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Approximaitly Semi-Prime Submodules and Some Related Concepts

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Abstract

We introduce in this paper the concept of approximaitly semi-prime submodules of unitary left R-module T over a commutative ring R with identity as a generalization of a prime submodules and semi-prime submodules, also generalization of quasi-prime submodules and approximaitly prime submodules. Various basic properties of an approximaitly semi-prime submodules are discussed, where a proper submodule L of an R-module T is called an approximaitly semi-prime submodule of , if whenever $a^n t \in L$, where $a \in R$, $t \in T$ and $n \in Z^+$, implies that $at \in L + soc(T)$. Furthermore the behaviors of approximaitly semi-prime submodule in some classes of modules are studied. On the other hand several characterizations of this concept are introduced.

Keywords: Prime submodules, Semi-prime submodules, Quasi-prime submodules, Approximaitly prime submodules, Approximaitly semi-prime submodules, Multiplication module, Socle of modules.

1. Introduction

Throughout this article, all rings are commutative rings with identity and all modules are unitary. Prime submodules play an important role in module theory over commutative ring with identity, where a proper submodule L of an R- module T is called a prime, if whenever $at \in L$, with $a \in R, t \in T$, implies that either $t \in L$ or $a \in [L:_R T]$ where $[L:_R T] = \{r \in R: rT \subseteq L\}$[1]. Recently several generalization of the concept of prime submodules are studied in [2-5]. The concept semi-prime submodule which was first introduced in [6]. and extensively studied in [7]. is given as a proper submodule L of an R-module T is called semi-prime submodule, if whenever$a^n t \in L$, where $a \in R$, $t \in T$ and $n \in Z^+$, implies that $at \in L$. [7]. characterized semi-prime submodules as follows: A proper submodule L of an R-module T is semi-prime if and only if whenever $a^2 t \in L$, where $a \in R$, $t \in T$, implies that $at \in L$. The concept quasi-prime submodule which introduced and studied in [8]. is a strong form of a semi-prime submodule, where a proper submodule L of an R- module T is called a quasi-prime, if whenever $abt \in L$, with $a, b \in R, t \in T$, implies that either $at \in L$ or $bt \in L$.

Recently extensive research has been done on generalizations of semi-prime submodules see for example [9, 10]. The socle of an R-module T (for short $soc(T)$) is defined by the intersection of all essential submodules of T [11]. where a non-zero submodule N of an R-module T is called essential if $N \cap K \neq (0)$ for all non-zero submodule K of T [12]. Recall that an ideal I of a ring R is a semi-prime ideal of R if $a^2 \in I$, implies that $a \in I$. Equivalent $I = \sqrt{I} = \{a \in R: a^n \in I \text{ for some } n \in Z^+\}$ [7]. Recall that a proper submodule L of an R-module T is called an approximaitly prime submodule of T, if whenever $at \in L$, where $a \in R$, $t \in T$, implies that either $t \in L + soc(T)$ or $a \in [L + soc(T): T]$[2].

2. Approximaitly Semi-prime Submodules

In this section, we introduce the definition of approximaitly semi-prime submodule and give it is basic properties, examples and characterizations.

Definition (1)

A proper submodule L of an R-module T is called an approximaitly semi-prime (for short app-semi-prime) submodule of T, if whenever $a^n t \in L$, where $a \in R$, $t \in T$ and $n \in Z^+$, implies that $at \in L + soc(T)$. An ideal I of a ring R is called an approximaitly semi-prime ideal of R if I is an approximaitly semi-prime submodule of R-module R.

Remarks and Examples (2)

1) It is clear that every semi-prime submodule of an R-module T is an app-semi-prime submodule while the convers is not true in general as the following example shows that.
Consider the Z-module Z_{12} and $L = \langle\bar{0}\rangle$ be a submodule of Z_{12}. $soc(Z_{12}) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$. L is not semi-prime in Z_{12} because $2^2.\bar{3} \in L$, but $2.\bar{3} = 6 \notin L$. But L is an app-semi-prime in Z_{12} since whenever $a^2\bar{t} \in L$, for $a \in R$, $\bar{t} \in Z_{12}$, implies that $a\bar{t} \in L + soc(Z_{12}) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$.
2) It is clear that every prime submodule of an R-module T is an app-semi-prime submodule while the convers is not true in general as the following example shows that.
Consider the Z-module Z_4 and $L = \langle\bar{0}\rangle$ be a submodule of Z_4. L is not prime but L is an app-semi-prime in Z_4 because $2^2.\bar{1} \in L$ but $2.1 \notin L$, while $2^2.\bar{1} \in L$, implies that $2.\bar{1} = 2 \in L + soc(Z_4) = \langle\bar{0}\rangle + \{\bar{0}, \bar{2}\} = \{\bar{0}, \bar{2}\}$. That is for all $a \in Z$, $\bar{t} \in Z_4$ with $a^2\bar{t} \in L$, implies that $a\bar{t} \in L + soc(Z_4)$.
3) It is clear that every quasi-prime submodule of an R-module T is an app-semi-prime submodule while the convers is not true in general as an example shows that.
Consider the Z-module Z_{24} and the submodule $L = \langle\bar{6}\rangle = \{\bar{0}, \bar{6}, \overline{12}, \overline{18}\}$. L is not quasi-prime in Z_{24} because $2.3.\bar{1} \in L$, but $2.\bar{1} \notin L$ and $3.\bar{1} \notin L$. But L is an app-semi-prime in Z_{24} since whenever $a^2\bar{t} \in L$ for $a \in R$, $\bar{t} \in Z_{24}$, implies that $a\bar{t} \in L + soc(Z_{24}) = \{\bar{0}, \bar{6}, \overline{12}, \overline{18}\} + \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}, \overline{12}, \overline{14}, \overline{16}, \overline{18}, \overline{20}, \overline{22}\} = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}, \overline{12}, \overline{14}, \overline{16}, \overline{18}, \overline{20}, \overline{22}\}$.
4) It is clear that every approximaitly prime submodule of an R-module T is an app-semi-prime submodule, but the convers is not true. The following example shows that.
The submodule $L = 6Z$ of a Z-module Z is an app-semi-prime submodule of Z (because L is a semi-prime submodule of Z), but L is not an approximaitly prime submodule of Z because, if $2,3 \in Z$ such that $2.3 \in 6Z$, but $3 \notin 6Z + soc(Z) = 6Z$ and $2 \notin [6Z + soc(Z): Z] = 6Z$, because $soc(Z) = (0)$.

The following results are characterizations of app-semi-prime submodules.

Proposition (3)

Let L be a proper submodule of an R-module T. Then L is an app-semi-prime submodule of T if and only if $J^n E \subseteq L$ where J is an ideal of R, E is a submodule of T and $n \in Z^+$, implies that $JE \subseteq L + soc(T)$.

Proof

($\Longrightarrow$) Suppose that $J^n E \subseteq L$, where J is an ideal of R, E is a submodule of T and $n \in Z^+$. Now, let $t \in JE$, then $t = a_1 t_1 + a_2 t_2 + \cdots + a_n t_n$, where $a_i \in J$, $t_i \in E$, $i = 1,2, \dots, n$, that is $a_i t_i \in JE$ for each $i = 1,2, \dots, n$, it follows that ${a^n}_i t_i \in J^n E \subseteq L$, that is ${a^n}_i t_i \in L$. But L is an app-semi-prime submodule of T, implies that $a_i t_i \in L + soc(T)$ for each $i = 1,2, \dots, n$, hence $t \in L + soc(T)$, it follows that $JE \subseteq L + soc(T)$.

($\Longleftarrow$) Let $a^n t \in L$, where $a \in R$, $t \in T$ and $n \in Z^+$, implies that $\langle a^n \rangle Rt \subseteq L$, that is $\langle a \rangle^n Rt \subseteq L$. Thus by hypothesis we have $\langle a \rangle Rt \subseteq L + soc(T)$, implies that $at \in \langle a \rangle Rt \subseteq L + soc(T)$, thus $at \in L + soc(T)$. Therefore L is an app-semi-prime submodule of T.

Corollary (4)

Let L be a proper submodule of an R-module T. Then L is an app-semi-prime submodule of T if and only if $J^n T \subseteq L$ where J is an ideal of R, $n \in Z^+$, implies that $JT \subseteq L + soc(T)$.

Proof

It follows by Proposition (2.3).

Corollary (5)

Let L be a proper submodule of an R-module T. Then L is an app-semi-prime submodule of T if and only if $J^2 E \subseteq L$ where J is an ideal of R and E is a submodule of T, implies that $JE \subseteq L + soc(T)$.

Proposition (6)

Let L be a proper submodule of an R-module T. Then $L + soc(T)$ is an app-semi-prime submodule of T if and only if $[L + soc(T){:}\,T]$ is a semi-prime ideal of R (hence an app-semi-prime).

Proof

($\Longrightarrow$) Let $r \in \sqrt{[L + soc(T){:}\,T]}$, implies that $r^n \in [L + soc(T){:}\,T]$ for some $n \in Z^+$, it follows that $r^n T \subseteq L + soc(T)$, then $r^n t \in L + soc(T)$ for all $t \in T$. But $L + soc(T)$ is an app-semi-prime, implies that $rt \in L + soc(T) + soc(T) = L + soc(T)$ for all $t \in T$. That is $rT \subseteq L + soc(T)$, implies that $r \in [L + soc(T){:}\,T]$. Thus $\sqrt{[L + soc(T){:}\,T]} \subseteq [L + soc(T){:}\,T]$, but $[L + soc(T){:}\,T] \subseteq \sqrt{[L + soc(T){:}\,T]}$, it follows that $[L + soc(T){:}\,T] = \sqrt{[L + soc(T){:}\,T]}$, hence $[L + soc(T){:}\,T]$ is a semi-prime ideal of R(hence $[L + soc(T){:}\,T]$ is an app-semi-prime).

($\Longleftarrow$) Let $a^n t \in L + soc(T)$, where $a \in R$, $t \in T$ and $n \in Z^+$, implies that $a^n T \subseteq L + soc(T)$, that is $a^n \in [L + soc(T){:}\,T]$. Since $[L + soc(T){:}\,T]$ is a semi-prime ideal of R then $a \in [L + soc(T){:}\,T]$, it follows that $aT \subseteq L + soc(T)$, implies that $at \in L + soc(T)$ for all $t \in T$. Therefore $L + soc(T)$ is an app-semi-prime submodule of T.

Proposition (7)

Let L be a proper submodule of an R-module T with $soc(T) \subseteq L$. Then L is an app-semi-prime submodule of T if and only if $[L{:}_T\, a^n] = [L{:}_T\, a]$ for $a \in R$ and some $n \in Z^+$.

Proof

($\Longrightarrow$) Let $t \in [L:_T a^n]$, implies that $a^n t \in L$. But L is an app-semi-prime submodule of T, then $at \in L + soc(T)$. But $soc(T) \subseteq L$, implies that $L + soc(T) = L$, thus $at \in L$, it follows that $t \in [L:_T a]$, hence $[L:_T a^n] \subseteq [L:_T a]$. But $[L:_T a] \subseteq [L:_T a^n]$, so we get $[L:_T a^n] = [L:_T a]$.

($\Longleftarrow$) Suppose that $a^n t \in L$, where $a \in R$, $t \in T$ and $n \in Z^+$, it follows that $t \in [L:_T a^n] = [L:_T a]$, implies that $t \in [L:_T a]$, so $at \in L \subseteq L + soc(T)$. That is $at \in L + soc(T)$, hence L is an app-semi-prime submodule of T.

We recall the following Lemmas before we introduce the next results.

Lemma (8) [11, Lemma 2.3.15].

Let T be R-module and L, K, E are submodules of T with $K \subseteq E$, then $(L + K) \cap E = (L \cap E) + (K \cap E) = (L \cap E) + K$.

Lemma (9) [13, Cor. 9.9].

Let T be R-module and L be a submodule of T, then $soc(L) = L \cap soc(T)$.

Proposition (10)

Let L and E are proper submodules of an R-module T withe $L \subsetneq E$ and L is an app-semi-prime submodule of T. Then L is an app-semi-prime submodule of E.

Proof

Suppose that $a^n t \in L$, where $a \in R$, $t \in E \subsetneq T$ and $n \in Z^+$. But L is an app-semi-prime submodule of T, implies that $at \in L + soc(T)$, but $t \in E$, implies that $at \in E$, then $at \in (L + soc(T)) \cap E$, thus by Lemma(2.8) we have $at \in (L \cap E) + (soc(T) \cap E) \subseteq L + soc(T) \cap E$. So by Lemma (2.9) we have $at \in L + soc(E)$. Thus L is an app-semi-prime submodule of E.

Proposition (11)

Let L and E are proper submodules of an R-module T withe $L \nsubseteq E$ and $soc(T) \subseteq E$. If L is an app-semi-prime submodule of T. Then $L \cap E$ is an app-semi-prime submodule of E.

Proof

Since $L \nsubseteq E$ then $L \cap E$ is a proper in E. Suppose that $a^n t \in L \cap E$, where $a \in R$, $t \in E \subsetneq T$ and $n \in Z^+$, implies that $a^n t \in L$. But L is an app-semi-prime submodule of T, then $at \in L + soc(T)$, it follows that $at \in (L + soc(T)) \cap E$, so by Lemma (2.8) we have $at \in (L \cap E) + (E \cap soc(T))$. But by lemma (2.9) we have $E \cap soc(T) = soc(E)$. Hence $at \in (L \cap E) + soc(E)$. Thus $L \cap E$ is an app-semi-prime submodule of E.

Remark (12)

If L and E are two app-semi-prime submodules of an R-module T, then $L \cap E$ is not necessary an app-semi-prime submodule of T as the following example shows that.

Let $= Z \oplus Z_4$, $R = Z$, $L = Z(1, \bar{0})$, $E = Z(1, \bar{1})$, where L and E are app-semi-prime submodules of T. Then $L \cap E = \{(0, \bar{0}), (4, \bar{0}), (8, \bar{0}), \dots \dots\}$ is not an app-semi-prime submodule of T, since $2^2(1, \bar{0}) \in L \cap E$, but $2(1, \bar{0}) \notin L \cap E + soc(T)$ where $soc(T) = (0)$.

Proposition (13)

Let L and E are two app-semi-prime submodules of an R-module T with $soc(T) \subseteq L$ or $soc(T) \subseteq E$. Then $L \cap E$ is an app-semi-prime submodule of T.

Proof

Suppose that $a^n t \in L \cap E$, where $a \in R$, $t \in T$ and $n \in Z^+$, implies that $a^n t \in L$ and $a^n t \in L$. But L and E are app-semi-prime submodules of T, then $at \in L + soc(T)$ and $at \in E + soc(T)$, it follows that $at \in (L + soc(T)) \cap (E + soc(T))$. If $soc(T) \subseteq E$, then $at \in \left(L + soc(T)\right) \cap E$ and by Lemma (2.8) we have $at \in (L \cap E) + soc(T)$. Similarly if $soc(T) \subseteq L$, we get $at \in (L \cap E) + soc(T)$. Hence $L \cap E$ is an app-semi-prime submodule of T.

Remark (14)

If L and E are submodules of an R-module T with $L \subseteq E$, and E ia an app-semi-prime submodule of T, then L is not an app-semi-prime submodule of T, the following example shows that.

Let $L = 12Z$ and $E = 6Z$ are submodules of a Z-module Z, $L \subseteq E$ and E is an app-semi-prime submodule of Z, but $L = 12Z$ is not an app-semi-prime submodule because $2^2.3 \in 12Z$, but $2.3 \notin 12Z + soc(Z)$.

Recall that an R-module T is multiplication if every submodule L of T is of the form $L = IT$ for some ideal I of R. In particular $L = [L:_R T]T$ [14]. Recall that for any submodules K and F of a multiplication R-module T with $K = IT$ and $F = JT$ for some ideals I and J of R. The product $KF = IT.JT = IJT$ that is $KF = IF$. In particular $KT = ITT = IT = K$. Also for any $t \in T$, we have $Kt = K\langle t\rangle = It$ [15].

The following propositions are characterizations of app-semi-prime submodules in the class of multiplication modules.

Proposition (15)

Let L be a proper submodule of a multiplication R-module T. Then L is an app-semi-prime submodule of T if and only if $K^n F \subseteq L$ implies that $KF \subseteq L + soc(T)$, where K, F are submodules of T, $n \in Z^+$.

Proof

($\Longrightarrow$) Suppose that $K^n F \subseteq L$, where K, F are submodules of T, $n \in Z^+$. Since T is a multiplication then $K = IT$ and $F = JT$ for some ideals I, J of R. Thus $K^n F = (IT)^n JT = I^n(JT) \subseteq L$. But L is an app-semi-prime, then by Proposition (2.3) we have $I(JT) \subseteq L + soc(T)$. That is $IF \subseteq L + soc(T)$, so $KF \subseteq L + soc(T)$.

($\Longleftarrow$) Suppose that $I^n F \subseteq L$, where I is an ideal of R, F is a submodule of T and $n \in Z^+$. Since T is a multiplication then $F = JT$ for some ideal J of R. That is $I^n JT \subseteq L$ implies that $K^n F \subseteq L$ so by hypothesis $KF \subseteq L + soc(T)$, thus $IF \subseteq L + soc(T)$. Hence by Proposition (2.3) L is an app-semi-prime submodule of T.

Proposition (16)

Let L be a proper submodule of a multiplication R-module T. Then the following statements are equivalent:

1) L is an app-semi-prime submodule of T.
2) $t^n \in L$ implies that $t \in L + soc(T)$ for every $t \in T$.
3) $\sqrt{L} \subseteq L + soc(T)$.
4) $F_1 F_2 \ldots \ldots F_j \subseteq L$, implies that $F_1 \cap F_2 \cap \ldots \ldots \cap F_j \subseteq L + soc(T)$ for every submodules $F_1, F_2, \ldots \ldots, F_j$ of T and $j \in Z^+$.

Proof

(1) ⟹(2) Let $t^n \in L$ where $t \in T$ and for some $n \in Z^+$, then $\langle t^n \rangle \subseteq L$. But T is a multiplication R-module, then $\langle t \rangle = IT$ for some ideal I of R, so $\langle t^n \rangle = I^n T \subseteq L$. Since L is an app-semi-prime submodule of T, then by Corollary (2.4) we have $IT \subseteq L + soc(T)$. That is $\langle t \rangle \subseteq L + soc(T)$, implies that $t \subseteq L + soc(T)$.

(2) ⟹(3) Let $t \in \sqrt{L}$, implies that $t^n \in L$ for some $n \in Z^+$, so by hypothesis $t \in L + soc(T)$. Thus $\sqrt{L} \subseteq L + soc(T)$.

(3) ⟹(4) Suppose that $F_1 F_2 \dots \dots F_j \subseteq L$ where $F_1, F_2, \dots \dots, F_j$ are submodules of T and $j \in Z^+$. Let $t \in F_1 \cap F_2 \cap \dots \dots \cap F_j$ then $t \in F_i$ for each $= 1,2, \dots \dots, j$, so $t^j \in F_1 F_2 \dots \dots F_j \subseteq L$, it follows that $t^j \in L$, so $t \in \sqrt{L}$. But by hypothesis $\sqrt{L} \subseteq L + soc(T)$, then $t \in L + soc(T)$. Thus $F_1 \cap F_2 \cap \dots \dots \cap F_j \subseteq L + soc(T)$.

(3) ⟹(4) Let $I^n E \subseteq L$, where I is an ideal of R, E is a submodule of T, and $n \in Z^+$. That is $(IT)(IT) \dots \dots (IT) \subseteq L$, so by hypothesis $(IT) \cap (IT) \dots \dots \cap (IT) \subseteq L + soc(T)$. Implies that $IT \subseteq L + soc(T)$. Thus by Corollary (2.4) L is an app-semi-prime submodule of T.

Remark (17)

If L is an app-semi-prime submodule of an R-module T, then $[L:_R T]$ is not necessary app-semi-prime ideal of R. The following example shows that.

Consider the Z-module Z_8 and a submodule $L = \langle \bar{0} \rangle$. L is an app-semi-prime submodule since $soc(Z_8) = \langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ and $2^2 . \bar{2} \in L$, implies that $2 . \bar{2} = 4 \in L + soc(Z_8) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$, where $2 \in Z, \bar{2} \in Z_8$. So for all $a \in R$, $\bar{t} \in Z_8$, such that $a^n \bar{t} \in L$ for some $n \in Z^+$, implies that $a\bar{t} \in L + soc(Z_8)$. But $[L:_Z T] = [\langle \bar{0} \rangle :_Z Z_8] = 8Z$ is not app-semi-prime ideal in Z, since $2^2 . 2 \in 8Z$ but $2.2 \notin 8Z + soc(Z) = 8Z + (0) = 8Z$.

Proposition (18)

Let L be an app-semi-prime submodule of an R-module T, with $soc(T) \subseteq L$. Then $[L:_R T]$ is an app-semi-prime ideal of R.

Proof

Suppose that $a^n s \in [L:_R T]$, where $a, s \in R$, $n \in Z^+$, then $a^n s\, T \subseteq L$. That is $a^n (s\, T) \subseteq L$, implies that $a^n (s\, t) \in L$ for all $t \in T$. But L is an app-semi-prime submodule of T, implies that $a(s\, t) \in L + soc(T)$, that is $as\, T \subseteq L + soc(T)$. But $soc(T) \subseteq L$, implies that $L + soc(T) = L$, thus $as\, T \subseteq L$, it follows that $as \in [L:_R T] \subseteq [L:_R T] + soc(R)$. Therefore $[L:_R T]$ is an app-semi-prime ideal of R.

We need to introduce the following Lemma which appear in [14].

Lemma (19)[14, Coro. 2.14].

Let T be a faithful multiplication R-module then $soc(R)T = soc(T)$.

Proposition (20)

Let T be a faithful multiplication R-module and L be a proper submodule of T. Then L is an app-semi-prime submodule of T if and only if $[L:_R T]$ is an app-semi-prime ideal of R.

Proof

($\Longrightarrow$) Suppose that L is an app-semi-prime submodule of T, to prove that $\sqrt{[L:_R T]} \subseteq [L:_R T] + soc(R)$ by proposition (2.16), we get $[L:_R T]$ is an app-semi-prime ideal of R. Let $a \in \sqrt{[L:_R T]}$, implies that $a^n \in [L:_R T]$ for some $n \in Z^+$, it follows that $a^n T \subseteq L$, that is $a^n t \in L$ for all $t \in T$. But L is an app-semi-prime submodule of T, then $at \in L + soc(T)$ for all $t \in T$. That is $aT \subseteq L + soc(T)$. Since T is a multiplication$R$-module, so $L = [L:_R T]T$, and since T is faithful multiplication, so by Lemma (2.19) $soc(T) = soc(R)T$. thus $aT \subseteq [L:_R T]T + soc(R)T$, it follows that $a \in [L:_R T] + soc(R)$. Hence $\sqrt{[L:_R T]} \subseteq [L:_R T] + soc(R)$, therefore $[L:_R T]$ is an app-semi-prime ideal of R.

($\Longleftarrow$) Suppose that $[L:_R T]$ is an app-semi-prime ideal of R, and let $a^n t \in L$, where $a \in R$, $t \in T$ and $n \in Z^+$, it follows that $a^n T \subseteq L$, implies that $a^n \in [L:_R T]$ so $a \in \sqrt{[L:_R T]}$. Since $[L:_R T]$ is an app-semi-prime ideal of R then by proposition (2.16) $\sqrt{[L:_R T]} \subseteq [L:_R T] + soc(R)$, it follows that $a \in [L:_R T] + soc(R)$, so $aT \subseteq [L:_R T]T + soc(R)T$. Since T is faithful multiplication, then by lemma (2.19) we get $aT \subseteq L + soc(T)$, hence $at \in L + soc(T)$ for all $t \in T$. Thus L is an app-semi-prime submodule of T.

Recall that an R-module T is a non-singular provided that $Z(T) = T$, where $Z(T) = \{x \in T : xI = 0$ for some essential ideal I of $R\}$ [12].

We need the following Lemma which appears in [12]. before we introduced the next result.

Lemma (21) [12, Coro. 1.26].

If T is a non-singular R-module, then $soc(R)T = soc(T)$.

Proposition (22)

Let L be a proper submodule of non-singular multiplication R-module T. Then L is an app-semi-prime submodule of T if and only if $[L:_R T]$ is an app-semi-prime ideal of R.

Proof

Follows by similar steps of proposition (2.20) and used lemma (2.21), and proposition (2.16).

We need the following Lemma which appear in [16]. before we introduce the next result.

Lemma (23)[16, Coro. Of Theo. 9].

Let T be a finitely generated multiplication R-module and I, J are ideals of a ring R. Then $IT \subseteq JT$ if and only if $I \subseteq J + ann(T)$.

Proposition (24)

Let T be a faithful finitely generated multiplication R-module and J be an app-semi-prime ideal of R. Then JT is an app-semi-prime submodule of T.

Proof

Let $a^n E \subseteq JT$, where $a \in R$, E be a submodule of T and $n \in Z^+$. Since T is a multiplication, then $E = IT$ for some ideal I of R. That is $a^n IT \subseteq JT$. But T is a finitely generated, so by Lemma (2.23) we have $a^n I \subseteq J + ann(T)$, but T is faithful, then $ann(T) = (0)$, hence $a^n I \subseteq J$, but J an app-semi-prime ideal of R, then by proposition (2.3) $aI \subseteq J + soc(R)$. That is $aIT \subseteq JT + soc(R)T$, then by Lemma (2.19) we get $aE \subseteq JT + soc(T)$. Hence JT is an app-semi-prime submodule of T.

Proposition (25)

Let T be a finitely generated multiplication non-singular R-module and J be an app-semi-prime ideal of R with $ann(T) \subseteq J$. Then JT is an app-semi-prime submodule of T.

Proof

Suppose that $a^n L \subseteq JT$, where $a \in R$, L be a submodule of T and $n \in Z^+$. Since T is a multiplication, then $L = IT$ for some ideal I of R. That is $a^n IT \subseteq JT$. But T is a finitely generated, so by Lemma (2.23) we have $a^n I \subseteq J + ann(T)$, but $ann(T) \subseteq J$, implies that $J + ann(T) = J$, so $a^n I \subseteq J$, but J an app-semi-prime ideal of R, we have $aI \subseteq J + soc(R)$. That is $aIT \subseteq JT + soc(R)T$, then by Lemma (2.19) we get $aL \subseteq JT + soc(T)$. Hence JT is an app-semi-prime submodule of T.

Theorem (26)

Let T be a faithful finitely generated multiplication R-module and L be a proper submodule of T. Then the following statements are equivalent.

1) L is an app-semi-prime submodule of T.
2) $[L:_R T]$ is an app-semi-prime ideal of R.
3) $L = IT$ for some app-semi-prime ideal I of R.

Proof

(1) $\Longleftrightarrow$ (2) Follows by proposition (2.20).

(2) $\Longrightarrow$ (3) Since $[L:_R T]$ is an app-semi-prime ideal of R, and $L = [L:_R T]T$ for T is a multiplication, implies that $L = IT$ where $[L:_R T]$ is an app-semi-prime ideal of R.

(3) $\Longrightarrow$ (2) Suppose that $L = IT$ for some app-semi-prime ideal I of R. But T is a faithful finitely generated multiplication, then by Lemma (2.23) $I = [L:_R T]$, hence $[L:_R T]$ is an app-semi-prime ideal of R.

Theorem (27)

Let T be non-singular finitely generated multiplication R-module and L be a proper submodule of T. Then the following statements are equivalent.

1) L is an app-semi-prime submodule of T.
2) $[L:_R T]$ is an app-semi-prime ideal of R.
3) $L = IT$ for some app-semi-prime ideal I of R with $ann(T) \subseteq I$.

Proof

(1) $\Longleftrightarrow$ (2) Follows by proposition (2.22).

(2) $\Longrightarrow$ (3) Since $[L:_R T]$ is an app-semi-prime ideal of R, and $L = [L:_R T]T$ for T is a multiplication, then $L = IT$ and $I = [L:_R T]$ is an app-semi-prime ideal of R with $[(0):_R T] = ann(T) \subseteq [L:_R T]$.

(3) $\Longrightarrow$ (2) Suppose that $L = IT$ for some app-semi-prime ideal I of R with $ann(T) \subseteq I$. But T is a multiplication, we have $I = [L:_R T]T = IT$, that is $[L:_R T] = I + ann(T) = I$, it follows that $[L:_R T]$ is an app-semi-prime ideal of R.

Recall that an envelope of a submodule L of an R-module denoted by $E_T(L)$ defined by $E_T(L) = \{at: a \in R, t \in T \text{ such that } a^n t \in L, n \in Z^+\}$ and $L \subseteq E_T(L)$ [17].

Proposition (28)

Let L be a proper submodule of an R-module T. Then L is an app-semi-prime submodule of T if and only if $E_T(L) \subseteq L + soc(T)$.

Proof

($\Longrightarrow$) Let $\in E_T(L)$, implies that $y = at$, where $a \in R$, $t \in T$ such that $a^n t \in L$ for some $n \in Z^+$. But L is an app-semi-prime submodule of T, then $at \in L + soc(T)$, that is $y \in L + soc(T)$ so $E_T(L) \subseteq L + soc(T)$.

($\Longleftarrow$) Suppose that $a^n t \in L$, where $a \in R$, $t \in T$ and $n \in Z^+$. Since $a^n t \in L$ then $at \in E_T(L) \subseteq L + soc(T)$ by hypothesis. It follows that $rt \in L + soc(T)$. Hence L is an app-semi-prime submodule of T.

Proposition (29)

Let L be a proper submodule of an R-module T with $soc(T) \subseteq L$. Then L is an app-semi-prime submodule of T if and only if $[L:_R T]$ is a semi-prime ideal of R.

Proof

($\Longrightarrow$) Let $a \in \sqrt{[L:_R T]}$, implies that $a^n \in [L:_R T]$ for some $n \in Z^+$, it follows that $a^n T \subseteq L$, that is $a^n t \in L$ for all $t \in T$. But L is an app-semi-prime submodule of T, then $at \in L + soc(T)$ for all $t \in T$. That is $aT \subseteq L + soc(T)$. Since $soc(T) \subseteq L$ we get $aT \subseteq L$ since $L + soc(T) = L$. That is $a \in [L:_R T]$, hence $\sqrt{[L:_R T]} \subseteq [L:_R T]$. But $[L:_R T] \subseteq \sqrt{[L:_R T]}$, it follows that $\sqrt{[L:_R T]} = [L:_R T]$, hence $[L:_R T]$ is a semi-prime ideal of R.

($\Longleftarrow$) Suppose that $a^n t \in L$, where $a \in R$, $t \in T$ and $n \in Z^+$, it follows that $a^n \in [L:_R T]$, implies that $a \in \sqrt{[L:_R T]}$. But $[L:_R T]$ is a semi-prime ideal of R, implies that $\sqrt{[L:_R T]} = [L:_R T]$, hence $a \in [L:_R T]$, it follows that $aT \subseteq L = L + soc(T)$. Thus $at \in L + soc(T)$ for all $t \in T$. Thus L is an app-semi-prime submodule of T.

Proposition (30)

Let $T = T_1 \oplus T_2$ be an R-module, where T_1 and T_2 R-modules, and $L = L_1 \oplus L_2$ be a submodule of T, where L_1 is a submodule of T_1 and L_2 is a submodule of T_2 with $L \subseteq soc(T) = soc(T_1) \oplus soc(T_2)$. If L is an app-semi-prime submodule of T, then L_1 is an app-semi-prime submodule of T_1 and L_2 is an app-semi-prime submodule of T_2.

Proof

Let $a^n t \in L_1$, where $a \in R$, $t \in T_1$ and $n \in Z^+$, it follows that $a^n(t, 0) \in L$. But L is an app-semi-prime submodule of T, then $a(t, 0) \in L + soc(T)$. But $L \subseteq soc(T)$, implies that $L + soc(T) = soc(T)$, thus $a(t, 0) \in soc(T) = soc(T_1) \oplus soc(T_2)$, it follows that $at \in soc(T_1) \subseteq L_1 + soc(T_1)$. Thus L_1 is an app-semi-prime submodule of T_1.
Similarly we can prove L_2 is an app-semi-prime submodule of T_2.

Proposition (31)

Let $T = T_1 \oplus T_2$ be an R-module, where each of T_1 and T_2 R-module. Then the following statements are satisfy:

1) L_1 is an app-semi-prime submodule of T_1 such that $L_1 \subseteq soc(T_1)$ and $T_2 = soc(T_2)$ if and only if $L_1 \oplus T_2$ is an app-semi-prime submodule of T.

2) L_2 is an app-semi-prime submodule of T_2 such that $L_2 \subseteq soc(T_2)$ and $T_1 = soc(T_1)$ if and only if $T_1 \oplus L_2$ is an app-semi-prime submodule of T.

Proof

1) ($\Longrightarrow$) Let $a^n(t_1,t_2) \in L_1 \oplus T_2$, where $a \in R$, $(t_1,t_2) \in T$ and $n \in Z^+$, then $a^n t_1 \in L_1$. But L_1 is an app-semi-prime submodule of T_1 and $L_1 \subseteq soc(T_1)$, then $at_1 \in L_1 + soc(T_1) = soc(T_1)$. Now, we have $T_2 = soc(T_2)$ then $a(t_1,t_2) \in soc(T_1) \oplus soc(T_2) = soc(T_1 \oplus T_2) \subseteq L_1 \oplus T_2 + soc(T_1 \oplus T_2)$. Thus $L_1 \oplus T_2$ is an app-semi-prime submodule of T.

($\Longleftarrow$) Suppose that $a^n t_1 \in L_1$, where $a \in R$, $t_1 \in T_1$ and $n \in Z^+$. Then for each $t_2 \in T_2$ $a^n(t_1,t_2) \in L_1 \oplus T_2$, but $L_1 \oplus T_2$ is an app-semi-prime submodule of T, so $a(t_1,t_2) \in L_1 \oplus T_2 + soc(T_1 \oplus T_2) = (L_1 \oplus T_2) + (soc(T_1) \oplus soc(T_2))$. Since $L_1 \subseteq soc(T_1)$ and $T_2 = soc(T_2)$, it follows that $a(t_1,t_2) \in (L_1 \oplus T_2) + \left(\left(L_1 + soc(T_1)\right) \oplus T_2\right)$, implies that $a(t_1,t_2) \in$
$\left(L_1 + soc(T_1)\right) \oplus T_2$ $\left[\text{because } L_1 \oplus T_2 \subseteq \left(L_1 + soc(T_1)\right) \oplus T_2 \text{ implies that } (L_1 \oplus T_2) + \left(\left(L_1 + soc(T_1)\right) \oplus T_2\right) = \left(L_1 + soc(T_1)\right) \oplus T_2\right]$. Thus $at_1 \in L_1 + soc(T_1)$. Hence L_1 is an app-semi-prime submodule of T_1.

2) Similarly we can prove (2).

Proposition (32)

Let $f: T \longrightarrow T'$ be an R-epimorphism and L is an app-semi-prime submodule of T'. Then $f^{-1}(L)$ is an app-semi-prime submodule of T.

Proof

It is clear that $f^{-1}(L)$ is a proper submodule of T. Now, let $a^n t \in f^{-1}(L)$, where $a \in R$, $t \in T$ and $n \in Z^+$, implies that $a^n f(t) \in L$. But L is an app-semi-prime submodule of T', implies that $af(t) \in L + soc(T')$, it follows that $at \in f^{-1}(L) + f^{-1}\left(soc(T')\right) \subseteq f^{-1}(L) + soc(T)$. Thus $at \in f^{-1}(L) + soc(T)$. Therefore $f^{-1}(L)$ is an app-semi-prime submodule of T.

Proposition (33)

Let $f: T \longrightarrow T'$ be an R-epimorphism and K be an app-semi-prime submodule of T with $Ker\, f \subseteq K$. Then $f(K)$ is an app-quasi-prime submodule of T'.

Proof

$f(K)$ is a proper submodule of T'. If not, $f(K) = T'$, that is $t \in T$, then $f(t) \in T' = f(K)$, implies that $f(t) = f(k)$ for some $k \in K$, that is $f(t-k) = 0$, thus $t - k \in Ker\, f \subseteq K$, it follows that $t \in K$, that is $T \subseteq K$, but $K \subseteq T$, so $T = K$ contradiction. Now let $a^n t' \in f(K)$, where $a \in R$, $t' \in T'$ and $n \in Z^+$. But f is an epimorphism, then $f(t) = t'$ for some $t \in T$. That is $a^n f(t) \in f(K)$, implies that $a^n f(t) = f(k)$ for some $k \in K$. That is $f(a^n t - k) = 0$, it follows that $a^n t - k \in Ker\, f \subseteq K$, hence $a^n t \in K$. But K is an app-semi-prime submodule of T, then $at \in K + soc(T)$. Thus $(t) \in f(K) + f(soc(T)) \subseteq f(K) + soc(T')$. That is $at' \in f(K) + soc(T')$. Hence $f(K)$ is an app-semi-prime submodule of T'.

3. Conclusion

In this paper we define the concept of approximaitly semi-prime (for short app-semi-prime) submodules, and we introduced several properties, characterizations of it. Also, we investigate the relationships of app-semi-prime submodules with prime submodules, semi-prime submodules, quasi-prime submodules and approximaitly prime submodules, we proved that app-semi-prime submodules are generalizations of above concepts, and we illistright the convers by examples. Also, we show by example that the resudule of an app-semi-prime submodule is not necessary app-semi-prime ideal of R, but we prove under certain conditions they are equivalents. A mange the main results we get are the following.

1) Let L be a proper submodule of an R-module T. Then $L + soc(T)$ is an app-semi-prime submodule of T if and only if $[L + soc(T): T]$ is a semi-prime ideal of R (hence an app-semi-prime).

2) Let L be a proper submodule of an R-module T with $soc(T) \subseteq L$. Then L is an app-semi-prime submodule of T if and only if $[L:_T a^n] = [L:_T a]$ for $a \in R$ and some $n \in Z^+$.

3) Let L and E are two app-semi-prime submodules of an R-module T with $soc(T) \subseteq L$ or $soc(T) \subseteq E$. Then $L \cap E$ is an app-semi-prime submodule of T.

4) Let L be a proper submodule of a multiplication R-module T. Then L is an app-semi-prime submodule of T if and only if $K^n F \subseteq L$ implies that $KF \subseteq L + soc(T)$, where K, F are submodules of $T, n \in Z^+$.

5) Let L be a proper submodule of a multiplication R-module T. Then the following statements are equivalent:

1) L is an app-semi-prime submodule of T.

2) $t^n \in L$ implies that $t \in L + soc(T)$ for every $t \in T$.

3) $\sqrt{L} \subseteq L + soc(T)$.

4) $F_1 F_2 \dots\dots F_j \subseteq L$, implies that $F_1 \cap F_2 \cap \dots\dots \cap F_j \subseteq L + soc(T)$ for every submodules $F_1, F_2, \dots\dots, F_j$ of T and $j \in Z^+$.

6) Let T be a faithful multiplication R-module and L be a proper submodule of T. Then L is an app-semi-prime submodule of T if and only if $[L:_R T]$ is an app-semi-prime ideal of R.

7) Let T be a faithful finitely generated multiplication R-module and J be an app-semi-prime ideal of R. Then JT is an app-semi-prime submodule of T.

8) Let T be a faithful finitely generated multiplication R-module and L be a proper submodule of T. Then the following statements are equivalent.

1) L is an app-semi-prime submodule of T.

2) $[L:_R T]$ is an app-semi-prime ideal of R.

3) $L = IT$ for some app-semi-prime ideal I of R.

9) Let T be non-singular finitely generated multiplication R-module and L be a proper submodule of T. Then the following statements are equivalent.

1) L is an app-semi-prime submodule of T.

2) $[L:_R T]$ is an app-semi-prime ideal of R.

3) $L = IT$ for some app-semi-prime ideal I of R with $ann(T) \subseteq I$.

10) Let $T = T_1 \oplus T_2$ be an R-module, where T_1 and T_2 R-modules, and $L = L_1 \oplus L_2$ be a submodule of T, where L_1 is a submodule of T_1 and L_2 is a submodule of T_2 with $L \subseteq soc(T) = soc(T_1) \oplus soc(T_2)$. If L is an app-semi-prime submodule of T, then L_1 is an app-semi-prime submodule of T_1 and L_2 is an app-semi-prime submodule of T_2.

11) Let $T = T_1 \oplus T_2$ be an R-module, where each of T_1 and T_2 R-module. Then the following statements are satisfy:

1) L_1 is an app-semi-prime submodule of T_1 such that $L_1 \subseteq soc(T_1)$ and $T_2 = soc(T_2)$ if and only if $L_1 \oplus T_2$ is an app-semi-prime submodule of T.

2) L_2 is an app-semi-prime submodule of T_2 such that $L_2 \subseteq soc(T_2)$ and $T_1 = soc(T_1)$ if and only if $T_1 \oplus L_2$ is an app-semi-prime submodule of T.

12) Let $f: T \longrightarrow T'$ be an R-epimorphism and L is an app-semi-prime submodule of T'. Then $f^{-1}(L)$ is an app-semi-prime submodule of T.

13) Let $f: T \longrightarrow T'$ be an R-epimorphism and K be an app-semi-prime submodule of T with $Ker\ f \subseteq K$. Then $f(K)$ is an app-quasi-prime submodule of T'.

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