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The Continuous Classical Optimal Control governing by Triple Linear Parabolic Boundary Value Problem

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Abstract

This paper deals with the continuous classical optimal control problem for triple partial differential equations of parabolic type with initial and boundary conditions; the Galerkin method is used to prove the existence and uniqueness theorem of the state vector solution for given continuous classical control vector. The proof of the existence theorem of a continuous classical optimal control vector associated with the triple linear partial differential equations of parabolic type is given. The derivation of the Fréchet derivative for the cost function is obtained. At the end, the theorem of the necessary conditions for optimality of this problem is stated and is proved.

Keywords: continuous classical optimal control, triple parabolic partial differential equations, Galerkin Method, the necessary conditions for optimality.

1. Introduction

Different applications for real life problems take a main place in the optimal control problems, for examples in medicine [1]. Robots [2]. Engineering [3]. Economic [4]. And many others fields. In the field of mathematics, optimal control problem (OCP) usually governing either by ordinary differential equations (ODEs) or partial differential equations(PDEs), examples for OCP which are governing by parabolic or hyperbolic or elliptic PDEs are studied by [5-7]. Respectively, while which are governing by couple of PDEs (CPDEs) of parabolic or of hyperbolic or of elliptic type are studied by [8-10]. On the other hand [11-13]. Rre studied boundary OCP associated with CPDEs of parabolic, hyperbolic and elliptic; while [14]. Studied the OCP for triple PDEs (TPDEs) of elliptic type. These works push us to seek about the OCP for TPDEs of parabolic type. This work consists of the study of the continuous classical optimal control problem (CCOCP) , starting with the state and prove the existence theorem of a unique solution (state vector solution SVS) for the triple state equations (TSE) of PDEs of parabolic type (TPPDEs) by using the Galerkin method (GM) when the continuous classical control vector (CCCV) is fixed then it deals with the state and proof of the existence theorem of a continuous classical optimal control vector (CCOCV) , the solution vector of the triple adjoint equations (TAPEs) associated the (TPPDEs) is studied . The derivation of the Fréchet derivative (FD) for the cost function is obtained, at the end; the theorem of the necessary conditions for optimality (NCO) of this OCP is sated and proved.

2. Description of the problem

Let $\Omega \subset R^2$, $x=(x_1,x_2)$, $Q=[0,T]\times\Omega$, $\tilde{I}=[0,T]$, $\Gamma=\partial\Omega$, $\Sigma=\Gamma\times\tilde{I}$, the CCOCP consists of TSE are given by the following TPDEs :

$$y_{1t}-\Delta y_1+y_1-y_2-y_3=f_1(x,t)+u_1 \quad \text{in } Q \tag{1}$$
$$y_{2t}-\Delta y_2+y_2+y_3+y_1=f_2(x,t)+u_2 \quad \text{in } Q \tag{2}$$
$$y_{3t}-\Delta y_3+y_3+y_1-y_2=f_3(x,t)+u_3 \quad \text{in } Q \tag{3}$$

with the following boundary conditions (BCs) and the initial conditions (ICs)

$$y_1(x,t)=0 \quad , \text{on } \Sigma \tag{4}$$
$$y_2(x,t)=0 \quad , \text{on } \Sigma \tag{5}$$
$$y_3(x,t)=0 \quad , \text{on } \Sigma \tag{6}$$
$$y_1(x,0)=y_1^0(x) \text{ , on } \Omega \tag{7}$$
$$y_2(x,0)=y_2^0(x) \text{ , on } \Omega \tag{8}$$
$$y_3(x,0)=y_3^0(x) \text{ , on } \Omega \tag{9}$$

where (f_1,f_2,f_3) is a vector of given function for each $(x_1,x_2) \in \Omega$, $\vec{u}=(u_1,u_2,u_3) \in \left(L^2(Q)\right)^3$ is a CCCV and $\vec{y}=(y_1,y_2,y_3) \in (H^2(\Omega))^3$, is its corresponding SVS .

The set of admissible CCCV is defined by

$\vec{W}_A=\{(u_1,u_2,u_3) \in (L^2(Q))^3 \mid (u_1,u_2,u_3) \in \vec{U}=U_1\times U_2\times U_3 \subset R^3$ a.e. in Q }, $\vec{U}$ is convex.

The cost function is defined for $\beta>0$by

$$G_0(\vec{u})=\tfrac{1}{2}\left(\|y_1-y_{1d}\|_Q^2+\|y_2-y_{2d}\|_Q^2+\|y_3-y_{3d}\|_Q^2\right)+\tfrac{\beta}{2}\left(\|u_1\|_Q^2+\|u_2\|_Q^2+\|u_3\|_Q^2\right) \tag{10}$$

$let\ \vec{V}=V_1\times V_2\times V_3; V_1=V_2=V_3=H^1(\Omega)$& $\vec{V}=\{\vec{v}:\vec{v}=(v_1,v_2,v_3)\in(H^1(\Omega))^3, \vec{v}=0 \text{ on} \partial\Omega\}$.

The weak form(wf) of problem (1- 9) when $\vec{y} \in \left(H^2(\Omega)\right)^3$ is given by

$$\langle y_{1t},v_1\rangle+(\nabla y_1,\nabla v_1)+(y_1,v_1)-(y_2,v_1)-(y_3,v_1)=(f_1+u_1,v_1), \forall\, v_1\in V_1 \tag{11.a}$$
$$(y_1^0,v_1)=(y_1(0),v_1), \forall\, v_1\in V_1 \tag{11.b}$$
$$\langle y_{2t},v_2\rangle+(\nabla y_2,\nabla v_2)+(y_2,v_2)+(y_3,v_2)+(y_1,v_2)=(f_2+u_2,v_2), \forall\, v_2\in V_2 \tag{12.a}$$
$$(y_2^0,v_2)=(y_2(0),v_2), \forall\, v_2\in V_2 \tag{12.b}$$
$$\langle y_{3t},v_3\rangle+(\nabla y_3,\nabla v_3)+(y_3,v_3)+(y_1,v_3)-(y_2,v_3)=(f_3+u_3,v_3), \forall\, v_3\in V_3 \tag{13.a}$$
$$(y_3^0,v_3)=(y_3(0),v_3), \forall\, v_3\in V_3 \tag{13.b}$$

The following assumption is important to study the CCOCV problem (CCOCVP)

2.1. Assumption (A): The function f_i ($\forall\, i=1,2,3$) is satisfied the following condition w.r.t. x & t , i.e. $|f_i|\le\eta_i(x,t)$, where $(x,t)\in Q, \eta_i\in L^2(Q,\mathbb{R})$.

3. The Solution for the wf:

Theorem 3.1: Existence of a Unique Solution for the wf: With assumption (A), for each given CCCV $\vec{u}\in\left(L^2(Q)\right)^3$, the wf(11−13)has a unique solution $\vec{y}=(y_1,y_2,y_3)$ with $\vec{y}\in\left(L^2(I,V)\right)^3$ and $\vec{y}_t=(y_{1t},y_{2t},y_{3t})\in\left(L^2(I,V^*)\right)^3$.

Proof: Let for each n, $\vec{V}_n = V_n \times V_n \times V_n \subset \vec{V}$ be the set of continuous and piecewise affine functions in Ω, let v_{ij}, $i = 1,2,3$ & $j = 1,2,\dots,n$ be a basis of $V_{in} = V_n$,and let $\vec{y}_n$ be an approximate solution for the solution $\vec{y}$, then by Gm:

$$y_{1n} = \sum_{j=1}^{n} c_{1j}(t)\, v_{1j}(x) \quad (14)$$

$$y_{2n} = \sum_{j=1}^{n} c_{2j}(t)\, v_{2j}(x) \quad (15)$$

$$y_{3n} = \sum_{j=1}^{n} c_{3j}(t)\, v_{3j}(x) \quad (16)$$

where $c_{ij}(t)$ is unknown function of t, $\forall\, i = 1,2,3, j = 1,2,\dots,n$.

The wf (11–13) is approximated by using (14−16) as ,

$$\langle y_{1nt}, v_1\rangle + (\nabla y_{1n}, \nabla v_1) + (y_{1n}, v_1) - (y_{2n}, v_1) - (y_{3n}, v_1) = (f_1 + u_1, v_1), \forall\, v_1 \in V_n \quad (17.a)$$

$$(y_{1n}^0, v_1) = (y_1^0, v_1), \forall\, v_1 \in V_n \quad (17.b)$$

$$\langle y_{2nt}, v_2\rangle + (\nabla y_{2n}, \nabla v_2) + (y_{2n}, v_2) + (y_{3n}, v_2) + (y_{1n}, v_2) = (f_2 + u_2, v_2), \forall\, v_2 \in V_n \quad (18.a)$$

$$(y_{2n}^0, v_2) = (y_2^0, v_2), \forall\, v_2 \in V_n \quad (18.b)$$

$$\langle y_{3nt}, v_3\rangle + (\nabla y_{3n}, \nabla v_3) + (y_{3n}, v_3) + (y_{1n}, v_3) - (y_{2n}, v_3) = (f_3 + u_3, v_3), \forall\, v_3 \in V_n \quad (19.a)$$

$$(y_{3n}^0, v_3) = (y_3^0, v_3), \forall\, v_3 \in V_n \quad (19.b)$$

where $y_{in}^0 = y_{in}^0(x) = y_{in}(x,0) \in V_n \subset V_i \subset L^2(\Omega)$ is the projection of y_i^0 , thus

$(y_{in}^0, v_i) = (y_i^0, v_i), \quad \forall\, v_i \in V_n \iff \| y_{in}^0 - y_i^0 \|_0 \le \| y_i^0 - v_i \|_0 , \forall\, v_i \in V_n$

Substituting (14−16) in (17−19) respectively and then setting $v_1 = v_{1l}$, $v_2 = v_{2l}$ & $v_3 = v_{3l}$ $\forall\, l = 1,2,\dots,n$, Then the obtained equations are equivalent to the following linear system (LS) of 1^{st} order ODEs with ICs (which has a unique solution), i.e.

$$AC_1'(t) + BC_1(t) - DC_2(t) - EC_3(t) = b_1 \quad (20.a)$$

$$AC_1(0) = b_1^0 \quad (20.b)$$

$$FC_2'(t) + GC_2(t) + HC_3(t) + KC_1(t) = b_2 \quad (21.a)$$

$$FC_2(0) = b_2^0 \quad (21.b)$$

$$MC_3'(t) + NC_3(t) + RC_1(t) - WC_2(t) = b_3 \quad (22.a)$$

$$MC_3(0) = b_3^0 \quad (22.b)$$

Where $A = (a_{lj})_{n\times n}$, $a_{lj} = (v_{1j}, v_{1l})$, $B = (b_{lj})_{n\times n}$, $b_{lj} = (\nabla v_{1j}, \nabla v_{1l}) + (v_{1j}, v_{1l})$, $D = (d_{lj})_{n\times n}$, $d_{lj} = (v_{2j}, v_{1l})$, $E = (e_{lj})_{n\times n}$, $e_{lj} = (v_{3j}, v_{1l})$, $F = (f_{lj})_{n\times n}$, $f_{lj} = (v_{2j}, v_{2l})$, $G = (g_{lj})_{n\times n}$ $g_{lj} = (\nabla v_{2j}, \nabla v_{2l}) + (v_{2j}, v_{2l})$, $H = (h_{lj})_{n\times n}$, $h_{lj} = (v_{3j}, v_{2l})$, $K = (k_{lj})_{n\times n}$, $k_{lj} = (v_{1j}, v_{2l})$, $M = (m_{lj})_{n\times n}$, $m_{lj} = (v_{3j}, v_{3l})$, $N = (n_{lj})_{n\times n}$, $n_{lj} = (\nabla v_{3j}, \nabla v_{3l}) + (v_{3j}, v_{3l})$ $R = (r_{lj})_{n\times n}$, $r_{lj} = (v_{1j}, v_{3l})$, $W = (w_{lj})_{n\times n}$, $w_{lj} = (v_{2j}, v_{3l})$, $b_{il}^0 = (y_i^0, v_{il})$, $b_i^0 = (b_{il}^0)$, $b_i = (b_{il})_{n\times 1}$, $b_{il} = (f_i + u_i, v_{il})$, $C_i'(t) = (c_{ij}'(t))_{n\times 1}$, $C_i(t) = (c_{ij}(t))_{n\times 1}$, $C_i(0) = (c_{ij}(0))_{n\times 1}$, $\forall\, l = 1,2,3,\dots,n$, $i = 1,2,3$.

To show the norm $\left\|\overrightarrow{y_n^0}\right\|_0$ is bounded

Since $y_1^0 \in L^2(\Omega)$, then there exists a sequence $\{v_{1n}^0\}$ with $v_{1n}^0 \in V_n$, such that $v_{1n}^0 \longrightarrow y_1^0$ strongly in $L^2(\Omega)$, then from the projection theorem [15]. And (17.b),

$\|y_{1n}^0 - y_1^0\|_0 \le \|y_1^0 - v_1\|_0$, $\forall\, v_1 \in V$, Then $\|y_{1n}^0 - y_1^0\|_0 \le \|y_1^0 - v_{1n}^0\|_0, \forall\, v_{1n}^0 \in V_n \subset V$,$\forall n$

$\Rightarrow y_{1n}^0 \longrightarrow y_1^0$ strongly in $L^2(\Omega)$ with $\|y_{1n}^0\|_0 \le b_1$,

by the same way, one can show that $\|y_{2n}^0\|_0 \le b_2$ & $\|y_{3n}^0\|_0 \le b_3$, then

$\left\|\overrightarrow{y_n^0}\right\|_0$ is bounded in $\left(L^2(\Omega)\right)^3$.

The norms $\|\overrightarrow{y_n}(t)\|_{L^\infty(\check{I},L^2(\Omega))}$ and $\|\overrightarrow{y_n}(t)\|_{Q}$ are bounded

Setting $v_1 = y_{1n}$, $v_2 = y_{2n}$ and $v_3 = y_{3n}$ in (17 –19) respectively, integrating w.r.t. t from 0 to T , adding the obtained three equations , one gets

$$\int_0^T \langle \vec{y}_{nt}, \vec{y}_n \rangle dt + \int_0^T \|\vec{y}_n\|_1^2 dt = \int_0^T [(f_1 + u_1, y_{1n}) + (f_2 + u_2, y_{2n}) + (f_3 + u_3, y_{3n})] dt \qquad (23)$$

Using Lemma (1.2) in [11]. For the 1^{st} term in the L.H.S. of (23) and since the 2^{nd} term is positive, using assumptions (A) for the R.H.S. of (23), it yields

$$\frac{1}{2}\int_0^t \frac{d}{dt}\|\vec{y}_n(t)\|_0^2\, dt \le \int_0^t \int_\Omega \eta_1 |y_{1n}| dx dt + \int_0^t \int_\Omega |u_1||y_{1n}| dx dt + \int_0^t \int_\Omega \eta_2 |y_{2n}| dx dt + \int_0^t \int_\Omega |u_2||y_{2n}| dx dt + \int_0^t \int_\Omega \eta_3 |y_{3n}| dx dt + \int_0^t \int_\Omega |u_3||y_{3n}| dx dt \Rightarrow$$

$$\int_0^t \frac{d}{dt}\|\vec{y}_n(t)\|_0^2\, dt \le \|\eta_1\|_Q^2 + \|\eta_2\|_Q^2 + \|\eta_3\|_Q^2 + \|u_1\|_Q^2 + \|u_2\|_Q^2 + \|u_3\|_Q^2 + 2\int_0^t \|y_{1n}\|_0^2\, dt + 2\int_0^t \|y_{2n}\|_0^2\, dt + 2\int_0^t \|y_{3n}\|_0^2\, dt$$

since $\|\eta_i\|_Q^2 \le \acute{b}_i$, $\|u_i\|_Q^2 \le c_i$, $\forall\ i = 1,2,3$, $\|\vec{y}_n(0)\|_0^2 \le b$.

$\Rightarrow\ \|\vec{y}_n(t)\|_0^2 \le c^* + 2\int_0^t \|\vec{y}_n\|_0^2\, dt$; $c^* = \acute{b}_1 + \acute{b}_2 + \acute{b}_3 + c_1 + c_2 + c_3 + b$,

using the Continuous Bellman Gronwall Inequality (BGI) , one gets

$\|\vec{y}_n(t)\|_0^2 \le c^* e^{2T} = b^2(c)$, $\forall\, t \in [0,T]$ or $\|\vec{y}_n(t)\|_{L^\infty(\check{I},L^2(\Omega))} \le b(c) \Rightarrow \|\vec{y}_n(t)\|_{\boldsymbol{Q}} \le b_1(c)$.

The norm $\|\vec{y}_n(t)\|_{L^2(\check{I},V)}$ is bounded

Again for (23) by using Lemma (1.2) in [11]. For the R.H.S. The same results will be obtained (from the above steps) and since $\|\vec{y}_n(t)\|_0^2$ is positive, equation (23) with $t = T$, becomes

$$\|\vec{y}_n(T)\|_0^2 - \|\vec{y}_n(0)\|_0^2 + 2\int_0^T \|\vec{y}_n\|_1^2 dt \le \|\eta_1\|_Q^2 + \|\eta_2\|_Q^2 + \|\eta_3\|_Q^2 + \|u_1\|_Q^2 + \|u_2\|_Q^2 + \|u_3\|_Q^2 + 2\|\vec{y}_n\|_Q^2 ,$$

which gives

$\int_0^T \|\vec{y}_n\|_1^2\, dt \le b_2^2(c)$, with $b_2^2(c) = \frac{\left(b + b_1' + b_2' + b_3' + c_1 + c_2 + c_3 + 2b_1(c)\right)}{2}$, thus $\|\vec{y}_n\|_{L^2(\check{I},V)} \le b_2(c)$.

The solution convergence

Let $\{\vec{V}_n\}_{n=1}^{\infty}$ be a sequence of subspaces of $\vec{V}$, s.t. $\forall\ \vec{v} \in \vec{V}$, there exists a sequence $\{\vec{v}_n\}$ with $\vec{v}_n \in \vec{V}_n$, $\forall$ n and $\vec{v}_n \longrightarrow \vec{v}$ strongly in $\vec{V} \Rightarrow \vec{v}_n \longrightarrow \vec{v}$ strongly in $(L^2(\Omega))^3$, since for each n , with $\vec{V}_n \subset \vec{V}$, (17−19) has a unique solution (y_{1n}, y_{2n}, y_{3n}) , hence corresponding to the sequence of subspaces $\{\vec{V}_n\}_{n=1}^{\infty}$, there exist a sequence of (approximation) problems like (17−19) now, by substituting $\vec{v} = \vec{v}_n = (v_{1n}, v_{2n}, v_{3n})$, in these equations for $n = 1,2,\ldots$, one has

$$\langle y_{1nt}, v_{1n}\rangle + (\nabla y_{1n}, \nabla v_{1n}) + (y_{1n}, v_{1n}) - (y_{2n}, v_{1n}) - (y_{3n}, v_{1n}) = (f_1 + u_1, v_{1n}) \qquad (24.a)$$

$$(y_{1n}^0, v_{1n}) = (y_1^0, v_{1n}) , \forall v_{1n} \in V_n \qquad (24.b)$$

$$\langle y_{2nt}, v_{2n}\rangle + (\nabla y_{2n}, \nabla v_{2n}) + (y_{2n}, v_{2n}) + (y_{3n}, v_{2n}) + (y_{1n}, v_{2n}) = (f_2 + u_2, v_{2n}) \qquad (25.a)$$

$$(y_{2n}^0, v_{2n}) = (y_2^0, v_{2n})), \forall v_{2n} \in V_n \qquad (25.b)$$

$$\langle y_{3nt}, v_{3n}\rangle + (\nabla y_{3n}, \nabla v_{3n}) + (y_{3n}, v_{3n}) + (y_{1n}, v_{3n}) - (y_{2n}, v_{3n}) = (f_3 + u_3, v_{3n}) \qquad (26.a)$$

$$(y_{3n}^0, v_{3n}) = (y_3^0, v_{3n}) , \forall v_{3n} \in V_n \qquad (26.b)$$

which has a sequence of solutions $\{\vec{y}_n\}_{n=1}^{\infty}$, with $\vec{y}_n = (y_{1n}, y_{2n}, y_{3n})$, but from the above steps we have $\|\vec{y}_n\|_{L^2(\boldsymbol{Q})}$ and $\|\vec{y}_n\|_{L^2(\check{I},V)}$ are bounded , then by Alaoglu's theorem (AT), there exists a subsequence of $\{\vec{y}_n\}_{n\in N}$,say again $\{\vec{y}_n\}_{n\in N}$ s.t. $\vec{y}_n \longrightarrow \vec{y}$ weakly in $(L^2(Q))^3$ and

$\vec{y}_n \longrightarrow \vec{y}$ weakly in $\left(L^2(\tilde{I},V)\right)^3$, multiplying both sides of (24.a), (25.a) & (26.a) by $\varphi_i(t) \in C^1[0,T]$ respectively, s.t. $\varphi_i(T)=0,\ \forall i=1,2,3$, then integrating both sides w.r.t. t on$[0,T]$, then integrating by parts the 1^{st} terms in the L.H.S. of each one obtained equation, one gets

$$-\int_0^T (y_{1n},v_{1n})\varphi_1'(t)dt+\int_0^T[(\nabla y_{1n},\nabla v_{1n})+(y_{1n},v_{1n})]\varphi_1(t)dt-\int_0^T(y_{2n},v_{1n})\varphi_1(t)dt-\int_0^T(y_{3n},v_{1n})\varphi_1(t)dt=\int_0^T(f_1+u_1,v_{1n})\varphi_1(t)+(y_{1n}^0,v_{1n})\varphi_1(0) \quad (27)$$

$$-\int_0^T (y_{2n},v_{2n})\varphi_2'(t)dt+\int_0^T[(\nabla y_{2n},\nabla v_{2n})+(y_{2n},v_{2n})]\varphi_2(t)dt+\int_0^T(y_{3n},v_{2n})\varphi_2(t)dt+\int_0^T(y_{1n},v_{2n})\varphi_2(t)dt=\int_0^T(f_2+u_2,v_{2n})\varphi_2(t)dt+(y_{2n}^0,v_{2n})\varphi_2(0) \quad (28)$$

$$-\int_0^T (y_{3n},v_{3n})\varphi_3'(t)dt+\int_0^T[(\nabla y_{3n},\nabla v_{3n})+(y_{3n},v_{3n})]\varphi_3(t)dt+\int_0^T(y_{1n},v_{3n})\varphi_3(t)dt-\int_0^T(y_{2n},v_{3n})\varphi_3(t)dt=\int_0^T(f_3+u_3,v_{3n})\varphi_3(t)+(y_{3n}^0,v_{3n})\varphi_3(0) \quad (29)$$

Since $\left.\begin{matrix} v_{in}\longrightarrow v_i \text{ strongly in } L^2(\Omega) \\ v_{in}\longrightarrow v_i \text{ strongly in } V_1\end{matrix}\right\} \Rightarrow \left\{\begin{matrix} v_{in}\varphi_i'\longrightarrow v_i\varphi_i' \text{ strongly in } L^2(Q) \\ v_{in}\varphi_i\longrightarrow v_i\varphi_i \text{ strongly in } L^2(\tilde{I},V)\end{matrix}\right.$

since $y_{in}\longrightarrow y_i$ weakly in $L^2(Q)$, also $y_{in}^0\longrightarrow y_i^0$ weakly in $L^2(\Omega), \forall\, i=1,2,3$. Then

$$\int_0^T (y_{1n},v_{1n})\varphi_1'(t)dt+\int_0^T[(\nabla y_{1n},\nabla v_{1n})+(y_{1n},v_{1n})]\varphi_1(t)dt-\int_0^T(y_{2n},v_{1n})\varphi_1(t)dt-\int_0^T(y_{3n},v_{1n})\varphi_1(t)dt\longrightarrow \int_0^T(y_1,v_1)\varphi_1'(t)dt+\int_0^T[(\nabla y_1,\nabla v_1)+(y_1,v_1)]\varphi_1(t)dt-\int_0^T(y_2,v_1)\varphi_1(t)dt-\int_0^T(y_3,v_1)\varphi_1(t)dt \quad (30.a)$$

$$\int_0^T (y_{2n},v_{2n})\varphi_2'(t)dt+\int_0^T[(\nabla y_{2n},\nabla v_{2n})+(y_{2n},v_{2n})]\varphi_2(t)dt+\int_0^T(y_{3n},v_{2n})\varphi_2(t)dt+\int_0^T(y_{1n},v_{2n})\varphi_2(t)dt\longrightarrow \int_0^T(y_2,v_2)\varphi_2'(t)dt+\int_0^T[(\nabla y_2,\nabla v_2)+(y_2,v_2)]\varphi_2(t)dt+\int_0^T(y_3,v_2)\varphi_2(t)dt+\int_0^T(y_1,v_2)\varphi_2(t)dt \quad (31.a)$$

$$\int_0^T (y_{3n},v_{3n})\varphi_3'(t)dt+\int_0^T[(\nabla y_{3n},\nabla v_{3n})+(y_{3n},v_{3n})]\varphi_3(t)dt+\int_0^T(y_{1n},v_{3n})\varphi_3(t)dt-\int_0^T(y_{2n},v_{3n})\varphi_3(t)dt\longrightarrow \int_0^T(y_3,v_3)\varphi_3'(t)dt+\int_0^T[(\nabla y_3,\nabla v_3)+(y_3,v_3)]\varphi_3(t)dt+\int_0^T(y_1,v_3)\varphi_3(t)dt-\int_0^T(y_2,v_3)\varphi_3(t)dt \quad (32.a)$$

$$(y_{1n}^0,v_{1n})\varphi_1(0)\longrightarrow(y_1^0,v_1)\varphi_1(0) \quad (30.b)$$
$$(y_{2n}^0,v_{2n})\varphi_2(0)\longrightarrow(y_2^0,v_2)\varphi_2(0) \quad (31.b)$$
$$(y_{3n}^0,v_{3n})\varphi_3(0)\longrightarrow(y_3^0,v_3)\varphi_3(0) \quad (32.b)$$

since $v_{i\,n}\longrightarrow v_i$ weakly in $L^2(\Omega)$, then

$$\int_0^T(f_1+u_1,v_{1n})\varphi_1(t)dt\longrightarrow\int_0^T(f_1+u_1,v_1)\,\varphi_1(t)dt \quad (30.c)$$
$$\int_0^T(f_2+u_2,v_{2n})\varphi_2(t)dt\longrightarrow\int_0^T(f_2+u_2,v_2)\,\varphi_2(t)dt \quad (31.c)$$
$$\int_0^T(f_3+u_3,v_{3n})\varphi_3(t)dt\longrightarrow\int_0^T(f_3+u_3,v_3)\,\varphi_3(t)dt \quad (32.c)$$

which means (30−32) converge to (33–35) respectively , with

$$-\int_0^T (y_1,v_1)\varphi_1'(t)dt+\int_0^T[(\nabla y_1,\nabla v_1)+(y_1,v_1)]\varphi_1(t)dt-\int_0^T(y_2,v_1)\varphi_1(t)dt-\int_0^T(y_3,v_1)\varphi_1(t)dt=\int_0^T(f_1+u_1,v_1)\,\varphi_1(t)dt+(y_1^0,v_1)\varphi_1(0) \quad (33)$$

$$-\int_0^T (y_2,v_2)\varphi_2'(t)dt+\int_0^T[(\nabla y_2,\nabla v_2)+(y_2,v_2)]\varphi_2(t)dt+\int_0^T(y_3,v_2)\varphi_2(t)dt+\int_0^T(y_1,v_2)\varphi_2(t)dt=\int_0^T(f_2+u_2,v_2)\,\varphi_2(t)dt+(y_2^0,v_2)\varphi_2(0) \quad (34)$$

$-\int_0^T (y_3, v_3)\varphi_3'(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\varphi_3(t)dt + \int_0^T (y_1, v_3)\varphi_3(t)dt -$
$\int_0^T (y_2, v_3)\varphi_3(t)dt = \int_0^T (f_3 + u_3, v_3)\varphi_3(t)dt + (y_3^0, v_3)\varphi_3(0)$ (35)

Case1: Choose $\varphi_i \in D[0,T]$, i.e. $\varphi_i(0) = \varphi_i(T) = 0$, $\forall\, i = 1,2,3$. Substituting in (33−35), using integration by parts for the 1^{st} terms in L.H.S. of each one of the obtained equations, to get

$\int_o^T \langle y_{1t}, v_1\rangle \varphi_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)\,dt -$
$\int_0^T (y_3, v_1)\varphi_1(t)\,dt = \int_0^T (f_1 + u_1, v_1)\varphi_1(t)dt$ (36)

$\int_0^T \langle y_{2t}, v_2\rangle \varphi_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\varphi_2(t)dt + \int_0^T (y_3, v_2)\varphi_2(t)\,dt +$
$\int_0^T (y_1, v_2)\varphi_2(t)\,dt = \int_0^T (f_2 + u_2, v_2)\varphi_2(t)dt$ (37)

$\int_o^T \langle y_{3t}, v_3\rangle \varphi_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\varphi_3(t)dt + \int_0^T (y_1, v_3)\varphi_3(t)\,dt -$
$\int_0^T (y_2, v_3)\varphi_3(t)\,dt = \int_0^T (f_3 + u_3, v_3)\varphi_3(t)dt$ (38)

i.e. (y_1, y_2, y_3) is solution of the wf (11−13).

case 2 : Choose $\varphi_i \in C^1[0,T]$ s.t. $\varphi_i(T) = 0$ & $\varphi_i(0) \neq 0$, $\forall\, i = 1,2,3$ using integration by parts for the 1^{st} term in the L.H.S. of (36) , one gets

$-\int_0^T (y_1, v_1)\varphi_1'(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)\,dt -$
$\int_0^T (y_3, v_1)\varphi_1(t)\,dt = \int_0^T (f_1 + u_1, v_1)\varphi_1(t)dt + (y_1(0), v_1)\varphi_1(0)$ (39)

subtracting (33) from (39) , one obtains that

$(y_1^0, v_1)\varphi_1(0) = (y_1(0), v_1)\varphi_1(0)$, $\varphi_1(0) \neq 0, \forall \varphi_1 \in [0,T] \Rightarrow (y_1^0, v_1) = (y_1(0), v_1)$

i.e. the IC (11.b) holds . By the same above way one can show that

$(y_2^0, v_2) = (y_2(0), v_2)$ & $(y_3^0, v_2) = (y_3(0), v_2)$ that means the ICs (12.b)&(13.b) are hold.

The strongly convergence for $\vec{y}_n$:

Substituting $v_1 = y_{1n}$, $v_2 = y_{2n}$ and $v_3 = y_{3n}$ in (17.a),(18.a)&(19.a) respectively, adding the three obtained equations together , and then integrating the obtained equation from 0 to T, on the other hand substituting $v_1 = y_1, v_2 = y_2$ & $v_3 = y_3$ in (11.a),(12.a)&(13.a) respectively, adding them and then integrating the three obtained equations from 0 to T , to get

$\int_0^T \langle \vec{y}_{nt}, \vec{y}_n\rangle dt + \int_0^T a(\vec{y}_n, \vec{y}_n)dt = \int_0^T [(f_1 + u_1, y_{1n}) + (f_2 + u_2, y_{2n}) + (f_3 + u_3, y_{3n})]dt$ (40)

and

$\int_0^T \langle \vec{y}_t, \vec{y}\rangle dt + \int_0^T a(\vec{y}, \vec{y})dt = \int_0^T [(f_1 + u_1, y_1) + (f_2 + u_2, y_2) + (f_3 + u_3, y_3)]dt$ (41)

using Lemma (1.2) in [11]. For the 1^{st} terms in the L.H.S. of (40) and (41), they become

$\frac{1}{2}\|\vec{y}_n(T)\|_0^2 - \frac{1}{2}\|\vec{y}_n(0)\|_0^2 + \int_0^T a(\vec{y}_n, \vec{y}_n)dt = \int_0^T [(f_1 + u_1, y_{1n}) + (f_2 + u_2, y_{2n}) +$
$(f_3 + u_3, y_{3n})]dt$ (42)

$\frac{1}{2}\|\vec{y}(T)\|_0^2 - \frac{1}{2}\|\vec{y}(0)\|_0^2 + \int_0^T a(\vec{y}, \vec{y})dt = \int_0^T [(f_1 + u_1, y_1) + (f_2 + u_2, y_2) +$
$(f_3 + u_3, y_3)]dt$ (43)

Since

$\frac{1}{2}\|\vec{y}_n(T) - \vec{y}(T)\|_0^2 - \frac{1}{2}\|\vec{y}_n(0) - \vec{y}(0)\|_0^2 + \int_0^T a(\vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})\,dt = A_1 - B_1 - C_1$ (44)

where

$A_1 = \frac{1}{2}\|\vec{y}_n(T)\|_0^2 - \frac{1}{2}\|\vec{y}_n(0)\|_0^2 + \int_0^T a(\vec{y}_n(t), \vec{y}_n(t))dt$

$B_1 = \frac{1}{2}(\vec{y}_n(T), \vec{y}(T)) - \frac{1}{2}(\vec{y}_n(0), \vec{y}(0)) + \int_0^T a(\vec{y}_n(t), \vec{y}(t))\,dt,$

$C_1 = \frac{1}{2}(\vec{y}(T), \vec{y}_n(T) - \vec{y}(T)) - \frac{1}{2}(\vec{y}(0), \vec{y}_n(0) - \vec{y}(0)) + \int_0^T a(\vec{y}(t), \vec{y}_n(t) - \vec{y}(t))dt$

since

$\vec{y}_n^0 = \vec{y}_n(0) \longrightarrow \vec{y}^0 = \vec{y}(0)$ strongly in $\left(L^2(\Omega)\right)^3$ (44.a)

$\vec{y}_n(T) \longrightarrow \vec{y}(T)$ strongly in $\left(L^2(\Omega)\right)^3$ (44.b)

Then

$(\vec{y}(0), \vec{y}_n(0) - \vec{y}(0)) \longrightarrow 0, \left(\vec{y}(T), \vec{y}_n(T) - \vec{y}(T)\right) \longrightarrow 0$ (44.c)

$\|\vec{y}_n(0) - \vec{y}(0)\|_0^2 \longrightarrow 0$ and $\|\vec{y}_n(T) - \vec{y}(T)\|_0^2 \longrightarrow 0$ (44.d)

Since

$\vec{y}_n \longrightarrow \vec{y}$ weakly in $\left(L^2(I,V)\right)^3$,then $\int_0^T a\left(\vec{y}(t), \vec{y}_n(t) - \vec{y}(t)\right)dt \longrightarrow 0$ (44.e)

also since $\vec{y}_n \longrightarrow \vec{y}$ weakly in $\left(L^2(Q)\right)^3$, then

$$\int_0^T [(f_1 + u_1, y_{1n}) + (f_2 + u_2, y_{2n}) + (f_3 + u_3, y_{3n})]\, dt \longrightarrow \int_0^T [\, (f_1 + u_1, y_1) + (f_2 + u_2, y_2) + (f_3 + u_3, y_3)]dt \quad (44.f)$$

i.e. when $n \to \infty$ in both sides of (44), one has the following results :

(1)The first two terms in the L.H.S. of (44) are tending to zero (from (44.d)),

(2) Eq. $A_1 \underset{(42)}{\overset{\text{from}}{=}} \int_0^T [\, (f_1 + u_1, y_{1n}) + (f_2 + u_2, y_{2n}) + (f_3 + u_3, y_{3n})]dt \underset{(44.\text{f})}{\overset{\text{from}}{\longrightarrow}}$

$\int_0^T [\, (f_1 + u_1, y_1) + (f_2 + u_2, y_2) + (f_3 + u_3, y_3)]dt$

(3) Eq. $B_1 \longrightarrow$ L.H.S. of (43)$= \int_0^T [\, (f_1 + u_1, y_1) + (f_2 + u_2, y_2) + (f_3 + u_3, y_3)]dt$

(4) The 1^{st} two terms in Eq. C_1 are tending to zero from (44.c), and the last one term also tend to zero from (44.e), from these results (44) gives when $n \to \infty$

$\int_0^T \|\vec{y}_n - \vec{y}\|_1^2\, dt = \int_0^T a(\vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})dt \longrightarrow 0 \Rightarrow \vec{y}_n \longrightarrow \vec{y}$ strongly in $\left(L^2(I,V)\right)^3$.

Uniqueness of the solution: Let $\vec{y}, \vec{\bar{y}}$ are two solutions of the wf (11–13), i.e. y_1 and $\bar{y}_1$ are satisfied (11.a), or

$\langle y_{1t}, v_1\rangle + a_1(y_1, v_1) - (y_2, v_1) - (y_3, v_1) = (f_1 + u_1, v_1) \quad , \ \forall\, v_1 \in V_1$

$\langle \bar{y}_{1t}, v_1\rangle + a_1(\bar{y}_1, v_1) - (\bar{y}_2, v_1) - (\bar{y}_3, v_1) = (f_1 + u_1, v_1) \quad , \ \forall\, v_1 \in V_1$

Subtracting the 2^{nd}equation from the 1^{st} one and substituting $v_1 = y_1 - \bar{y}_1$in the obtained equation, one gets that

$$\langle (y_1 - \bar{y}_1)_t\, , y_1 - \bar{y}_1\rangle + a_1(y_1 - \bar{y}_1, y_1 - \bar{y}_1) - (y_2 - \bar{y}_2, y_1 - \bar{y}_1) - (y_3 - \bar{y}_3, y_1 - \bar{y}_1) = 0 \quad (45)$$

By the similar manner, one gets

$$\langle (y_2 - \bar{y}_2)_t\, , y_2 - \bar{y}_2\rangle + a_2(y_2 - \bar{y}_2, y_2 - \bar{y}_2) + (y_3 - \bar{y}_3, y_2 - \bar{y}_2) + (y_1 - \bar{y}_1, y_2 - \bar{y}_2) = 0 \quad (46)$$

$$\langle (y_3 - \bar{y}_3)_t\, , y_3 - \bar{y}_3\rangle + a_3(y_3 - \bar{y}_3, y_3 - \bar{y}_3) + (y_1 - \bar{y}_1, y_3 - \bar{y}_3) - (y_2 - \bar{y}_2, y_3 - \bar{y}_3) = 0 \quad (47)$$

Adding (45−47), using Lemma(1.2) in [11]. In the 1^{st} term of the obtained equation , to get

$$\tfrac{1}{2}\tfrac{d}{dt}\left\|\vec{y} - \vec{\bar{y}}\right\|_0^2 + \left\|\vec{y} - \vec{\bar{y}}\right\|_1^2 = 0 \quad (48)$$

since the 2^{nd} term of the L.H.S. of (48) is positive, integrating both sides of (48) w.r.t. t from 0 to t, one gets

$\int_0^t \frac{d}{dt}\left\|\vec{y} - \vec{\bar{y}}\right\|_0^2 dt \leq 0 \Rightarrow \left\|\left(\vec{y} - \vec{\bar{y}}\right)(t)\right\|_0^2 \leq 0 \Rightarrow \left\|\vec{y} - \vec{\bar{y}}\right\|_0^2 = 0 \quad , \quad \forall\ t \in \tilde{I}$

integrating both sides of (48) from 0 to T, using the given ICs and the above result, one has

$\int_0^T \|\vec{y}-\vec{\bar{y}}\|_1^2\,dt = 0 \Rightarrow \|\vec{y}-\vec{\bar{y}}\|_{L^2(\bar{I},V)} = 0 \Rightarrow \vec{y} = \vec{\bar{y}}$.

4. Existence of a CCOCP

Theorem 4.1: In addition to assumptions (A), assume that $\vec{y}$ and $\vec{y}+\delta\vec{y}$ are the SVS corresponding to the CVS $\vec{u}$ and $\vec{u}+\delta\vec{u}$ respectively with $\vec{u}$ and $\delta\vec{u}$ are bounded in $(L^2(Q))^3$ then

$\|\delta\vec{y}\|_{L^\infty(\bar{I},L^2(\Omega))} \le M\|\delta\vec{u}\|_Q$, $M \in \mathbb{R}^+$

$\|\delta\vec{y}\|_{L^2(Q)} \le \bar{M}\|\delta\vec{u}\|_Q$, $\bar{M} \in \mathbb{R}^+$

$\|\delta\vec{y}\|_{L^2(\bar{I},V)} \le \bar{M}_1\|\delta\vec{u}\|_Q$, $\bar{M}_1 \in \mathbb{R}^+$

Proof : let $\vec{u} = (u_1,u_2,u_3) \in (L^2(Q))^3$ then by Theorem 3.1 there exists $\vec{y} = \big(y_1 = y_{u_1}, y_2 = y_{u_2}, y_3 = y_{u_3}\big)$ which is satisfied (11−13) and also let $\vec{\bar{y}} = (\bar{y}_1, \bar{y}_2, \bar{y}_3)$be the solution of (11−13) corresponds to the cv $\vec{\bar{u}} = (\bar{u}_1,\bar{u}_2,\bar{u}_3) \in (L^2(Q))^3$ i.e.

$\langle \bar{y}_{1t}, v_1\rangle + (\nabla\bar{y}_1,\nabla v_1) + (\bar{y}_1,v_1) - (\bar{y}_2,v_1) - (\bar{y}_3,v_1) = (f_1+\bar{u}_1,v_1)$ (49.a)

$(\bar{y}_1(0), v_1) = (y_1^0, v_1)$ (49.b)

$\langle \bar{y}_{2t}, v_2\rangle + (\nabla\bar{y}_2,\nabla v_2) + (\bar{y}_2,v_2) + (\bar{y}_3,v_2) + (\bar{y}_1,v_2) = (f_2+\bar{u}_2,v_2)$ (50.a)

$(\bar{y}_2(0), v_2) = (y_2^0, v_2)$ (50.b)

$\langle \bar{y}_{3t}, v_3\rangle + (\nabla\bar{y}_3,\nabla v_3) + (\bar{y}_3,v_3) + (\bar{y}_1,v_3) - (\bar{y}_2,v_3) = (f_3+\bar{u}_3,v_3)$ (51.a)

$(\bar{y}_3(0), v_3) = (y_3^0, v_3)$ (51.b)

subtracting (11.a&b) from (49.a&b) , (12.a&b) from (50.a&b) ,and (13.a&b) from (51.a&b) and setting $\delta y_1 = \bar{y}_1 - y_1$, $\delta y_2 = \bar{y}_2 - y_2$, $\delta y_3 = \bar{y}_3 - y_3$,$\delta u_1 = \bar{u}_1 - u_1$, $\delta u_2 = \bar{u}_2 - u_2$ and $\delta u_3 = \bar{u}_3 - u_3$ in the obtained equations , they give

$\langle \delta y_{1t}, v_1\rangle + (\nabla\delta y_1,\nabla v_1) + (\delta y_1,v_1) - (\delta y_2,v_1) - (\delta y_3,v_1) = (\delta u_1,v_1)$ (52.a)

$(\delta y_1(0), v_1) = 0$ (52.b)

$\langle \delta y_{2t}, v_2\rangle + (\nabla\delta y_2,\nabla v_2) + (\delta y_2,v_2) + (\delta y_3,v_2) + (\delta y_1,v_2) = (\delta u_2,v_2)$ (53.a)

$(\delta y_2(0), v_2) = 0$ (53.b)

$\langle \delta y_{3t}, v_3\rangle + (\nabla\delta y_3,\nabla v_3) + (\delta y_3,v_3) + (\delta y_1,v_3) - (\delta y_2,v_3) = (\delta u_3,v_3)$ (54.a)

$(\delta y_3(0), v_3) = 0$ (54.b)

substituting $v_1 = \delta y_1$, $v_2 = \delta y_2$ & $v_3 = \delta y_3$ in (52.a&b) , (53.a&b) and (54.a&b) respectively, adding the obtained equations, using Lemma (1.2) in [11]. They give

$\Rightarrow \frac{1}{2}\frac{d}{dt}\|\delta\vec{y}\|_0^2 + \|\delta\vec{y}\|_1^2 = (\delta u_1,\delta y_1) + (\delta u_2,\delta y_2) + (\delta u_3,\delta y_3)$ (55)

since the 2^{nd} term of (55) is positive , integrating w.r.t. t From 0 to t , and then using the Cauchy Schwartz inequality (CSI), it becomes

$\int_0^t \frac{d}{dt}\|\delta\vec{y}\|_0^2 dt \le 2\int_0^t\int_\Omega |\delta u_1||\delta y_1|dxdt + 2\int_0^t\int_\Omega |\delta u_2||\delta y_2|dxdt + 2\int_0^t\int_\Omega |\delta u_3||\delta y_3|dxdt$

$\le \int_0^T \|\delta\vec{u}\|_0^2\,dt + \int_0^t \|\delta\vec{y}\|_0^2\,dt,\ t \in [0,T],$

by the BGI , once get

$\|\delta\vec{y}(t)\|_0^2 \le M^2\|\delta\vec{u}\|_Q^2 \Rightarrow \|\delta\vec{y}(t)\|_0 \le M\|\delta\vec{u}\|_Q$, where $M^2 = e^T$, $M > 0$, $t \in [0,T]$

$\Rightarrow \|\delta\vec{y}\|_{L^\infty(\bar{I},L^2(\Omega))} \le M\|\delta\vec{u}\|_Q$

since $\|\delta\vec{y}\|_{L^2(Q)}^2 \le TM^2\|\delta\vec{u}\|_Q^2$, then

$\|\delta\vec{y}\|_{L^2(Q)} \le \bar{M}\|\delta\vec{u}\|_Q$, $\bar{M}^2 = TM^2$

Using a similar way which is used in above steps, gives

$\int_0^T \frac{d}{dt}\|\delta\vec{y}\|_0^2 + 2\int_0^T \|\delta\vec{y}\|_1^2\,dt \le \|\delta\vec{u}\|_Q^2 + \int_0^T \|\delta\vec{y}\|_0^2\,dt \le \bar{M}^2\|\delta\vec{u}\|_Q^2$

$\Rightarrow \|\delta\vec{y}\|^2_{L^2(\tilde{I},V)} \le \bar{M}_1^2\|\delta\vec{u}\|^2_Q$ where $\bar{M}_1^2 = (1+\bar{M}^2)/2$

$\Rightarrow \|\delta\vec{y}\|_{L^2(\tilde{I},V)} \le \overline{M}_1\|\delta\vec{u}\|_Q$.

Theorem 4.2: With assumptions (A), the operator $\vec{u} \mapsto \vec{y}_{\vec{u}}$ is continuous from $(L^2(Q))^3$ in to $(L^\infty{}_{(\tilde{I},L^2(\Omega))})^3$ or in to $(L^2(\tilde{I},V))^3$ or in to $(L^2(Q))^3$.

Proof: Let $\delta\vec{u} = \vec{\vec{u}} - \vec{u}$ and $\delta\vec{y} = \vec{\vec{y}} - \vec{y}$, where $\vec{\vec{y}}$ *and* $\vec{y}$ are the correspond SVS to the CVS $\vec{\vec{u}}$ and $\vec{u}$ using the first result in Theorem 4.1 , one has $\|\vec{\vec{y}} - \vec{y}\|_{L^\infty(\tilde{I},L^2(\Omega))} \le M\|\vec{\vec{u}} - \vec{u}\|_Q$, If $\vec{\vec{u}} \xrightarrow[L^2(Q)]{} \vec{u}$ then $\vec{\vec{y}} \xrightarrow[L^\infty(\tilde{I},L^2(\Omega))]{} \vec{y}$, i.e. The operator $\vec{u} \mapsto \vec{y}_{\vec{u}}$ is Lipschitz continuous (LC) from $(L^2(Q))^3$ in to$(L^\infty{}_{(\tilde{I},L^2(\Omega))})^3$. By a similar way this operator is also LC from $(L^2(Q))^3$ into $(L^2(Q))^3$and into $(L^2(\tilde{I},V))^3$.

Lemma *4.1* [10]: The norm $\|.\|_0$ is weakly lower semi continuous (W.L.S.C.).

Lemma (4.2) : The cost function which is given by (10) is W.L.S.C.

Proof: From Lemma (3.1) $\|\vec{u}\|_{L^2(Q)}$, is W.L.S.C. when $\vec{u}_k \to \vec{u}$ weakly in $(L^2(Q))^3$ then $\vec{y}_k \to \vec{y}$ weakly in $(L^2(Q))^3$ by Theorem 4.1 then $\|\vec{y} - \vec{y}_d\| \le \lim_{k\to\infty}\inf_{y_k\in V_k}\|\vec{y}_k - \vec{y}_d\|$

Then $\|\vec{y} - \vec{y}_d\|$ is W.L.S.C., hence $G_0(\vec{u})$ is W.L.S.C.

Lemma 4.3 [13]: The norm $\|\cdot\|_0^2$ is strictly convex.

Theorem 4.3: Consider the cost function (10) , if $G_0(\vec{u})$ is coercive, then there exists CCOC **.**

Proof: Since $G_0(\vec{u}) \ge 0$ and $G_0(\vec{u})$ is coercive, then there exists a minimizing sequence$\{\vec{u}_k\} = \{(u_{1k}, u_{2k}, u_{3k})\} \in \overrightarrow{W}_A$, $\forall k$ such that

$\lim_{n\to\infty} G_0(\vec{u}_k) = \inf_{\vec{\vec{u}}_k\in\overrightarrow{W}_A} G_0(\vec{\vec{u}})$, and $\|\vec{u}_k\| \le c$, then by AT there exists a subsequence of $\{\vec{u}_k\}$, for simplicity say again $\{\vec{u}_k\}$ s.t. $\vec{u}_k \longrightarrow \vec{u}$ weakly in $(L^2(Q))^3$, as $k \to \infty$. From Theorem 3.1 we got that for each control $\vec{u}_k$ there exists a unique solution $\vec{y}_k = \vec{y}_{\vec{u}_k}$ then corresponding to the sequence of control $\{\vec{u}_k\}$ there exists a sequence of solutions $\{\vec{y}_k\}$ such that the norms $\|\vec{y}_k\|_{L^\infty(\tilde{I},L^2(\Omega))}$, $\|\vec{y}_k\|_{L^2(Q)}$ & $\|\vec{y}_k\|_{L^2(\tilde{I},V)}$ are bounded ,then by AT there exists a subsequence of $\{\vec{y}_k\}$, for simplicity say again $\{\vec{y}_k\}$, such that

$\vec{y}_k \longrightarrow \vec{y}$ weakly in $\left(L^\infty\left(\tilde{I}, L^2(\Omega)\right)\right)^3$, $\vec{y}_k \longrightarrow \vec{y}$ weakly in $(L^2(Q))^3$, and

$\vec{y}_k \longrightarrow \vec{y}$ weakly in $(L^2(\tilde{I},V))^3$, to show the norm $\|\vec{y}_{kt}\|_{(L^2(\tilde{I},V^*))^3}$ is bounded , let (2.19.a),(2.20.a)&(2.21.a) be rewritten as

$\langle y_{1kt}, v_1\rangle = -(\nabla y_{1k}, \nabla v_1) - (y_{1k}, v_1) + (y_{2k}, v_1) + (y_{3k}, v_1) + (f_1 + u_{1k}, v_1)$

$\langle y_{2kt}, v_2\rangle = -(\nabla y_{2k}, \nabla v_2) - (y_{2k}, v_2) - (y_{3k}, v_2) - (y_{1k}, v_2) + (f_2 + u_{2k}, v_2)$

$\langle y_{3kt}, v_3\rangle = -(\nabla y_{3k}, \nabla v_3) - (y_{3k}, v_3) - (y_{1k}, v_3) + (y_{2k}, v_3) + (f_3 + u_{3k}, v_3)$.

by adding the above equations and integrating both sides of the obtained equation from 0 to T, taking the absolute value then using the CSI , and finally using assumptions (A) , it yields

$$\left|\int_0^T \langle\vec{y}_{kt}, \vec{v}\rangle dt\right| = \Big|\int_0^T [-(\nabla y_{1k}, \nabla v_1) - (y_{1k}, v_1) + (y_{2k}, v_1) + (y_{3k}, v_1) + (f_1 + u_{1k}, v_1) - (\nabla y_{2k}, \nabla v_2) - (y_{2k}, v_2) - (y_{3k}, v_2) - (y_{1k}, v_2) + (f_2 + u_{2k}, v_2) - (\nabla y_{3k}, \nabla v_3) - (y_{3k}, v_3) - (y_{1k}, v_3) + (y_{2k}, v_3) + (f_3 + u_{3k}, v_3)]dt\Big|$$, gives

$$\left|\int_0^T \langle\vec{y}_{kt}, \vec{v}\rangle dt\right| \le \|\nabla y_{1k}\|_Q\|\nabla v_1\|_Q + \|y_{1k}\|_Q\|v_1\|_Q + \|y_{2k}\|_Q\|v_1\|_Q + \|y_{3k}\|_Q\|v_1\|_Q + \|\nabla y_{2k}\|_Q\|\nabla v_2\|_Q + \|y_{2k}\|_Q\|v_2\|_Q + \|y_{3k}\|_Q\|v_2\|_Q + \|y_{1k}\|_Q\|v_2\|_Q + \|\nabla y_{3k}\|_Q\|\nabla v_3\|_Q + \|y_{3k}\|_Q\|v_3\|_Q + \|y_{1k}\|_Q\|v_3\|_Q + \|y_{2k}\|_Q\|v_3\|_Q +$$

$\|\eta_1\|_Q\|v_1\|_Q + \|\eta_2\|_Q\|v_2\|_Q + \|\eta_3\|_Q\|v_3\|_Q + \|u_{1k}\|_Q\|v_1\|_Q +$
$\|u_{2k}\|_Q\|v_2\|_Q + \|u_{3k}\|_Q\|v_3\|_Q$.

Since for each i = 1,2,3 the following inequalities are satisfied

$\|\nabla y_{ik}\|_{\boldsymbol{Q}} \le \|\nabla\vec{y}_k\|_{\boldsymbol{Q}}$, $\|v_i\|_{\boldsymbol{Q}} \le \|\vec{v}\|_{\boldsymbol{Q}}$, $\|u_{ik}\|_{\boldsymbol{Q}} \le \|\vec{u}_k\|_{\boldsymbol{Q}}$, $\|\vec{u}_k\|_{\boldsymbol{Q}} \le C$, $\|\nabla v_i\|_{\boldsymbol{Q}} \le \|\nabla\vec{v}\|_{\boldsymbol{Q}}$,
$\|y_{ik}\|_{\boldsymbol{Q}} \le \|\vec{y}_k\|_{\boldsymbol{Q}}$, $\|\nabla\vec{y}_k\|_{\boldsymbol{Q}} \le \|\vec{y}_k\|_{\boldsymbol{L^2}(\check{I},\boldsymbol{V})}$, $\|\eta_i\|_{\boldsymbol{Q}} \le b_i'$, $\|\vec{y}_k\|_{\boldsymbol{Q}} \le \|\vec{y}_k\|_{\boldsymbol{L^2}(\check{I},\boldsymbol{V})}$, $\|\vec{v}\|_{\boldsymbol{Q}} \le$
$\|\vec{v}\|_{\boldsymbol{L^2}(\check{I},\boldsymbol{V})}$, $\|\nabla\vec{v}\|_{\boldsymbol{Q}} \le \|\vec{v}\|_{\boldsymbol{L^2}(\check{I},\boldsymbol{V})}$, then

$$\left|\int_0^T\langle\vec{y}_{kt},\vec{v}\rangle dt\right| \le [\,3\|\vec{y}_k\|_{\boldsymbol{L^2}(\check{I},\boldsymbol{V})} + 9\|\vec{y}_k\|_{\boldsymbol{L^2}(\check{I},\boldsymbol{V})} + \left(b_1' + b_2' + b_3' + 3C\right)]\|\vec{v}\|_{\boldsymbol{L^2}(\check{I},\boldsymbol{V})}$$

Or

$$\left|\int_0^T\langle\vec{y}_{kt},\vec{v}\rangle dt\right| \le \left(12b_2(c) + b'(c)\right)\|\vec{v}\|_{\boldsymbol{L^2}(\check{I},\boldsymbol{V})} ,$$

where $\|\vec{y}_k\|_{\boldsymbol{L^2}(\check{I},\boldsymbol{V})} \le b_2(c)$ and $b'(c) = b_1' + b_2' + b_3' + 3C$

$$\Rightarrow \frac{\left|\int_0^T\langle\vec{y}_{kt},\vec{v}\rangle dt\right|}{\|\vec{v}\|_{\boldsymbol{L^2}(\check{I},\boldsymbol{V})}} \le b_3(c) \quad , \quad \text{with } b_3(c) = 12b_2(c) + b'(c) \Rightarrow \|\vec{y}_{kt}\|_{\boldsymbol{L^2}(\check{I},\boldsymbol{V}^*)} \le b_3(c)$$

since $\vec{y}_k$ is solution of the SEs (1−9) , then

$$\langle y_{1kt},v_1\rangle + (\nabla y_{1k},\nabla v_1) + (y_{1k},v_1) - (y_{2k},v_1) - (y_{3k},v_1) = (f_1 + u_{1k},v_1) \quad (56)$$
$$\langle y_{2kt},v_2\rangle + (\nabla y_{2k},\nabla v_2) + (y_{2k},v_2) + (y_{3k},v_2) + (y_{1k},v_2) = (f_2 + u_{2k},v_2) \quad (57)$$
$$\langle y_{3kt},v_3\rangle + (\nabla y_{3k},\nabla v_3) + (y_{3k},v_3) + (y_{1k},v_3) - (y_{2k},v_3) = (f_3 + u_{3k},v_3) \quad (58)$$

let $\varphi_i \in C^1[0,T]$, s.t. $\varphi_i(T) = 0$, $\forall\, i = 1,2,3$, rewriting the 1^{st} terms in the L.H.S. of (56–58) multiplying their both sides by $\varphi_1(t)$, $\varphi_2(t)$ & $\varphi_3(t)$ respectively , integrating both sides w.r.t. t from 0 to T ,and integration by parts for the 1^{st} terms in the L.H.S. of each obtained equation, one gets that

$$-\int_0^T(y_{1k},v_1)\varphi_1'(t)dt + \int_0^T[(\nabla y_{1k},\nabla v_1) + (y_{1k},v_1)]\varphi_1(t)dt - \int_0^T(y_{2k},v_1)\varphi_1(t)dt - \int_0^T(y_{3k},v_1)\,\varphi_1(t)dt = \int_0^T(f_1 + u_{1k},v_1)\,\varphi_1(t)dt + (y_{1k}(0),v_1)\,\varphi_1(0) \quad (59)$$

$$-\int_0^T(y_{2k},v_2)\varphi_2'(t)dt + \int_0^T[(\nabla y_{2k},\nabla v_2) + (y_{2k},v_2)]\varphi_2(t)dt + \int_0^T(y_{3k},v_2)\,\varphi_2(t)dt + \int_0^T(y_{1k},v_2)\,\varphi_2(t)dt = \int_0^T(f_2 + u_{2k},v_2)\varphi_2(t)dt + (y_{2k}(0),v_2)\varphi_2(0) \quad (60)$$

$$-\int_0^T(y_{3k},v_3)\varphi_3'(t)dt + \int_0^T[(\nabla y_{3k},\nabla v_3) + (y_{3k},v_3)]\varphi_3(t)dt + \int_0^T(y_{1k},v_3)\,\varphi_3(t)dt - \int_0^T(y_{2k},v_3)\,\varphi_3(t)dt = \int_0^T(f_3 + u_{3k},v_3)\varphi_3(t)dt + (y_{3k}(0),v_3)\varphi_3(0) \quad (61)$$

since $\vec{y}_k \longrightarrow \vec{y}$ weakly in $\left(L^2(Q)\right)^3$ and $\vec{y}_k \longrightarrow \vec{y}$ weakly in $\left(L^2(\tilde{I},V)\right)^3$, then the following convergences are hold

$$-\int_0^T(y_{1k},v_1)\varphi_1'(t)dt + \int_0^T[(\nabla y_{1k},\nabla v_1) + (y_{1k},v_1)]\varphi_1(t)dt - \int_0^T(y_{2k},v_1)\,\varphi_1(t)dt - \int_0^T(y_{3k},v_1)\,\varphi_1(t)dt \longrightarrow$$
$$-\int_0^T(y_1,v_1)\varphi_1'(t)dt + \int_0^T[(\nabla y_1,\nabla v_1) + (y_1,v_1)]\varphi_1(t)dt - \int_0^T(y_2,v_1)\,\varphi_1(t)dt - \int_0^T(y_3,v_1)\,\varphi_1(t)dt \quad (62)$$

$$-\int_0^T(y_{2k},v_2)\varphi_2'(t)dt + \int_0^T[(\nabla y_{2k},\nabla v_2) + (y_{2k},v_2)]\varphi_2(t)dt + \int_0^T(y_{3k},v_2)\,\varphi_2(t)dt + \int_0^T(y_{1k},v_2)\,\varphi_2(t)dt \longrightarrow$$
$$-\int_0^T(y_2,v_2)\varphi_2'(t)dt + \int_0^T[(\nabla y_2,\nabla v_2) + (y_2,v_2)]\varphi_2(t)dt + \int_0^T(y_3,v_2)\,\varphi_2(t)dt + \int_0^T(y_1,v_2)\,\varphi_2(t)dt \quad (63)$$

$-\int_0^T (y_{3k}, v_3)\varphi_3'(t)dt + \int_0^T [(\nabla y_{3k}, \nabla v_3) + (y_{3k}, v_3)]\varphi_3(t)dt + \int_0^T (y_{1k}, v_3)\,\varphi_3(t)dt - \int_0^T (y_{2k}, v_3)\varphi_3(t)dt \quad \longrightarrow$

$-\int_0^T (y_3, v_3)\varphi_3'(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\varphi_3(t)dt + \int_0^T (y_1, v_3)\,\varphi_3(t)dt - \int_0^T (y_2, v_3)\,\varphi_3(t)dt$ (64)

since $(y_{1k}(0)\,, y_{2k}(0),\ y_{3k}(0)\,)$ bounded in$(L^2(\Omega)\,)^3$and from the projection theorem one has

$(y_{1k}^0, v_1)\varphi_1(0) \longrightarrow (y_1^0, v_1)\varphi_1(0)$ (65)

$(y_{2k}^0, v_2)\varphi_2(0) \longrightarrow (y_2^0, v_2)\varphi_2(0)$ (66)

$(y_{3k}^0, v_3)\varphi_3(0) \longrightarrow (y_3^0, v_3)\varphi_3(0)$ (67)

and since $\vec{u}_k \longrightarrow \vec{u}$ weakly in $(L^2(Q)\,)^3$, then

$\int_0^T (f_1 + u_{1k}, v_1)\varphi_1(t)dt \longrightarrow \int_0^T (f_1 + u_1, v_1)\,\varphi_1(t)dt$ (68)

$\int_0^T (f_2 + u_{2k}, v_2)\varphi_2(t)dt \longrightarrow \int_0^T (f_2 + u_2, v_2)\,\varphi_2(t)dt$ (69)

$\int_0^T (f_3 + u_{3k}, v_3)\varphi_3(t)dt \longrightarrow \int_0^T (f_3 + u_3, v_3)\,\varphi_3(t)dt$ (70)

finally using(62 −64) ,(65−67), (68−70) in (59−61) respectively, one gets

$-\int_0^T (y_1, v_1)\varphi_1'(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\varphi_1(t)dt - \int_0^T (y_2, v_1)\,\varphi_1(t)dt - \int_0^T (y_3, v_1)\,\varphi_1(t)dt = \int_0^T (f_1 + u_1, v_1)\varphi_1(t)dt + (y_1^0, v_1)\varphi_1(0)$ (71)

$-\int_0^T (y_2, v_2)\varphi_2'(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\varphi_2(t)dt + \int_0^T (y_3, v_2)\,\varphi_2(t)dt + \int_0^T (y_1, v_2)\,\varphi_2(t)dt = \int_0^T (f_2 + u_2, v_2)\varphi_2(t)dt + (y_2^0, v_2)\varphi_2(0)$ (72)

$-\int_0^T (y_3, v_3)\varphi_3'(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\varphi_3(t)dt + \int_0^T (y_1, v_3)\,\varphi_3(t)dt - \int_0^T (y_2, v_3)\,\varphi_3(t)dt = \int_0^T (f_3 + u_3, v_3)\,\varphi_3(t)dt + (y_3^0, v_3)\varphi_3(0)$ (73)

Case 1: We choose $\varphi_i \in D[0,T]$, i.e. $\varphi_i(0) = \varphi_i(T) = 0$, $\forall i = 1,2,3$ .now using integration by parts for the 1^{st} terms in the L.H.S. of (71−73), one gets that

$\int_0^T \langle y_{1t}, v_1\rangle\varphi_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\varphi_1(t)dt - \int_0^T (y_2, v_1)\,\varphi_1(t)dt - \int_0^T (y_3, v_1)\,\varphi_1(t)dt = \int_0^T (f_1 + u_1, v_1)\varphi_1(t)dt\ ,\ \forall v_1 \in V_1\,, \forall\varphi_i \in D[0,T]$ (74)

$\int_0^T \langle y_{2t}, v_2\rangle\varphi_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\varphi_2(t)dt + \int_0^T (y_3, v_2)\,\varphi_2(t)dt + \int_0^T (y_1, v_2)\,\varphi_2(t)dt = \int_0^T (f_2 + u_2, v_2)\varphi_2(t)dt\ , \forall v_2 \in V_2\,, \forall\varphi_i \in D[0,T]$ (75)

$\int_0^T \langle y_{3t}, v_3\rangle\varphi_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\varphi_3(t)dt + \int_0^T (y_1, v_3)\,\varphi_3(t)dt - \int_0^T (y_2, v_3)\,\varphi_3(t)dt = \int_0^T (f_3 + u_3, v_3)\varphi_3(t)dt\ , \forall v_3 \in V_3, \forall\varphi_i \in D[0,T]$ (76)

Then

$\langle y_{1t}, v_1\rangle + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) - (y_3, v_1) = (f_1 + u_1, v_1)$, $\forall v_1 \in V_1$, a.e. on $\tilde{I}$

$\langle y_{2t}, v_2\rangle + (\nabla y_2, \nabla v_2) + (y_2, v_2) + (y_3, v_2) + (y_1, v_2) = (f_2 + u_2, v_2)$, $\forall v_2 \in V_2$, a.e. on $\tilde{I}$

$\langle y_{3t}, v_3\rangle + (\nabla y_3, \nabla v_3) + (y_3, v_3) + (y_1, v_3) - (y_2, v_3) = (f_3 + u_3, v_3)$, $\forall v_3 \in V_3$, a.e. on $\tilde{I}$

i.e. $(y_1,\ y_2, y_3)$ satisfies the wf of the SEs

Case 2 : we choose $\varphi_i \in C^1[\tilde{I}]$, s.t. $\varphi_i(T) = 0$, $\varphi_i(0) \neq 0$, $\forall i = 1,2,3$, using integration by parts for the 1^{st} terms in the L.H.S. of (74−76),one has

$-\int_0^T (y_1, v_1)\varphi_1'(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\varphi_1(t)dt - \int_0^T (y_2, v_1)\,\varphi_1(t)dt - \int_0^T (y_3, v_1)\,\varphi_1(t)dt = \int_0^T (f_1 + u_1, v_1)\varphi_1(t)dt + (y_1(0), v_1)\varphi_1(0)$ (77)

$$-\int_0^T (y_2, v_2)\varphi_2'(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\varphi_2(t)dt + \int_0^T (y_3, v_2)\,\varphi_2(t)dt + \int_0^T (y_1, v_2)\,\varphi_2(t)dt = \int_0^T (f_2 + u_2, v_2)\varphi_2(t)dt + (y_2(0), v_2)\varphi_2(0) \quad (78)$$

$$-\int_0^T (y_3, v_3)\varphi_3'(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\varphi_3(t)dt + \int_0^T (y_1, v_3)\,\varphi_3(t)dt - \int_0^T (y_2, v_3)\,\varphi_3(t)dt = \int_0^T (f_3 + u_3, v_3)\,\varphi_3(t)dt + (y_3(0), v_3)\varphi_3(0) \quad (79)$$

by subtracting(77−79) from (71−73) respectively, one obtain

$(y_1^0, v_1)\,\varphi_1(0)$ =$(y_1(0), v_1)\,\varphi_1(0)$, , $\forall\, \varphi_1 \in [0, T]$

$\Rightarrow y_1^0 = y_1(0) = y_1^0(x)$,

by the same way show that $y_2^0 = y_2(0) = y_2^0(x)$ and $y_3^0 = y_3(0) = y_3^0(x)$

$\Rightarrow$ (y_1, y_2, y_3) is a solution of the wf of the SE ,since $G_0(\vec{u})$ is W.L.S.C. from Lemma4.1 and since $\vec{u}_k \longrightarrow \vec{\vec{u}}$ weakly in $\left(L^2(\Omega)\right)^3$, then

$G_0(\vec{u}) \le \lim_{k\to\infty} \inf_{\vec{u}_k \in \vec{W}_A} G_0(\vec{u}_k) = \lim_{k\to\infty} G_0(\vec{u}_k) = \inf G_0(\vec{\vec{u}})$

$\Rightarrow \quad G_0(\vec{u}) \le \inf_{\vec{\vec{u}} \in \vec{W}_A} G_0(\vec{\vec{u}}) = \min_{\vec{\vec{u}} \in \vec{W}_A} G_0(\vec{\vec{u}})$.

Then $\vec{u}$ is a CCOC.

5. The NCO:

In order to state the NCs for CCOC, the FD of the cost function (10) is derived and the theorem for the NCO is proved.

Theorem 5.1: Consider $G_0(\vec{u})$ is given by (10) and the TAEs of the STE (1-9) are given by

$$-z_{1t} - \Delta z_1 + z_1 + z_2 + z_3 = (y_1 - y_{1d}) \quad (80)$$
$$-z_{2t} - \Delta z_2 + z_2 - z_1 - z_3 = (y_2 - y_{2d}) \quad (81)$$
$$-z_{3t} - \Delta z_3 + z_3 - z_1 + z_2 = (y_3 - y_{3d}) \quad (82)$$
$$z_1(T) = 0 \quad (83)$$
$$z_2(T) = 0 \quad (84)$$
$$z_3(T) = 0 \quad (85)$$

Then $(u_1, u_2, u_3) \in \vec{W}_A$ and the FD of G_0 is given by $\left(G_0'(\vec{u}), \delta\vec{u}\right) = (\vec{z} + \beta\vec{u}, \delta\vec{u})$

proof: The wf of (80–85) for $v_i \in V_i$,$\forall\ i = 1,2,3$ is given by

$$-\langle z_{1t}, v_1\rangle + (\nabla z_1, \nabla v_1) + (z_1, v_1) + (z_2, v_1) + (z_3, v_1) = (y_1 - y_{1d}, v_1) \quad (86)$$
$$-\langle z_{2t}, v_2\rangle + (\nabla z_2, \nabla v_2) + (z_2, v_2) - (z_1, v_2) - (z_3, v_2) = (y_2 - y_{2d}, v_2) \quad (87)$$
$$-\langle z_{3t}, v_3\rangle + (\nabla z_3, \nabla v_3) + (z_3, v_3) - (z_1, v_3) + (z_2, v_3) = (y_3 - y_{3d}, v_3) \quad (88)$$

The existence of a unique solution of (86−88) can be proved by the same manner which is used in the proof of Theorem 3.1 , now substituting $v_1 = z_1,\ v_2 = z_2\ and\ v_3 = z_3$ in (52.a) ,(53.a) and (54.a) respectively, these equations become ,

$$\langle \delta y_{1t}, z_1\rangle + (\nabla\delta y_1, \nabla z_1) + (\delta y_1, z_1) - (\delta y_2, z_1) - (\delta y_3, z_1) = (\delta u_1, z_1) \quad (89)$$
$$\langle \delta y_{2t}, z_2\rangle + (\nabla\delta y_2, \nabla z_2) + (\delta y_2, z_2) + (\delta y_3, z_2) + (\delta y_1, z_2) = (\delta u_2, z_2) \quad (90)$$
$$\langle \delta y_{3t}, z_3\rangle + (\nabla\delta y_3, \nabla z_3) + (\delta y_3, z_3) + (\delta y_1, z_3) - (\delta y_2, z_3) = (\delta u_3, z_3) \quad (91)$$

also, substituting $v_1 = \delta y_1, v_2 = \delta y_2$ and $v_3 = \delta y_3$ in (86−88) respectively, to get

$$-\langle z_{1t}, \delta y_1\rangle + (\nabla z_1, \nabla\delta y_1) + (z_1, \delta y_1) + (z_2, \delta y_1) + (z_3, \delta y_1) = (y_1 - y_{1d}, \delta y_1) \quad (92)$$
$$-\langle z_{2t}, \delta y_2\rangle + (\nabla z_2, \nabla\delta y_2) + (z_2, \delta y_2) - (z_1, \delta y_2) - (z_3, \delta y_2) = (y_2 - y_{2d}, \delta y_2) \quad (93)$$
$$-\langle z_{3t}, \delta y_3\rangle + (\nabla z_3, \nabla\delta y_3) + (z_3, \delta y_3) - (z_1, \delta y_3) + (z_2, \delta y_3) = (y_3 - y_{3d}, \delta y_3) \quad (94)$$

Integrating both sides of equations (89− 94) w.r.t. t from 0 to T, using integration by parts for the 1^{st}terms of the L.H.S. of each of the obtained equations from (92–94), then subtracting each one of the obtained equations from it's corresponding equation of (89–91), add all three result get

$\langle (y_1 + \delta y_1)_t, v_1 \rangle + (\nabla(y_1 + \delta y_1), \nabla v_1) + (y_1 + \delta y_1, v_1) - (y_2 + \delta y_2, v_1) -$
$(y_3 + \delta y_3, v_1) = (f_1 + u_1 + \delta u_1, v_1)$ (95)
$\langle (y_2 + \delta y_2)_t, v_2 \rangle + (\nabla(y_2 + \delta y_2), \nabla v_2) + (y_2 + \delta y_2, v_2) + (y_3 + \delta y_3, v_2) +$
$(y_1 + \delta y_1, v_2) = (f_2 + u_2 + \delta u_2, v_2)$ (96)
$\langle (y_3 + \delta y_3)_t, v_3 \rangle + (\nabla(y_3 + \delta y_3), \nabla v_3) + (y_3 + \delta y_3, v_3) + (y_1 + \delta y_1, v_3) -$
$(y_2 + \delta y_2, v_3) = (f_3 + u_3 + \delta u_3, v_3)$ (97)
which means the CV ($u_1 + \delta u_1, u_2 + \delta u_2$, $u_3 + \delta u_3$) gives the solution ($y_1 + \delta y_1$, $y_2 + \delta y_2$, $y_3 + \delta y_3$) of (95−97).
Now, from the cost function, we have
$G_0(\vec{u} + \delta\vec{u}) - G_0(\vec{u}) = (\delta u_1, z_1) + (\delta u_2, z_2) + (\delta u_3, z_3) + (\beta u_1, \delta u_1) + (\beta u_2, \delta u_2) + (\beta u_3, \delta u_3) + \frac{1}{2}\|\delta\vec{y}\|_Q^2 + \frac{\beta}{2}\|\delta\vec{u}\|_Q^2$,
Or
$G_0(\vec{u} + \delta\vec{u}) - G_0(\vec{u}) = (\vec{z} + \beta\vec{u}, \delta\vec{u}) + \frac{1}{2}\|\delta\vec{y}\|_Q^2 + \frac{\beta}{2}\|\delta\vec{u}\|_Q^2$
from the first results of Theorem 4.1, we have
$\frac{1}{2}\|\delta\vec{y}\|_{\boldsymbol{Q}}^2 = \varepsilon_1(\delta\vec{u})\|\delta\vec{u}\|_{\boldsymbol{Q}}$ and $\frac{\beta}{2}\|\delta\vec{u}\|_{\boldsymbol{Q}}^2 = \varepsilon_2(\delta\vec{u})\|\delta\vec{u}\|_{\boldsymbol{Q}}$
with $\varepsilon_1(\delta\vec{u}) = \frac{1}{2}M^2\|\delta\vec{u}\|_{\boldsymbol{Q}}$, where $\varepsilon_1(\delta\vec{u}), \varepsilon_2(\delta\vec{u}) \longrightarrow 0$, as $\|\delta\vec{u}\|_{\boldsymbol{Q}} \longrightarrow 0$
Then
$G_0(\vec{u} + \delta\vec{u}) - G_0(\vec{u}) = (\vec{z} + \beta\vec{u}, \delta\vec{u}) + \varepsilon(\delta\vec{u})\|\delta\vec{u}\|_{\boldsymbol{Q}}$
with $\varepsilon(\delta\vec{u}) = \varepsilon_1(\delta\vec{u}) + \varepsilon_2(\delta\vec{u})$, where $\varepsilon(\delta\vec{u}) \longrightarrow 0$,as $\|\delta\vec{u}\|_{\boldsymbol{Q}} \longrightarrow 0$
using the definition of FD of G_0 ,one has
$\left(G_0'(\vec{u}), \delta\vec{u}\right) = (\vec{z} + \beta\vec{u}, \delta\vec{u})$

Theorem 5.2:
The CCOC of the above problem is $G_0'(\vec{u}) = \vec{z} + \beta\vec{u} = 0$ with $\vec{y} = \vec{y}_{\vec{u}}$ and $\vec{z} = \vec{z}_{\vec{u}}$.
Proof : If $\vec{u}$ is an CCOC of the problem, then
$G_0(\vec{\vec{u}}) = \min_{\vec{u} \in \overrightarrow{W}_A} G_0(\vec{u})$, $\forall \vec{u} \in \left(L^2(\Omega)\right)^3$,
i.e. $G_0'(\vec{\vec{u}}) = 0 \Rightarrow \vec{z} + \beta\vec{\vec{u}} = 0$ ⇒ The NCO is
$\left(\vec{z} + \beta\vec{\vec{u}}, \delta\vec{u}\right) \geq 0$, $\forall\, \delta\vec{u} = \vec{w} - \vec{\vec{u}} \Rightarrow \left(\vec{z} + \beta\vec{\vec{u}}, \vec{w}\right) \geq \left(\vec{z} + \beta\vec{\vec{u}}, \vec{\vec{u}}\right)$, $\forall \vec{w} \in \left(L^2(\Omega)\right)^2$.

6. Conclusions

The GM is employed to prove the existence and unique theorem for a SVS of the TSPDEs of parabolic type for fixed CCCV. The existence of a CCOCV governing with the considered TLPDEs of parabolic type is proved. The existence and uniqueness solution of the TAEs associated with the TSPDEs. The derivation of the Fréchet derivative for the cost function is obtained. The theorem of the NCO is stated and proved.

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