2-Regular Modules

Nuhad S. AL-Mothafar
Dept. of Mathematic/ College of Science/ University of Baghdad, Iraq
Ghaleb A. Humod
Dept. of Mathematic/ College of Education for Pure Science (Ibn Al-Haitham)/ University of Baghdad

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Abstract

In this paper we introduced the concept of 2-pure submodules as a generalization of pure submodules, we study some of its basic properties and by using this concept we define the class of 2-regular modules, where an R-module M is called 2-regular module if every submodule is 2-pure submodule. Many results about this concept are given.

Key Words: 2-pure submodules, 2-regular modules, pure submodules, regular modules.

Introduction

Throughout this paper, R denotes a commutative ring with identity and every R-module is a unitary. It is well-known that the pure submodules were given by several authors. For example [1] and [2].

Definition (0.1): [1]

Let M be an R-module. A submodule N of M is called pure if the sequence $0 \longrightarrow E\otimes N \longrightarrow E\otimes M$ is exact for every R-module E.

Proposition (0.2): [1]

Let N be a submodule of M. The following statements are equivalent:

(1) N is a pure submodule of M.

(2) For each $\sum_{i=1}^{n} r_{ji}m_i \in N$, $r_{ji} \in R$, $m_i \in M$, j = 1,2,…,k, there exists $x_i \in N$, i = 1,2,…,n such that $\sum_{i=1}^{n} r_{ji}m_i = \sum_{i=1}^{n} r_{ji}x_i$ for each j.

(3)

Proposition (0.3): [2]

Let N be an R-submodule of M. Consider the following statements:

(1) N is a pure submodule of M.

(2) $N \cap IM = IN$ for each ideal I of R.

(3) $N \cap IM = IN$ for each finitely generated ideal I of R.

(4) $N \cap (r)M = (r)N$ for each principal ideal (r) of R.

(5) $N \cap rM = rN$ for each $r \in R$.

Then (1) ⇒ (2) ⇔ (3) ⇒ (4) ⇔ (5). And if M is flat then (1) ⇔ (2).

Notice that: Anderson was called the submodule N of M pure if it satisfies (2), see [3].

Recall that an R-module M is called regular module if every submodule of M is pure [2]. M is called a Von Neumman regular module if every cyclic submodule of M is a direct summand of M, [4].

This paper is structured in three sections. In section one we give a comprehensive study of 2-pure submodules. Some results are analogous to the properties of pure submodules. In section two, we study the concept of 2-regular modules. It is clear that every regular module is 2-regular, but the converse is not true (see Remarks and Examples (2.2)(1)). Section three is concerned with the direct sum of 2-regular modules. It is shown under certain condition, the direct sum of 2-regular modules is 2-regular (see corollary 3.3). Also we show that the 2-regular property of a module is inherited by its submodules (see Corollary 3.7). Other results are given in this section.

0- 2-Pure Submodules

In this section we introduce the concept of 2-pure submodules. We investigate the basic properties of this type of submodules, some of these properties are analogous to the properties of pure submodules.

Definition (1.1):

Let M be an R-module. A submodule N of M is called a **2-pure submodule** of M if for each ideal I of R, $I^2M \cap N = I^2N$.

Remarks and Examples (1.2):

(1) It is clear that every pure submodule is a 2-pure, but not the converse. For example: the submodule $\{\bar{0},\bar{2}\}$ of the module Z_4 as Z-module is 2-pure submodule since if I= 2Z is an

ideal of Z, then $I^2Z_4 \cap \{\bar{0},\bar{2}\} = 4Z_4 \cap \{\bar{0},\bar{2}\} = \{\bar{0}\}$. On the other hand $I^2 \{\bar{0},\bar{2}\} = 4 \{\bar{0},\bar{2}\} = \{\bar{0}\}$.
By the similar simple calculation one can easily to show that $I^2Z_4 \cap \{\bar{0},\bar{2}\} = I^2 \{\bar{0},\bar{2}\}$ for every ideal I = nZ of Z where n is any positive integer. Thus $\{\bar{0},\bar{2}\}$ is a 2-pure submodule of Z_4 but is not pure since if I = 2Z, then $IZ_4 \cap \{\bar{0},\bar{2}\} = 2Z_4 \cap \{\bar{0},\bar{2}\} = \{\bar{0},\bar{2}\}$ and $I \{\bar{0},\bar{2}\} = 2 \{\bar{0},\bar{2}\} = \{\bar{0}\}$.

(2) In any R-module M, the submodules M and {0} are always 2-pure submodules in M.

(3) In the module Z as Z-module, the only 2-pure submodules are {0} and Z. To see this, for every submodule nZ of Z, $n^2 = n^2 1 \in \langle n \rangle^2 Z \cap nZ$, but $n^2 \notin n^2(nZ) = n^3Z$.

(4) Every nonzero cyclic submodule of the module Q as Z-module is a non 2-pure submodule.

Proof:

Let N be a cyclic submodule of Q as Z-module, generated by an element $\frac{a}{b}$ where a and b are two nonzero elements in Z. If we take an ideal <n> of Z where n is greater than one, then $\langle n^2 \rangle \cdot \frac{a}{b} = \langle \frac{n^2 a}{b} \rangle$.

Also, $Q = \langle n^2 \rangle \cdot Q$, because for any element $\frac{c}{d} \in Q$ we have $\frac{c}{d} = \frac{c}{n^2 d} \cdot n^2 \in \langle n^2 \rangle \cdot Q$, thus $Q = \langle n^2 \rangle \cdot Q$. Therefore $\langle n^2 \rangle \cdot Q \cap \langle \frac{a}{b} \rangle = \langle \frac{a}{b} \rangle$, implies that $\langle n^2 \rangle \cdot Q \cap \langle \frac{a}{b} \rangle \neq \langle n^2 \rangle \cdot \langle \frac{a}{b} \rangle$.

(5) It is clear every direct summand is 2-pure since every direct summand is pure submodule, hence is a 2-pure submodule, but the converse is not true, for example: the submodule $\{\bar{0},\bar{3},\bar{6}\}$ of the module Z_9 as Z-module. It is easily to check that $I^2Z_9 \cap \{\bar{0},\bar{3},\bar{6}\} = I^2 \{\bar{0},\bar{3},\bar{6}\}$ for each I of Z. So, $\{\bar{0},\bar{3},\bar{6}\}$ is 2-pure in Z_9 but not pure and hence not direct summand. Since if we take I = 3Z, then $IZ_9 \cap \{\bar{0},\bar{3},\bar{6}\} = \{\bar{0},\bar{3},\bar{6}\}$ and $I \cdot \{\bar{0},\bar{3},\bar{6}\} = \{\bar{0}\}$.

(6) Let N be a 2-pure submodule of M such that N ≅ K for some submodule K of M, then K may not be a 2-pure. For example: consider the module Z as Z-module. Let N = Z and K = 2Z. It is clear Z ≅ 2Z but 2Z is not 2-pure in Z.

The following propositions give some properties of 2-pure submodules.

Proposition (1.3):

Let M be an R-module and N be a 2-pure submodule of M. If A is a 2-pure submodule in N, then A is a 2-pure submodule in M.

Proof:

Let I be an ideal of R. Since N is a 2-pure submodule in M and A is a 2-pure submodule in N, then $I^2M \cap N = I^2N$ and $I^2N \cap A = I^2A$. But $A \subseteq N$, implies $I^2A = I^2N \cap A = (I^2M \cap N) \cap A = I^2M \cap (N \cap A) = I^2M \cap A$.

Proposition (1.4):

Let M be an R-module and N is a 2-pure submodule of M. If A is a submodule of M containing N, then N is a 2-pure submodule in A.

Proof:

Let I be an ideal of R. Since N is a 2-pure submodule in M, hence $I^2M \cap N = I^2N$ and since $N \subseteq A \subseteq M$ implies $I^2A \cap N = (I^2A \cap I^2M) \cap N = I^2A \cap (I^2M \cap N) = I^2A \cap I^2N = I^2N$.

Proposition (1.5):

Let M be an R-module and N is a 2-pure submodule of M. If H is a submodule of N, then $\frac{N}{H}$ is a 2-pure submodule in $\frac{M}{H}$.

Proof:

Let I be an ideal of R. Since

$$I^2\left(\frac{M}{H}\right) \cap \frac{N}{H} = \frac{I^2M + H}{H} \cap \frac{N}{H}$$

$$= \frac{(I^2M + H) \cap N}{H}$$

$$= \frac{(I^2M \cap N) + (H \cap N)}{H} \quad \text{by Modular law}$$

$$= \frac{I^2N + H}{H}$$

$$= I^2\left(\frac{N}{H}\right)$$

Recall that a ring R is called an arithmetical ring if every finitely generated ideal of R is a multiplication ideal, where an ideal I of R is called a multiplication ideal if every ideal $J \subseteq I$ there exists an ideal K of R such that J = IK, see [5].

The following proposition gives a characterization of 2-pure submodules of modules over some classes of rings. First let us state the following theorem, which can be found in [6].

Theorem (1.6):

Let $I = (a_1, a_2, \ldots, a_n)$ be a multiplication ideal in the ring R. Then for each positive integer k, $(a_1, a_2, \ldots, a_n)^k = (a_1^k, a_2^k, \ldots, a_n^k)$.

Proof: see [6].

Proposition (1.7):

Let M be a module over arithmetical ring R. The following statements are equivalent:

(1) N is a 2-pure submodule of M.

(2) For each $\sum_{i=1}^{n} r_{ij}^2 x_i \in N$, $r_{ij} \in R$, $x_i \in M$, $j = 1,2,\ldots,m$, there exists $x_i' \in N$, $i = 1,2, \ldots, n$ such that $\sum_{i=1}^{n} r_{ij}^2 x_i = \sum_{i=1}^{n} r_{ij}^2 x_i'$ for each j.

Proof:

(1) ⇒ (2) Assume that N is a 2-pure submodule of M, let $y_i = \sum_{i=1}^{n} r_{ij}^2 x_i \in N$ for any finite sets, $\{x_i\}_{i=1}^{n}$ in M, $\{y_j\}_{j=1}^{m}$ in N and $\{r_{ij}\}$ in R where $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$. Let I be an ideal of R generated by the finite set $\{r_{1j},r_{2j},\ldots,r_{nj}\}$, then $r_{ij} \in I$ and $r_{ij}^2 \in I^2$ imply $r_{ij}^2 x_i \in I^2M$. Thus $y_j = \sum_{i=1}^{n} r_{ij}^2 x_i \in I^2M$, therefore $y_j \in I^2M \cap N$. But $I^2M \cap N = I^2N$, implies $y_i \in I^2N$. Since R is arithmetical ring, hence by theorem (1.6), $I^2 = (r_{1j}^2, r_{2j}^2,\ldots,r_{nj}^2)$. Therefore $y_j = \sum_{i=1}^{n} r_{ij}^2 x'_i$ for some $x'_i \in N$.

(2) ⇒ (1) Let N be any submodule of M. Let $y_j \in I^2M \cap N$, $y_j = \sum_{i=1}^{n} r_{ij}^2 x_i$ where $\{x_i\}_{i=1}^{n} \subseteq M$, $\{y_j\}_{j=1}^{m} \subseteq N$, i = 1,2,…,n, j = 1,2,…,m. Therefore by hypothesis, there exists $x'_i \in N$ such that $y_j = \sum_{i=1}^{n} r_{ij}^2 x_i = \sum_{i=1}^{n} r_{ij}^2 x'_i \in I^2N$ implies $y_j \in I^2N$. Then $I^2M \cap N \subseteq I^2N$. The reverse inclusion is clear. Thus $I^2M \cap N = I^2N$, and hence N is a 2-pure submodule of M.

1- 2-Regular Modules

In this section, we introduce and study the class of 2-regular modules.

Definition (2.1):

An R-module M is called **2-regular module** if every submodule of M is 2-pure.

Remarks and Examples (2.2):

(1) It is clear that the following implications hold:

Von Neumman regular ⇒ regular ⇒ 2-regular

But non of these implications is reversible. For example: the module Z_4 as Z-module is 2-regular since every submodule of Z_4 is 2-pure submodule in Z_4, but Z_4 is not regular since the submodule $\{\bar{0},\bar{2}\}$ of Z_4 is not pure, see remark and example (1.2)(1).

(2) The modules Z and Q as Z-modules are not 2-regular modules, see remarks and examples (1.2)(3), and (4).

The following theorem shows that the cyclic 2-pure submodules is enough to make the module be 2-regular.

Theorem (2.3):

Let M be an R-module. The following statements are equivalent:

(1) M is 2-regular module.
(2) Every cyclic submodule of M is 2-pure submodule of M.
(3) Every finitely generated submodule of M is 2-pure submodule.
(4) Every submodule of M is a 2-pure submodule of M.

Proof:

(1) ⇒ (2) it follows by definition (2.1).

(2) ⇒ (1) Assume that every cyclic submodule of M is 2-pure. Let N be a submmodule of M and I is an ideal of R. Let $x \in I^2M \cap N$ implies $x \in I^2M$ and $x \in N$. Therefore $x \in I^2M \cap \langle x\rangle = I^2 \langle x\rangle \subseteq I^2 N$.

(1) ⇒ (3) It follows by definition (2.1), and the proof of (2) ⇒ (1).

(3) ⇒ (2) It is clear.

(1) ⇒ (4) It follows by definition (2.1).

2- The Direct Sum of 2-Regular Modules-Basic Results

In this section, we study the direct sum and the epimorphic image of 2-regular module; various properties of 2-regular modules are discussed and illustrated.

We start with the following proposition.

The following proposition shows that the factor module of a 2-regular module is also 2-regular module.

Proposition (3.1):

Let M be an R-module. Then M is a 2-regular if and only if $\frac{M}{N}$ is 2-regular for every submodule N of M.

Proof:

(⇒) Let N be a submodule of M and K is any submodule of M containing N. Since M is 2-regular then K is 2-pure in M. Thus $\frac{K}{N}$ is 2-pure in $\frac{M}{N}$ by proposition (1.5), therefore $\frac{M}{N}$ is 2-regular.

(⇐) It is easily by taking N = 0.

Now, we have several consequences of the proposition (3.1), the first result shows that the epimorphic image of 2-regular module is 2-regular.

Corollary (3.2):

Let M and M' be R-modules and f: M ⟶ M' be an R-epimorphism. If M is 2-regular module then M' is 2-regular.

Proof: Since f:M⟶M' is an R-epimorphism and M is 2-regular. Then $\frac{M}{\ker f}$ is 2-regular module by proposition (3.1).But $\frac{M}{\ker f} \cong M'$ by the first isomorphism theorem. Therefore M' is 2-regular.

Corollary (3.3):

Let M_1 and M_2 be R-modules. If $M = M_1 \oplus M_2$ is 2-regular R-module, then M_1 and M_2 are 2-regular R-modules. The converse is true provided ann (M_1) + ann (M_2) = R.
The following statements are equivalent:

Proof:

For the first assertion, assume that $M = M_1 \oplus M_2$ is 2-regular R-module. Let ρ_i:M ⟶ M_i be the natural projective map of M onto M_i for each i = 1, 2. Since ρ_I is an R-epimorphism then the epimorphic image of M is 2-regular, implies that M_i is 2-regular.
Conversely, assume M_1 and M_2 are 2-regular R-modules and $M = M_1 \oplus M_2$. Let be a submodule of $M = M_1 \oplus M_2$. Since ann(M_1) + ann(M_2) = R then by the same way of the proof of [7,prop.(4.2),CH.1], $N = N_1 \oplus N_2$ where N_1 is a submodule in M_1 and N_2 is a submodule in M_2. Let I be an ideal of R. To show $I^2M \cap N = I^2N$. Since $I^2M_1 \cap N_1 = I^2N_1$ and $I^2M_2 \cap N_2 = I^2N_2$ implies that $(I^2M_1 \cap N_1) \oplus (I^2M_2 \cap N_2) = I^2N_1 \oplus I^2N_2$. Then $(I^2M_1 \oplus I^2M_1) \cap (N_1 \oplus N_2) = I^2(N_1 \oplus N_2)$, therefore M is 2-regular module.

The proof of the following result is similar to that of corollary (3.3).

Corollary (3.4):

Let M_1 and M_2 be R-modules. If N_1 is a 2-pure submodule in M_1 and N_2 is a 2-pure submodule in M_2, then $N_1 \oplus N_2$ is a 2-pure submodule in $M_1 \oplus M_2$.

Corollary (3.5):

Let M_1 and M_2 be R-modules and $M_1 \oplus M_2$ is 2-regular R-module, then M_1+M_2 is 2-regular.

Proof:

Define $f : M_1 \oplus M_2 \longrightarrow M_1+M_2$ by $f(m_1,m_2) = m_1 + m_2$. It is easily to check that f is an epimorphism. Since $M_1 \oplus M_2$ is 2-regular, thus the epimorphic image of $M_1 \oplus M_2$ is 2-regular by corollary (3.2). Therefore M_1+M_2 is 2-regular.

Corollary (3.6):

Let M_1 and M_2 be 2-regular R-modules such that ann (M_1) + ann (M_2) = R, then M_1+M_2 is a 2-regular R-module.

Proof:

Since M_1 and M_2 are 2-regular R-modules then $M_1 \oplus M_2$ is 2-regular by corollary (3.3) implies that M_1+M_2 is a 2-regular by corollary (3.5).

The following result shows that every submodule of a 2-regular module inherits the 2-regular property.

Corollary (3.7):

Every submodule of a 2-regular module is a 2-regular module.

Proof:

Let N be a submodule of a 2-regular R-module M. To show that N is 2-regular R-module. Let K be a submodule in N and I is an ideal of R. Thus we have:

$I^2N \cap K = (I^2M \cap N) \cap K$ since N is 2-pure in M

$= I^2M \cap (N \cap K)$

$= I^2M \cap K$

$= I^2K$ since K is 2-pure in M

Therefore K is 2-pure in N implies N is 2-regular.

We end this paper by the following remark.

Remark (3.8):

If all proper submodules of an R-module M are 2-regular then M may not be 2-regular, for example: the module Z_8 as Z-module is not 2-regular. Since $<\bar{4}>$ is not 2-pure submodule of Z_8 because $2^2 \cdot Z_8 \cap <\bar{4}> = <\bar{4}>$ but $2^2 \cdot <\bar{4}> = <\bar{0}>$, while every proper submodule of Z_8 is 2-regular, since $<\bar{2}> \cong Z_4$ and $<\bar{4}> \cong Z_2$ are 2-regular modules.

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المقاسات المنتظمة من النمط -2

نـهاد سالم عبد الكريم
قسم الرياضيات/ كلية العلوم/ جامعة بغداد
غالب أحمد حمود
قسم الرياضيات/كلية التربية للعلوم الصرفة (ابن الهيثم) / جامعة بغداد

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الخلاصة

في بحثنا هذا نقدم مفهوم المقاسات الجزئية النقية من النمط – 2 كتعميم لمفهوم المقاسات الجزئية النقية وباستعمال هذا المفهوم نعرف المقاسات المنتظمة من النمط– 2 إذ يقال ان المقاس M على الحلقة R بأنه منتظم من النمط – 2 اذا كان كل مقاس جزئي فيه يكون نقياً من النمط – 2. أعطينا العديد من النتائج حول هذا المفهوم.

الكلمات المفتاحية: المقاسات الجزئية النقية من النمط – 2، المقاسات النتظمة من النمط – 2، المقاسات الجزئية النقية، المقاسات المنتظمة.