tawfiq, L.N.M. Using collocation neural network to solve Eigenvalue problems. *MJ Journal on Applied Mathematics*.**2016,** *1*, *1*, 1-8.