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Pseudo Weakly Closed Submodules and Related Concepts

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Abstract

Let R be a commutative ring with identity, and M be a unitary left R-module. In this paper we introduce the concept pseudo weakly closed submodule as a generalization of W-closed submodules, where a submodule E of an R-module M is called a pseudo weakly closed submodule, if for all $m \in M - E$, there exists a W-closed submodule K of M with E is a submodule of K such that $m \notin K$. Several basic properties, examples and results of pseudo weakly closed submodules are given. Furthermore the behavior of pseudo weakly closed submodules in class of multiplication modules are studied. On the other hand modules with chain conditions on pseudo weakly closed submodules are established. Also, the relationships of pseudo weakly closed submodules with other classes of modules are discussed.

Keywords: Closed submodules, ω-closed submodules, pseudo weakly closed submodules, semi-prime submodules, fully semi-prime submodules, weakly essential submodules.

1. Introduction

A proper submodule L of an R-module M is called closed in M, provided that L has no proper essential extensions in M [1]. Where a non-zero submodule K of an R-module M is called essential in M if $K \cap F \neq (0)$ for each non-zero submodule F of M [1]. And a non-zero submodule L of M is called weakly essential submodule of M if $L \cap S \neq (0)$ for each non-zero semi-prime submodule S of M [2]. Equivalently L is weak essential, if whenever $L \cap S = (0)$, then $S = (0)$ for every semi-prime submodule S of M [3]. Where a submodule S of M is called semi-prime if whenever $r^2 m \in S$, for $r \in R$, $m \in M$, implies that $rm \in S$ [4]. The concept of closed submodule recently extended by [5]. To ω-closed submodule, where a submodule L of M is called ω-closed submodule of M if L has no proper weak-essential extensions in M, that is if L is a weak essential submodule of K, where K is a submodule of M, then $L = K$ [6,7]. This concept is generalized in this article to a pseudo weakly closed submodule. Many basic properties of this concept are discussed. Finally, we notes that throughout this paper all rings are commutative with identity and all modules are unitary left R-modules, unless otherwise. Also, in this paper all R-module under study contains semi-prime submodules.

2. Pseudo Weakly Closed Submodules

In this section we introduce the notion of pseudo weakly closed submodule as a generalization of ω-closed submodule and give some basic properties and examples of this class.

Definition 2.1

A submodule E of an R-module M is called pseudo weakly closed submodule(for a short $\rho\omega$-closed), if for each $m \in M - E$, there exists a ω-closed L of M with $E \subseteq L$ such that $m \notin L$. An ideal J of a ring R is called $\rho\omega$-closed if it is $\rho\omega$-closed submodule of an R-module R.

Remarks and Examples 2.2

1. It is clear that every ω-closed submodule of an R-module M is $\rho\omega$-closed submodule of M, but the converse is not true in general as the following example explain that:
Consider the Z-module Z_{12} . The proper submodules of Z_{12} are: $N_1 = \langle\bar{2}\rangle$, $N_2 = \langle\bar{3}\rangle$, $N_3 = \langle\bar{4}\rangle$ and $N_4 = \langle\bar{6}\rangle$. The submodule $N_3 = \{\bar{0}, \bar{4}, \bar{8}\}$ is $\rho\omega$-closed but not closed because $\bar{1}, \bar{3}, \bar{5}, \bar{7}, \bar{9}, \overline{11}$ are in Z_{12} but not in N_3, then there exists a ω-closed submodule $N_1 = \langle\bar{2}\rangle$ in Z_{12} such that $N_3 \subseteq N_1$ and $\bar{1}, \bar{3}, \bar{5}, \bar{7}, \bar{9}, \overline{11}$ are not in N_1. Now N_3 is not a ω-closed since N_3 is a weak essential submodule of N_1.

2. A direct summand of an R-module M is not necessary $\rho\omega$-closed submodule in M, for example:
Consider the Z-module Z_{60}. In this module there are ten proper submodules $H_1 = \langle\bar{2}\rangle$, $H_2 = \langle\bar{3}\rangle$, $H_3 = \langle\bar{4}\rangle$, $H_4 = \langle\bar{5}\rangle$, $H_5 = \langle\bar{6}\rangle$, $H_6 = \langle\overline{10}\rangle$, $H_7 = \langle\overline{12}\rangle$, $H_8 = \langle\overline{15}\rangle$, $H_9 = \langle\overline{20}\rangle$, $H_{10} = \langle\overline{30}\rangle$ where $Z_{60} = \langle\bar{6}\rangle \oplus \langle\overline{10}\rangle = H_5 \oplus H_6$. That is both H_5 and H_6 are a direct summands in Z_{60}. But $H_5 = \langle\bar{6}\rangle$ is not $\rho\omega$-closed submodule in Z_{60}, since $\bar{4} \in Z_{60}$, $\bar{4} \notin H_5 = \langle\bar{6}\rangle$, then there exists a ω-closed submodule $H_1 = \langle\bar{2}\rangle$ in Z_{60} with $\langle\bar{6}\rangle \subseteq \langle\bar{2}\rangle$, but $\bar{4} \in \langle\bar{2}\rangle$. Also $\langle\overline{10}\rangle$ is not $\rho\omega$-closed submodule in Z_{60}, since $\bar{8} \in Z_{60}$, $\bar{8} \notin \langle\overline{10}\rangle$, then there exists a ω-closed submodule $\langle\bar{2}\rangle$ in Z_{60} such that $\langle\overline{10}\rangle \subseteq \langle\bar{2}\rangle$, but $\bar{8} \in \langle\bar{2}\rangle$.

3. If K is a $\rho\omega$-closed submodule in an R-module M, then $[K :_R M]$ need not be a $\rho\omega$-closed ideal in R. For example:
Consider the Z-module Z_{28}. In this module there are four proper submodules, which are: $K_1 = \langle\bar{2}\rangle$, $K_2 = \langle\bar{4}\rangle$, $K_3 = \langle\bar{7}\rangle$, $K_4 = \langle\overline{14}\rangle$. Not that the submodule K_3 is a $\rho\omega$-closed submodule in Z_{28} (since K_3 is a ω-closed submodule in Z_{28} and hence by (1) K_3 is a $\rho\omega$-closed). While $[K_3 :_Z Z_{28}] = 7Z$ is not a $\rho\omega$-closed ideal in Z. Since there is no a ω-closed ideal in Z containing $7Z$.

4. If M is a module and E is a $\rho\omega$-closed submodule in M with D is a submodule of M such that $E \cong D$, then it is not necessary that D is a $\rho\omega$-closed submodule in M. For example:
The Z-module Z is a $\rho\omega$-closed submodule in Z (since Z as a Z-module is a ω-closed in Z) and $Z \cong 2Z$, but $2Z$ is not a $\rho\omega$-closed submodule in Z.

5. If M is an R-module, L and K are submodules of an R-module M such that $L \subseteq K \subseteq M$ and K is a $\rho\omega$-closed submodule in M, then L need not to be a $\rho\omega$-closed submodule in M. For example:

Consider the Z-module Z is a $\rho\omega$-closed submodule in Z and $3Z \subseteq Z$, while $3Z$ is not $\rho\omega$-closed submodule in Z, , since there is no a ω-closed submodule in Z containing $3Z$.

6. If M is an R-module, L and K are submodules of an R-module M such that $L \subseteq K \subseteq M$ and L is a $\rho\omega$-closed submodule in M, then K need not to be a $\rho\omega$-closed submodule in M. For example:

Consider the Z-module Z and the submodules $L = (0)$ and $K = 6Z$. Notes that (0) is a $\rho\omega$-closed submodule in Z (since (0) is a ω-closed in Z), but $6Z$ is not $\rho\omega$-closed submodule in Z, since there is no a ω-closed submodule in Z containing $6Z$.

7. $[1\,, \text{Exc}\,(1.6), \text{p}.\,15]$ shows that the intersection of two closed submodules need not to be closed submodule". Also, the intersection of two $\rho\omega$-closed submodules need not to be $\rho\omega$-closed submodule as the following example shows:

The Z-module $M = Z_6 \oplus Z_2$ the submodules $\langle(\bar{0}, \bar{1})\rangle$ and $\langle(\bar{3}, \bar{1})\rangle$ are $\rho\omega$-closed submodules in M (since they are ω-closed submodules in M), but $\langle(\bar{0}, \bar{1})\rangle \cap \langle(\bar{3}, \bar{1})\rangle = \langle(\bar{0}, \bar{0})\rangle$ is not $\rho\omega$-closed submodule in M since $\langle(\bar{0}, \bar{0})\rangle$ is closed submodule in M.

8. Closed submodules and $\rho\omega$-closed submodules are independent concepts, as the following examples show that:

Consider the Z-module Z_{36}. In this module the proper submodules of Z_{36} are: $L_1 = \langle\bar{2}\rangle$, $L_2 = \langle\bar{3}\rangle$, $L_3 = \langle\bar{4}\rangle$, $L_4 = \langle\bar{6}\rangle$, $L_5 = \langle\bar{9}\rangle$, $L_6 = \langle\overline{12}\rangle$, $L_7 = \langle\overline{18}\rangle$. We notes that the submodule $L_5 = \langle\bar{9}\rangle$ is closed in Z_{36}, since L_5 has no proper essential extension in Z_{36}, while $L_5 = \langle\bar{9}\rangle$ is not $\rho\omega$-closed submodule in Z_{36} since $\bar{6} \in Z_{36}$, $\bar{6} \notin L_5$, then there exists a ω-closed submodule $L_2 = \langle\bar{3}\rangle$ in Z_{36} with $L_5 \subseteq L_2$, but $\bar{6} \in L_2$. That is $L_5 = \langle\bar{9}\rangle$ closed submodule in Z_{36}, but not $\rho\omega$-closed submodule in Z_{36}. In the Z-module Z_{28}, the submodule $\langle\bar{2}\rangle$ is a $\rho\omega$-closed submodule in Z_{28} since$\langle\bar{2}\rangle$ is a ω-closed in Z_{28}, while $\langle\bar{2}\rangle$ is not closed in Z_{28}, since $\langle\bar{2}\rangle$ is an essential submodule of Z_{28}.

We start this section by the following proposition.

Proposition 2.3

If M is an R-module, E and B are submodules of M with $E \subseteq B$ and E is a $\rho\omega$-closed submodule in B and B is a ω-closed submodule in M, then E is a $\rho\omega$-closed submodule in M, provided that $B \subseteq K$ for any weak-essential extensions K of E.

Proof:

To prove that E is a $\rho\omega$-closed submodule in M, suppose that $x \in M$ with $x \notin E$, then either $x \in B$ or $x \notin B$. If $x \in B$ and since E is a $\rho\omega$-closed submodule in B, so there exists a ω-closed submodule K in B such that $E \subseteq K$ and $x \notin K$. Since K is a ω-closed in B and B is a ω-closed in M, then by $[5, \text{prop}\,(2.9)]$ K is a ω-closed in M. Thus we have a ω-closed submodule K in M such that $E \subseteq K$ and $x \notin K$. i.e. E is a $\rho\omega$-closed submodule in M. If $x \notin B$, then nothing to prove since B is a ω-closed submodule in M such that $E \subseteq B$ and $x \notin B$. Therefore E is a $\rho\omega$-closed submodule in M.

Proposition 2.4

Let E and D be submodules of a module M with $E \subseteq D$. If E is a $\rho\omega$-closed submodule in D, and D is a ω-closed in M, then E is a $\rho\omega$-closed submodule in M, provided that $K \subseteq D$ is for any weak essential extensions K of E.

Proof:

Similar as in proposition 2.3.

Proposition 2.5

If M is a uniserial R-module, E and D are submodules of M such that $E \subseteq D$ and E is a $\rho\omega$-closed submodule in D and D is a ω-closed submodule in M, then E is a $\rho\omega$-closed submodule in M.

Proof:

Prove is direct.

Recall that an R-module M is completely essential if every non-zero weak essential submodule of M is essential [5, 3].

As we mention in Example and Remarks (2.2)(8) closed submodules and $\rho\omega$-closed submodules
are independent, then the following propositions show that, the class of closed submodules is contained in the class of a $\rho\omega$-closed submodules under certain condition.

Proposition 2.6

Let E be a non zero closed submodule of a module M such that every weak essential extension of E is a completely essential submodule of M, then E is a $\rho\omega$-closed submodule in M.

Proof

Assume that E be a non zero closed submodule of M, then by $[5, \mathrm{prop}(2.13)]$ we get E is a ω-closed submodule in M, so by remarks and examples (2.2)(1) we get E is a $\rho\omega$-closed submodule of M.

Recall that an R-module M is called fully semi-prime, if every proper submodule of M is a semi-prime submodule [3].

Proposition 2.7

Let E be a non zero closed submodule of a fully semi-prime R-module M. Then E is a $\rho\omega$-closed submodule in M.

Proof

Suppose that E is a non zero closed submodule of M, then by $[5, \mathrm{prop}(2.14)]$ we get E is a ω-closed submodule in M. Hence by remarks and examples (2.2)(1) E is a $\rho\omega$-closed submodule in M.

Proposition 2.8

If E and D are submodules of an R-module M, with $E \subseteq D \subseteq M$, D containing every ω-closed submodule of M and E is a $\rho\omega$-closed submodule in M, then E is a a $\rho\omega$-closed submodule in D.

Proof

Let $x \in D$ and $x \notin E$, then $x \in M$. Since E is a $\rho\omega$-closed submodule in M, then $\exists$ a ω-closed submodule Q in M with $E \subseteq Q$ and $x \notin Q$. Hence Q is a ω-closed submodule in D with $E \subseteq Q$ and $x \notin Q$. Thus E is a a $\rho\omega$-closed submodule in D.

Proposition 2.9

If $M = M_1 \oplus M_2$ is a module where M_1 and M_2 are submodules of M, provided that $annM_1 + annM_2 = R$ and each weak essential extension of $E \oplus D$ is completely essential module, where E is a non-zero $\rho\omega$-closed submodule in M_1 and D is a non-zero $\rho\omega$-closed submodule in M_2, then $E \oplus D$ is a $\rho\omega$-closed submodule in $M_1 \oplus M_2$.

Proof

Let $x = x_1 + x_2 \in M_1 \oplus M_2$ with $x \notin E \oplus D$, then either $x_1 \notin E$ or $x_2 \notin D$. If $x_1 \notin E$ and since E is a $\rho\omega$-closed submodule in M_1, so there exists a ω-closed submodule U in M_1 with $E \subseteq U$ and $x_1 \notin U$. By $[5, \mathrm{Rem}(2.5)(1)]$ M_2 is a ω-closed in M_2, then by $[5, \mathrm{prop}(2.26)]$, we have $U \oplus M_2$ is a ω-closed in M such that $E \oplus D \subseteq U \oplus M_2$ and $x \notin U \oplus M_2$. Similarly if $x_2 \notin D$, then $\exists$ a ω-closed submodule U in M containing $E \oplus D$ and does not contain x. Thus $E \oplus D$ is a $\rho\omega$-closed submodule in M.

Proposition 2.10

If $M = M_1 \oplus M_2$ is an R-module where M_1 and M_2 are submodules of M, provided that $annM_1 + annM_2 = R$ and all submodules of M are completely essential submodules. If E is a non-zero $\rho\omega$-closed submodule in M_1 and D is a non-zero $\rho\omega$-closed submodule in M_2, then $E \oplus D$ is a $\rho\omega$-closed submodule in M if and only if E is a $\rho\omega$-closed submodule in M_1 and D is a $\rho\omega$-closed submodule in M_2.

Proof

($\Rightarrow$) To prove E is a $\rho\omega$-closed submodule in M_1. If $x \in M_1$ with $x \notin E$. Then $(x, 0) \notin E \oplus D$. But $E \oplus D$ is a $\rho\omega$-closed submodule in M, so there exists a ω-closed submodule L in M with $E \oplus D \subseteq L$ and $(x, 0) \notin L$. Since $annM_1 + annM_2 = R$, then by $[7, \mathrm{prop}(4.2)]$ any submodule of $M = M_1 \oplus M_2$ can be written as $L = L_1 \oplus L_2$, where $L_1 \subseteq M_1$ and $L_2 \subseteq M_2$. Hence by $[5, \mathrm{prop}(4.27)]$ it follows that L_1 is a $\rho\omega$-closed submodule in M_1 and L_2 is a $\rho\omega$-closed submodule in M_2. Since $E \oplus D \subseteq L$ and $(x, 0) \notin L$, then $x \notin L_1$. Therefore E is a $\rho\omega$-closed submodule in M_1. Similarly D is a $\rho\omega$-closed submodule in M_2.
($\Leftarrow$) Suppose that E is a $\rho\omega$-closed submodule in M_1 and D is a $\rho\omega$-closed submodule in M_2. Let $x = (x_1, x_2) \in M_1 \oplus M_2$ with $x \notin E \oplus D$, then either $x_1 \notin E$ or $x_2 \notin D$. If $x_1 \notin E$ and E is a $\rho\omega$-closed submodule in M_1, so $\exists$ a ω-closed submodule L_1 in M_1 such that $E \subseteq L_1$ and $x_1 \notin L_1$. But L_1 is a ω-closed submodule in M_1 and by $[5, \mathrm{Rem}(2.5)(1)]$ M_2 is a ω-closed in M_2, hence by $[5, \mathrm{prop}(2.26)]$, we have $L_1 \oplus M_2$ is a $\rho\omega$-closed submodule in M. Also $E \oplus D \subseteq L_1 \oplus M_2$ and $x \notin L_1 \oplus M_2$. Similarly if $x_2 \notin D$, then there exists a ω-closed submodule in M containing $E \oplus D$ and does not containing x. Thus $E \oplus D$ is a $\rho\omega$-closed submodule in M.
It is well known that a fully semi-prime module is a completely essential $[3, \mathrm{Coro}\ (1.4)]$ so we get the following corollary.

Corollary 2.11

If $M = M_1 \oplus M_2$ is an R-module where M_1 and M_2 are submodules of M, with $annM_1 + annM_2 = R$ and all submodules of M are fully semi-prime and E is a non-zero $\rho\omega$-closed submodule in M_1 and D is a non-zero $\rho\omega$-closed submodule in M_2 such that $E \oplus D$ is a $\rho\omega$-

closed submodule in M, then E is a $\rho\omega$-closed submodule in M_1 and D is a $\rho\omega$-closed submodule in M_2.

Proposition 2.12

If $f: M_1 \to M_2$is an epimorphism and let E be a submodule of M_1 such that $\ker(f) \subseteq Srad(M_1) \cap E$ and E is a $\rho\omega$-closed submodule in M_1, then $f(E)$ is a $\rho\omega$-closed submodule in M_2, where $Srad(M_1)$ is the intersection of all semi-prime submodule of M_1.

Proof

Suppose that E is a $\rho\omega$-closed submodule in M_1 and $x \in M_2$ with $x \notin f(E)$. Since f is an epi-morphism then there exists $x_1 \in M_1$ with $f(x_1) \in f(E)$ and $x_1 \notin E$. Since E is a $\rho\omega$-closed submodule in M_1, then there exists a ω-closed submodule K in M_1 with $E \subseteq K$ and $x_1 \notin K$. Thus by $[5, \text{prop}(2.31)]$ $f(K)$ is a ω-closed submodule in M_2. Since f is an epi-morphism, then $f(E) \subseteq f(K)$ and $x \notin f(K)$. Thus $f(E)$ is a $\rho\omega$-closed submodule in M_2.

Corollary 2.13

If E and D are submodules of a module M with $\mathrm{E} \subseteq Srad(M) \cap D$ and D is a $\rho\omega$-closed submodule in M, then $\frac{D}{E}$ is a $\rho\omega$-closed submodule in $\frac{M}{E}$.

Proof

It is clear.

Recall that a submodule N of an R-module M is called y-closed submodule of M, if $\frac{M}{N}$ is a non-singular module [1].

The following Proposition gives a relationships between $\rho\omega$-closed submodule of M and y-closed submodule of M.

Proposition 2.14

If M is a fully semi-prime module, then every non- zero y-closed submodule is a $\rho\omega$-closed submodule in M.

Proof

Suppose that Q is a non-zero y-closed submodule of M, then by $[5, \text{prop}(2.33)]$ Q is a ω-closed submodule in M. Hence by remarks and examples (2.2)(1) N is a $\rho\omega$-closed submodule in M.

Proposition 2.15

If M is an R-module, N and K are non-zero submodules of M with $N \subseteq K$ and every weak essential extension of N is a completely essential submodule of M such that N is a $\rho\omega$-closed submodule in K and K is a ω-closed submodule in M, then N is a $\rho\omega$-closed submodule in M.

Proof

Assume that $x \in M$ with $x \notin N$, then either $x \in K$ with $x \notin K$. If $x \in K$ and since N is a $\rho\omega$-closed submodule in K, then $\exists$ a ω-closed submodule L in K with $N \subseteq L$ and $x \notin L$. Since L is a ω-closed submodule in K and K is a ω-closed submodule in M, then by $[5, \text{prop}(2.16)]$, we get L is a ω-closed submodule in M. Thus N is a $\rho\omega$-closed submodule in M. If $x \notin K$ and K is a ω-closed submodule in M such that $N \subseteq K$ and $x \notin K$. Thus N is a $\rho\omega$-closed submodule in M.

Proposition 2.16

If M is a fully semi-prime R-module and N be a non-zero $\rho\omega$-closed submodule in K and K is a ω-closed submodule in M, then N is a $\rho\omega$-closed submodule in M.

Proof

By using $[5, \text{prop}(2.17)]$ and similarly as in proposition (2.15), we get the result.

Proposition 2.17

Let M is a fully semi-prime module, A and E are submodules of M, if A is a $\rho\omega$-closed submodule in M and E is a weak essential submodule in M. Then $A \cap E$ is a $\rho\omega$-closed submodule in E.

Proof

Suppose that $x \in E$ with $x \notin A \cap E$ implies that $x \notin A$. Since $x \in E \subseteq M$, then $x \in M$. Since A is a $\rho\omega$-closed submodule in M, so there exists a ω-closed submodule in M with $A \subseteq K$ and $x \notin K$. Hence K is a closed submodule in M. And by $[3, \text{prop}(1.4)]$, E is essential in M. Therefore by $[1, \text{Exc}(17), \text{p}. 20]$ we get $K \cap E$ is a closed submodule in E. Hence by $[5, \text{prop}(2.14)]$ we have $K \cap E$ is a ω-closed submodule in E with $A \cap E \subseteq K \cap E$ and $x \notin K \cap E$. Hence $A \cap E$ is a $\rho\omega$-closed submodule in E.

Recall that an R-module M is called fully prime, if every proper submodule of M is a prime submodule [8].

It is well- known every fully prime R-module is a fully semi-prime we get the following result.

Corollary 2.18

Let M be a fully prime module, A and E are submodules of M, if A is a $\rho\omega$-closed submodule in M and E is a weak essential submodule in M. Then $A \cap E$ is a $\rho\omega$-closed submodule in E.

Since ω-closed submodule is a $\rho\omega$-closed submodule, then we get the following result.

Corollary 2.19

Let M be a fully semi-prime module, A and E are submodules of M, if A is a ω-closed submodule in M and E is a weak essential submodule in M. Then $A \cap E$ is a $\rho\omega$-closed submodule in E.

Since in the class of a fully semi-prime modules, y-closed submodule is a ω-closed submodule we get the following result.

Corollary 2.20

If M is a fully semi-prime module, A and E are submodules of M, if A is a y-closed submodule in M and E is a weak essential submodule in M. Then $A \cap E$ is a $\rho\omega$-closed submodule in E.

3. $\rho\omega$-Closed Submodules in Multiplication Modules

This section is devoted to study the behavior of $\rho\omega$-closed submodules in the class of multiplication modules.

Recall that an R-module M is a multiplication if every submodule of M is of the form IM for some ideal I of R [9]". And an R-module M is called faithful if for any non-zero $r \in R$, there is an element $x \in M$ such that $rx \neq 0$ [1].

Recall that for any R-module M and any ideals I and J of R, if I is a semi-prime ideal of J, then IM is a semi-prime submodule of JM. This is called condition(*) [3].

Proposition 3.1

If M is a faithful multiplication module satisfies condition(*), L and K are ideals of a ring R such that L is a $\rho\omega$-closed ideal in K, then LM is a $\rho\omega$-closed submodule in HM.

Proof

Let L be a $\rho\omega$-closed ideal in H, and let $x \in HM$ where $x = rm$, $r \in H$ and $m \in M$ with $x \notin LM$ i.e. $x = rm$, $r \in H$ but $r \notin L$. But L is a $\rho\omega$-closed ideal in H and $r \in H$, then there exists a ω-closed ideal J in H such that $L \subseteq J$ and $r \notin J$. But M is a faithful and multiplication then $LM \subseteq JM$ and $x = rm \notin JM$. But by $[5, \text{prop}(3.6)]$, JM is a ω-closed submodule in HM. Hence LM is a $\rho\omega$-closed submodule in HM.

Proposition 3.2

If M is a finitely generated, faithful and multiplication R- module, L and K are ideals of a ring R such that LM is a $\rho\omega$-closed submodule in KM, then L is a $\rho\omega$-closed ideal in K.

Proof

Suppose that LM is a $\rho\omega$-closed submodule in KM. Let $r \in K$ with $r \notin L$, there for each $m \in M$, $rm \in KM$ and $rm \notin LM$. But LM is a $\rho\omega$-closed submodule in KM then there exists a ω-closed submodule JM in KM, where J is an ideal in K such that $LM \subseteq JM$ and $rm \notin JM$. Hence by $[5, \text{prop}(3.7)]$ we get J is a ω-closed ideal in K, and $L \subseteq J$ with $r \notin L$ and $r \notin J$. Hence L is a $\rho\omega$-closed ideal in K.

From proposition (3.1) and proposition (3.2), we get the following corollaries

Corollary 3.3

Let M be a finitely generated, faithful and multiplication R- module which satisfies condition (*). Then L is a $\rho\omega$-closed ideal in K if and only if LM is a $\rho\omega$-closed submodule in KM.

Corollary 3.4

Let M be a finitely generated multiplication module and let E be a submodule of M such that M satisfies condition (*). Then the next statements are equivalents.

1. E is a $\rho\omega$-closed submodule in M.

2. $[E:_R M]$ is a $\rho\omega$-closed ideal in R.

3. $E = JM$ for some $\rho\omega$-closed ideal J in R.

Proof

(1)⇒(2) Since M is a multiplication module and E is a $\rho\omega$-closed submodule in M, then by [9] $E = [E:_R M]M$, hence by proposition (3.1) $[E:_R M]$ is a $\rho\omega$-closed ideal in R.

(2)⇒(3) Following by [9] and by proposition (3.2).

(3)⇒(1) Since J is a $\rho\omega$-closed ideal in R, then by proposition (3.2) JM is a $\rho\omega$-closed submodule in RM, but $E = JM$ and $RM = M$. Hence E is a $\rho\omega$-closed submodule in M.

Proposition 3.5

Let M be a non-zero multiplication module, with only one non-zero maximal submodule E, then E can not be a $\rho\omega$-closed submodule in M.

Proof

Let E be a $\rho\omega$-closed submodule in M, and $x \in M$, $x \notin E$, then there exists a ω-closed submodule C in M such that $E \subseteq C$ and $x \notin C$. Thus by $[3, \mathrm{prop}(1.20)]$, E is a weak essential submodule in M. But $E \subseteq C \subseteq M$, then by $[2, \mathrm{Rem}(1.5)(2)]$ C is a weak essential submodule in M. Hence by $[5, \mathrm{prop}(2.4)]$ $C = M$, thus $x \notin C = M$, implies that $x \notin M$ contradiction. Hence E dose not be a $\rho\omega$-closed submodule in M.

Proposition 3.6

If M is a multiplication module over regular ring R, and E is a non-zero closed submodule in M, then E is a $\rho\omega$-closed submodule in M.

Proof

Since M is a multiplication module over regular ring R, then by $[10, \mathrm{Rem}(2.6)]$ M is fully semi-prime. Hence E is a $\rho\omega$-closed submodule in M.

Proposition 3.7

If M is a multiplication module over regular ring R, and E is a non-zero y-closed submodule in M, then E is a $\rho\omega$-closed submodule in M.

Proof

Follows by $[10, \mathrm{Rem}(2.6)]$ and proposition (2.14).

Proposition 3.8

If M is a multiplication modulr over regular ring R, and N be a non-zero $\rho\omega$-closed submodule in K and K is a ω-closed submodule in M, then N is a $\rho\omega$-closed submodule in M.

Proof

Follows by $[10, \mathrm{Rem}(2.6)]$ and proposition (2.16).

Proposition 3.9

If M is a multiplication modulr over regular ring R, A and E are submodules of M, with A is a $\rho\omega$-closed submodule in M and E is a weak essential submodule in M, then $A \cap E$ is a $\rho\omega$-closed submodule in E.

Proof

Follows by $[10, \mathrm{Rem}(2.6)]$ and proposition (2.17).

Since ω-closed submodule is $\rho\omega$-closed M, we get the following:

Corollary 3.10

If M is a multiplication modulr over regular ring R, A and E are submodules of M, with A is a ω-closed submodule in M and E is a weak essential submodule in M, then $A \cap E$ is a $\rho\omega$-closed submodule in E.

Proposition 3.11

If M is a multiplication modulr over regular ring R, A and E are submodules of M, with A is a y-closed submodule in M and E is a weak essential submodule in M, then $A \cap E$ is a $\rho\omega$-closed submodule in E.

Proof

Follows by [10, Rem(2.6)] and corollary (2.20).

4. Chain Conditions On $\rho\omega$-Closed Submodules

Module with chain condition on $\rho\omega$-closed submodules are studied in this section.

Definition 4.1

A module M is called a module with ascending (respectively, descending) chain condition (briefly ACC respectively DCC) on $\rho\omega$-closed submodules, if every ascending (respectively, descending) chain of $\rho\omega$-closed submodules of M is finite, i.e. $\exists\ m \in Z_+$ such that $E_n = E_m \forall\ n \geq m$.

Remarks and Examples 4.2

1. Every Noetherian module has ACC on $\rho\omega$-closed submodules.
2. Every Artirian module has DCC on $\rho\omega$-closed submodules.

Proposition 4.3

If M is a module satisfies ACC (DCC) on $\rho\omega$-closed submodules of M, then M satisfiesACC (DCC) on ω-closed submodules of M.

Proof

Let $E_1 \subseteq E_2 \subseteq \ldots\ldots.$, be an ascending chain of ω-closed submodule E_i of M for each i. But every ω-closed submodule is $\rho\omega$-closed, then E_i is a $\rho\omega$-closed submodule for each $i = 1,2,\ldots.$ Since M has ACC on $\rho\omega$-closed submodules, then $\exists\ m \in Z_+$ such that $E_n = E_m \forall\ n \geq m$. Thus M has ACC on ω-closed submodules. For DCC in similarly way.

Proposition 4.4

If M is a fully semi-prime R-module satisfies ACC (DCC) on $\rho\omega$-closed submodules E_i of M, then M satisfies ACC (DCC) on closed submodules of M.

Proof

Let $E_1 \subseteq E_2 \subseteq \ldots\ldots.$,be an ascending chain of closed submodule. where E_i is a closed submodule in M for each $i = 1,2,\ldots.$ But M is a fully semi-prime module, then by proposition (2.7) E_i is a $\rho\omega$-closed submodule in M for each $i = 1,2,\ldots.$ But M has ACC on $\rho\omega$-closed submodules. Hence $\exists\ m \in Z_+$ such that $E_n = E_m \forall\ n \geq m$. Thus M has ACC on closed submodules. Similarly for DCC.

Proposition 4.5

If M is a module satisfies ACC (DCC) on $\rho\omega$-closed submodules E_i for each $i = 1,2,\ldots.$, with each weak essential extension of E_i is completely essential for each $i = 1,2,\ldots.$, then M satisfies ACC (DCC) on closed submodules E_i for each $i = 1,2,\ldots..$

Proof

Let $E_1 \subseteq E_2 \subseteq \ldots\ldots.$, be an ascending chain of closed submodule E_i for each $i = 1,2,\ldots.$ Then by proposition (2.6) E_i is a $\rho\omega$-closed submodule for each $i = 1,2,\ldots.$ But M has ACC on $\rho\omega$-closed submodules, then $\exists\, m \in Z_+$ such that $E_n = E_m \forall\, n \geq m$. Hence M has ACC on closed submodules E_i for each $i = 1,2,\ldots$.. Similarly for DCC.

Proposition 4.6

Let M be an R-module and D be a submodule of M such that $D \subseteq S\,rad(M) \cap K$, where K is any $\rho\omega$-closed submodule in M. If $\frac{M}{D}$ satisfies DCC (ACC) on $\rho\omega$-closed submodules of $\frac{M}{D}$. Then M satisfies ACC (DCC) on $\rho\omega$-closed submodules of M.

Proof

Let $E_1 \supseteq E_2 \supseteq \ldots\ldots$, be a descending chain of $\rho\omega$-closed submodules E_i in M for each $i = 1,2,\ldots$, and $D \subseteq S\,rad(M) \cap E_i$ for each $i = 1,2,\ldots$ Then by corollary (2.13) we have $\frac{E_i}{D}$ is a $\rho\omega$-closed submodules in $\frac{M}{D}$ for each $i = 1,2,\ldots$ Hence $\frac{E_1}{D} \supseteq \frac{E_2}{D} \supseteq \ldots\ldots$, is a descending chain of $\rho\omega$-closed submodules in $\frac{M}{D}$. But $\frac{M}{D}$ has (DCC) on $\rho\omega$-closed submodules, so $\exists\, m \in Z_+$ such that $\frac{E_n}{D} = \frac{E_m}{D} \forall\, n \geq m$. Hence M satisfies (DCC) on $\rho\omega$-closed submodules. Similarly for ACC.

Proposition 4.7

If M is a fully semi-prime R-module such that M satisfies ACC (DCC) on a non-zero $\rho\omega$-closed submodules of M, then M satisfies ACC (DCC) on non-zero y-closed submodules of M.

Proof

Let $E_1 \subseteq E_2 \subseteq \ldots\ldots$, be an ascending chain of a non-zero y-closed submodules E_i of M for each $i = 1,2,\ldots$.. Then by proposition (2.14) E_i is a $\rho\omega$-closed submodules in M for each $i = 1,2,\ldots$ Hence $E_1 \subseteq E_2 \subseteq \ldots$, be an ascending chain of a $\rho\omega$ -closed submodules in M. But M has ACC on $\rho\omega$-closed submodules, then $\exists\, m \in Z_+$ such that $E_n = E_m \forall\, n \geq m$. Hence M has ACC on y-closed submodules. Similarly for DCC.

5. Conclusions

In this article we introduce and study the notion of a pseudo weakly closed submodules as a generalization of a ω-closed submodules. Among the main results we get are the following.

1. If M is an R-module, E and B are submodules of M with $E \subseteq B$ and E is a $\rho\omega$-closed submodule in B and B is a ω-closed submodule in M, then E is a $\rho\omega$-closed submodule in M, provided that B contained in any weak-essential extensions of E.
2. If M is a uniserial R-module, E and D are submodules of M such that $E \subseteq D$ and E is a $\rho\omega$-closed submodule in D and D is a ω-closed submodule in M, then E is a $\rho\omega$-closed submodule in M.
3. If $M = M_1 \oplus M_2$ is an R-module where M_1 and M_2 are submodules of M, provided that $annM_1 + annM_2 = R$ and all submodules of M are completely essential submodules. If E is a non-zero $\rho\omega$-closed submodule in M_1 and D is a non-zero $\rho\omega$-closed submodule in M_2, then $E \oplus D$ is a $\rho\omega$-closed submodule in M if and only if E is a $\rho\omega$-closed submodule in M_1 and D is a $\rho\omega$-closed submodule in M_2.

4. Let M be a finitely generated, faithful and multiplication R- module which satisfies condition (*). Then L is a $\rho\omega$-closed ideal in K if and only if LM is a $\rho\omega$-closed submodule in KM.
5. If M is a module satisfies ACC (DCC) on $\rho\omega$-closed submodules of M, then M satisfiesACC (DCC) on ω-closed submodules of M.
6. If M is a fully semi-prime R-module and M satisfies ACC (DCC) on $\rho\omega$-closed submodules of M, then M satisfies ACC (DCC) on closed submodules of M.

References

1. Gooderal, K. R. *Ring Theory, Nonsingular Ring and Modules*. Marcel. Dekker, New York. **1976**.
2. Muna, A. A. Weak Essential Submodules. *Um-Science Journal*.**2009**, *6*, *1*, 214-222.
3. Muna, A. A. Some Results on Weak Essential Submodules. *Baghdad Science Journal*. **2016**, *13*, *3*, 599-606.
4. Athab, E. A.; AL-Hashimi, B. A. Anote on Seme-Prime Submodules in Multiplication Modules. *Iraq J.SC*.**2000**, *41*, *1*, 88-93.
5. Haibat., K. M.; Mohammad, E. D. W-Closed Submodules and Related Concepts. *Ibn-Al-Haitham Journal, for Pure and Apple. Sci*.**2018**, *31*, *2*, 164-178.
6. Osfsky, B.L. Aconstruction of Nonstandares Uniserial Modules over Valuation Domain. *Bull.Amer, Math.Soc*.**1991**, *25*, 89-97.
7. Abbas, M. S. On Fully Stable Modules. Thesis, University of Baghdad, **1991**.
8. Behoodi, M ; Karamzadeh, O. ; Koohy, H. Modules Whose Certain Submodules are Primes. *Vietnam Journal of Mathematics*.**2004**, *32*, *3*, 303-317.
9. Barnard, A. Multiplication Modules. *J.Algebra*.**1981**, *71*, 174-178.
10. Jabber, A.K.; Tawfiq, L.N.M. New Transform Fundamental Properties and its Applications. *Ibn Alhaitham Journal for Pure and Applied Science*.**2018**, *31*, *1*, 151-163.