Ibn Al-Haitham Jour. for Pure & Appl. Sci. 33 (1) 2020

Ibn Al Haitham Journal for Pure and Applied Science

Journal homepage: http://jih.uobaghdad.edu.iq/index.php/j/index

The Continuous Classical Optimal Control Problems for Triple Elliptic Partial Differential Equations

J. A. Al-Hawasy **Doaa Kateb Jasim**

Department of Mathematics, College of Science, University of Mustansiriyah

Jhawassy17@uomustansiriyah.edu.iq hawasy20@yahoo.com

Article history: Received 13 May 2019, Accepted 11 June 2019, Publish January 2020.

Doi: 10.30526/33.1.2380

Abstract

In this paper the Galerkin method is used to prove the existence and uniqueness theorem for the solution of the state vector of the triple linear elliptic partial differential equations for fixed continuous classical optimal control vector. Also, the existence theorem of a continuous classical optimal control vector related with the triple linear equations of elliptic types is proved. The existence of a unique solution for the triple adjoint equations related with the considered triple of the state equations is studied. The Fréchet derivative of the cost function is derived. Finally the theorem of necessary conditions for optimality of the considered problem is proved.

Keyword: Triple linear equations of elliptic type, optimal control (vector) of continuous classical type.

1. Introduction

Optimal control problems are a fundamental tool in many fields of applied mathematics and taken an important role in many aspects of life, for example in an electric power [1]. In robotics [2]. In biology [3]. In economic [4]. In medicine as [5]. In heat condition [6]. And in many others aspects. This importance encouraged researchers to study problems for the optimal control related with nonlinear ordinary differential equations [7]. Or related with different types of nonlinear partial differential equation as hyperbolic, parabolic, elliptic [8-10]. Or related with couple of nonlinear hyperbolic, parabolic and elliptic partial differential equation [11-13]. While many others researchers studied the Numann boundary optimal control problems related with couple of nonlinear hyperbolic, parabolic and elliptic partial differential equation [14-16]. This article deals with; the existence theorem for a unique solution (continuous state vector (CSV)) for the triple linear elliptic partial differential equations (TLEPDEqs) is sated, studied and proved by using the Galerkin Method (GM) for fixed continuous classical optimal control vector (CCOCV). The existence theorem for a continuous classical optimal control vector (CCOCV) related with the TLEPDEqs is state and proved. The existence for the unique solution of the triple adjoint equations (TAEqs) which corresponds to the TLEPDEqs is studied. The Fréchet derivative (FD) of the cost function is derived; finally the theorem for necessary conditions of optimality (NCO) is stated and proved.

2. Problem Description

Let Λ be a bounded and open connected subset in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Lambda$. Consider the CCOCV of the TLEPDEqs

$$-B_1\,\xi_1 + \xi_1 - \xi_2 - \xi_3 = a_1 + v_1 \tag{1}$$
$$-B_2\,\xi_2 + \xi_1 + \xi_2 + \xi_3 = a_2 + v_2 \tag{2}$$
$$-B_3\,\xi_3 + \xi_1 - \xi_2 + \xi_3 = a_3 + v_3 \tag{3}$$

with the Dirchlet boundary condition

$$\xi_1 = \xi_2 = \xi_3 = 0\ , \text{ in } \partial\Lambda \tag{4}$$

where $B_r\xi_r = \sum_{i,j}^{2} \frac{\partial}{\partial x_i}\left(b_{ij}\frac{\partial \xi_r}{\partial x_j}\right)$, $r = 1,2,3$, $b_{ij} = b_{ij}(x_{ij}) \in L^\infty(\Lambda)$, $\forall i,j = 1,2$, $(\xi_1,\xi_2,\xi_3) = (\xi_1(x_1,x_2),\xi_2(x_1,x_2),\xi_3(x_1,x_2)) \in \left(H(\bar{\Lambda})\right)^3$ is the state vector (classical solution of the system (1-4)), $(v_1,v_2,v_3) = \left(v_1(x_1,x_2),v_2(x_1,x_2),v_3(x_1,x_2)\right) \in \left(L^2(\Lambda)\right)^3$ is the classical control vector and $(a_1,a_2,a_3) = (a_1(x_1,x_2),a_2(x_1,x_2),a_3(x_1,x_2)) \in (L^2(\Lambda))^3$ is a vector of a given function , for all $(x_1,x_2) \in \Lambda$.

The Set of Admissible Control is $\vec{U} \in \left(L^2(\Lambda)\right)^3$, such that

$$\vec{U} = \left\{(v_1,v_2,v_3) \in \left(L^2(\Lambda)\right)^3 \middle| (v_1,v_2,v_3) \in V_1 \times V_2 \times V_3 = \vec{V} \subset \mathbb{R}^3 \text{ a.e. in } \Lambda\right\}$$

where $V_1 \times V_2 \times V_3$ is convex set.

The Cost Functional is

$$Y_0(\vec{v}) = \frac{1}{2}\|\xi_1 - \xi_{1d}\|_0^2 + \frac{1}{2}\|\xi_2 - \xi_{2d}\|_0^2 + \frac{1}{2}\|\xi_3 - \xi_{3d}\|_0^2$$
$$+\frac{\alpha}{2}\|v_1\|_0^2 + \frac{\alpha}{2}\|v_2\|_0^2 + \frac{\alpha}{2}\|v_3\|_0^2 \quad ,\vec{v} \in \vec{U} \tag{5}$$

Where α is a positive real number, $\vec{\xi}$ is the solution vector of (1-4) corresponding to the continuous classical control vector (CCV) $\vec{v}$ and $(\xi_{1d},\xi_{2d},\xi_{3d})$ is a vector of desired date.

The CCOCV Problem is to minimize $Y_0(\vec{v})$ (5) subject to $\vec{V} = (v_1,v_2,v_3) \in \vec{U}$.

Let $\vec{W} = W_1 \times W_2 \times W_3 = H_0^1(\Lambda) \times H_0^1(\Lambda) \times H_0^1(\Lambda)$.We denote by (w,w) and $\|w\|_1$ the inner product and the norm in $H^1(\Lambda)$, by $(\vec{w},\vec{w})$, $\|\vec{w}\|_0$ the inner product and the norm in $L^2(\Lambda)$ by $(\vec{w},\vec{w}) = (w_1,w_1) + (w_2,w_2) + (w_3,w_3)$ and $\|\vec{w}\|_0 = \|w_1\|_0 + \|w_2\|_0 + \|w_3\|_0$ the inner product and the norm in $\vec{W}$ and $\vec{W^*}$ (the dual of $\vec{W}$).

3. Weak Formulation of the TLEPDEqs

The weak form (WF) of problem (1-4) are obtained by multiplying both sides of Equations (1-3) by $w_1 \in W_1, w_2 \in W_2$ and $w_3 \in W_3$ respectively, integrating the obtained Equations and finally using the generalize Green's theorem for the 1st term in the Left hand side (L.H.S) of the three obtained equations , to get

$$b_1(\xi_1,w_1) + (\xi_1,w_1) - (\xi_2,w_1) - (\xi_3,w_1) = (a_1,w_1) + (v_1,w_1) \tag{6}$$
$$b_2(\xi_2,w_2) + (\xi_1,w_2) + (\xi_2,w_2) + (\xi_3,w_2) = (a_2,w_2) + (v_2,w_2) \tag{7}$$

$b_3(\xi_3, w_3) + (\xi_1, w_3) - (\xi_2, w_3) + (\xi_3, w_3) = (a_3, w_3) + (v_3, w_3)$ (8)

where $b_r(\xi_r, w_r) = \iint_\Lambda \sum_{i,j=1}^{2} b_{ij} \frac{\partial \xi_r}{\partial x_i} . \frac{\partial w_r}{\partial x_j}\, dx_1\, dx_2$, $(\Theta_r, w_p) = \iint_\Lambda \Theta_r\, w_p\; dx_1\, dx_2$

Θ_r=(a_p or v_p), $r = p = 1,2,3$ or $\Theta_r = \xi_r$, $r = 1,2,3$.

blending to gather Equations (6), (7) and (8), once get

$B(\vec{\xi}, \vec{w}) = \breve{A}(\vec{w})$ (9)

where $B(\vec{\xi}, \vec{w}) = b_1(\xi_1, w_1) + (\xi_1, w_1) - (\xi_2, w_1) - (\xi_3, w_1) + b_2(\xi_2, w_2) + (\xi_1, w_2) + (\xi_2, w_2)$

$+(\xi_3, w_2) + b_3(\xi_3, w_3) + (\xi_1, w_3) - (\xi_2, w_3) + (\xi_3, w_3)$

and for fixed $\vec{v}$,

$\breve{A}(\vec{w}) = (a_1, w_1) + (v_1, w_1) + (a_2, w_2) + (v_2, w_2) + (a_3, w_3) + (v_3, w_3)$

The following hypotheses are useful to study the existence of unique solution for the WF (9) .

Hypotheses:

a) $B(\vec{\xi}, \vec{w})$ is coercive, i.e. $B(\vec{\xi}, \vec{\xi}) \geq \epsilon \|\xi\|_1^{\,2}$, $\epsilon > 0$

b) $|B(\vec{\xi}, \vec{w})| \leq \epsilon_1 \|\vec{\xi}\|_1 \|\vec{w}\|_1$, $\epsilon_1 > 0$.

c) $|\ \breve{A}(\vec{w})\ | \leq \epsilon_2 \parallel \vec{w} \parallel_1$, $\forall \vec{w} \in \vec{W}$, $\epsilon_2 > 0$.

The GM is used here to find the solution of (9), This is doing through choosing a finite subspace $\vec{W}_n \subset \vec{W}$ ($\vec{W}_n = W_n \times W_n \times W_n$, where W_n contains the continuous and piecewise affine functions in Λ), hence the problem reduces to find an approximate solution of the following an approximation problem

$B(\vec{\xi_n}, \vec{w}) = \breve{A}(\vec{w})$, $\forall \vec{\xi}_n$, $\vec{w} \in \vec{W}_n$ (10)

Theorem 3.1:

For every fixed control vector $\vec{v} \in (L^2(\Lambda))^3$, the WF (10) has a unique approximation solution $\vec{\xi}_n \in \vec{W}_n$.

Proof: Let $\{\vec{\psi}_1, \vec{\psi}_2, \dots, \vec{\psi}_n\}$ be a finite basis of $\vec{W}_n$ and let

$\vec{\xi}_n = \vec{\xi}_n(x_1, x_2) = \sum_{j=1}^{N} d_j \vec{\psi}_j(x_1, x_2) = (\sum_{j=1}^{N} d_j \psi_{1j}, \sum_{j=1}^{N} d_j \psi_{2j}, \sum_{j=1}^{N} d_j \psi_{3j})$ (11)

Where $\vec{\psi}_j = ((1 - \frac{L - L \bmod 2}{2})\, \psi_k,\ (1 - P \bmod 2)\, \psi_k\ ,\ \frac{1}{2}(P \bmod 2 \,.\, L)\, \psi_k)$,

for $L = 0,1,2$, $P = L + 1 = 1,2,3$, $n = 3N$, $k = 1,2, \dots, N$

$j = k + N[((P-1)L) \bmod 3] + N\left[\frac{L(L-1)}{2}\right]$, and d_j with $j = 1,2, \dots, n$ are unknown constants.

By using $\vec{\xi}_n = \sum_{j=1}^{N} d_j \vec{\psi}_j$ and $\vec{w} = \vec{\psi}_i$, in (10), to get

$B(\sum_{j=1}^{N} d_j \vec{\psi}_j, \vec{\psi}_i) = \breve{A}(\vec{\psi}_i)$, $\forall i = 1,2, \dots, n$ (12)

which can be rewritten as a linear algebraic system, i.e.

$\Psi_{n\times n} d_{n\times 1} = a_{n\times 1}$ (13)

From hypothesis (a), easily once obtained the uniqueness of the solution of problem (13), which gives also the uniqueness of the solution of problem (10).

Theorem 3.2 (**Galarkin approch**)[**17**]. For each $\vec{w} \in \vec{W}$,there exists a sequence $\{\vec{\psi}_n\}$ with $\vec{\psi}_n \in \vec{W}_n$ for each n, such that $\vec{\psi}_n \longrightarrow \vec{w}$ strongly in $\vec{W}$.
Now from the WF (10) and theorem(3.2),once get that there exists a sequence of WF

$$B(\vec{\xi}_n,\vec{\psi}_n) = \breve{A}(\vec{\psi}_n) \ ,\forall \vec{\xi}_n,\vec{\psi}_n \in \vec{W}_n, \forall n \tag{14}$$

which has a sequence of solutions $\{\vec{\xi}_n\}_{n=1}^{\infty}$ and the sequence $\vec{\psi}_n \longrightarrow \vec{w}$ strongly in $\vec{W}$.

Theorem 3.3:
The sequence of solutions $\{\vec{\xi}_n\}_{n=1}^{\infty}$converges strongly to the solution$\vec{\xi}$ of (9).

Proof: Since for each n, $\vec{\xi}_n$ is a solution of (14), then from hypotheses (a&c), $\therefore \|\vec{\xi}_n\|_1 \le \epsilon_2$, $\forall$n, with $\epsilon_2 > 0$
Then by using Alaoglu theorem, there exists a subsequence of $\{\vec{\xi}_n\}$ (for simplicity say again $\{\vec{\xi}_n\}$), such that $\vec{\xi}_n \rightarrow \vec{\xi}$ weakly in $\vec{W}$. To prove, that the sequence $\{\vec{\xi}_n\}_{n=1}^{\infty}$of solution of (14) converges to a vector which is the solution of problem (9).
First, from hypothesis (b), the above weakly convergences and since $\vec{\psi}_n \longrightarrow \vec{w}$ strongly in $\vec{W}$**,** then

$$\left|B(\vec{\xi}_n,\vec{\psi}_n) - B(\vec{\xi},\vec{w})\right| \le \left|B(\vec{\xi}_n,\vec{\psi}_n - \vec{w})\right| + \left|B(\vec{\xi}_n - \vec{\xi},\vec{w})\right|$$
$$\le \epsilon_1\|\vec{\xi}_n\|_1\|\vec{\psi}_n - \vec{w}\|_1 + \epsilon_1\|\vec{\xi}_n - \vec{\xi}\|_1\|\vec{w}\|_1 \longrightarrow 0$$

Which means

$$B(\vec{\xi}_n,\vec{\psi}_n) \longrightarrow B(\vec{\xi},\vec{w})$$

Second, from theorem (3.2) $\vec{\psi}_n \longrightarrow \vec{w}$ weakly in $\vec{W}$**,** then $\breve{A}(\vec{\psi}_n) \longrightarrow \breve{A}(\vec{w})$
to prove $\vec{\xi}_n \longrightarrow \vec{\xi}$ strongly in $\vec{W}$, from hypothesis (1-a), one has

$$\epsilon\|\vec{\xi} - \vec{\xi}_n\|_1^2 \le B(\vec{\xi} - \vec{\xi}_n,\vec{\xi} - \vec{\xi}_n) = B(\vec{\xi} - \vec{\xi}_n,\vec{\xi}) - B(\vec{\xi},\vec{\xi}_n) + B(\vec{\xi}_n,\vec{\xi}_n) = B(\vec{\xi} - \vec{\xi}_n,\vec{\xi})$$
$$= \breve{A}(\vec{\xi} - \vec{\xi}_n) = \breve{A}(\vec{\xi}) - \breve{A}(\vec{\xi}_n) \longrightarrow 0$$

Which complete the proof of $\{\vec{\xi}_n\}$converges strongly to $\vec{\xi}$ with respect to$\|.\|_1$.
The uniqueness of solution is obtained easily through using hypothesis (a).

4. Existence of a CCOCV:

Lemma 4.1: The operator $\vec{v} \longmapsto \vec{\xi}_{\vec{v}}$ from $\vec{U}$ to $(L^2(\Lambda))^3$ is Lipschitz continuous (LC), i.e. $\| \overrightarrow{\delta\xi} \|_0 \le \breve{\epsilon} \| \overrightarrow{\delta v} \|_0$, for $\breve{\epsilon} > 0$.

Proof: Let $\overrightarrow{v'} = (v'_1, v'_2, v'_3)$ be a given control vector of the WF(6-8) and $\overrightarrow{\xi'} = (\xi'_1, \xi'_2, \xi'_3)$ be the corresponding state vector solution, we get new equations for $\overrightarrow{v'}$ and $\overrightarrow{\xi'}$, then by subtracting these new equations from their corresponding Equations (6-8) and then substituting $\delta\xi_1 = \xi'_1 - \xi_1$, $\delta v_1 = v'_1 -$

v_1, $\delta\xi_2$=$\xi'_2 - \xi_2$, $\delta v_2 = v'_2 - v_2$, $\delta\xi_3$=$\xi'_3 - \xi_3$ and $\delta v_3 = v'_3 - v_3$ in the obtained equations, to get

$$b_1(\delta\xi_1, w_1) + (\delta\xi_1, w_1) - (\delta\xi_2, w_1) - (\delta\xi_3, w_1) = (\delta v_1, w_1) \quad (15)$$
$$b_2(\delta\xi_2, w_2) + (\delta\xi_1, w_2) + (\delta\xi_2, w_2) + (\delta\xi_3, w_2) = (\delta v_2, w_2) \quad (16)$$
$$b_3(\delta\xi_3, w_3) + (\delta\xi_1, w_3) - (\delta\xi_2, w_3) + (\delta\xi_3, w_3) = (\delta v_3, w_3) \quad (17)$$

Next blending together the equations which obtained by substituting $w_1 = \delta\xi_1$, $w_2 = \delta\xi_2$ and $w_3 = \delta\xi_3$ in (15-17)) respectively, to give

$$b_1(\delta\xi_1, \delta\xi_1) + (\delta\xi_1, \delta\xi_1) + b_2(\delta\xi_2, \delta\xi_2) + (\delta\xi_2, \delta\xi_2) + b_3(\delta\xi_3, \delta\xi_3) + (\delta\xi_3, \delta\xi_3)$$
$$= (\delta v_1, \delta\xi_1) + (\delta v_2, \delta\xi_2) + (\delta v_3, \delta\xi_3) \quad (18)$$

After using Cauch-Schwarz inequality (C-S-I) and applying hypothesis (1-a),once has

$$\epsilon\left\|\overrightarrow{\delta\xi}\right\|_1^2 \le \|\delta v_1\|_0\|\delta\xi_1\|_0 + \|\delta v_2\|_0\|\delta\xi_2\|_0 + \|\delta v_3\|_0\|\delta\xi_3\|_0 \quad (19)$$

Since $\|\delta\xi_i\|_0 \le \left\|\overrightarrow{\delta\xi}\right\|_0 \le c\left\|\overrightarrow{\delta\xi}\right\|_1$ and $\|\delta v_i\|_0 \le \left\|\overrightarrow{\delta v}\right\|_0$, $\forall i = 1,2,3$, then (19) becomes

$$\left\|\delta\vec{\xi}\right\|_1 \le \breve{c}\left\|\overrightarrow{\delta v}\right\|_0, \text{ with } \breve{c} = \frac{3c}{\epsilon}$$

So $\vec{v} \mapsto \vec{\xi}_{\vec{v}}$ is LC on $(L^2(\Lambda))^3$.

Lemma 4.2[14]: The norm $\|\cdot\|_0$ is weakly lower semicontinuous (W.L.S.).

Lemma 4.3: The cost function in (5) is W.L.S. . .

Proof: the proof easily obtained through applying lemma (4.2), the weakly converge of $\vec{v}_n \longrightarrow \vec{v}$ in $L^2(\Lambda)$ and lemma (4.1).

Lemma 4.4[14]: The norm $\| \|_0^2$ is strictly convex.

Remark 4.1: The cost function $Y_0(\vec{v})$ is strictly convex by using Lemma (4.4).

Theorem 4.1: If $Y_0(\vec{v})$ is coercive and $\vec{V}$ is convex, then there exists CCOCV for the problem (5).

Proof: $\vec{U}$ is convex since $\vec{V}$ is convex with $Y_0(\vec{v}) \ge 0$, and $Y_0(\vec{v})$ is coercive then there exist a minimization sequence $\{\vec{v}_n\} \in \vec{U}, \forall n$ such that

$$\lim_{n\to\infty} Y_0(\vec{v}_n) = \inf_{\vec{u}\in\vec{U}} Y_0(\vec{u})$$

Therefore :

$$\|\vec{v}_n\|_0 \le c, \forall n\ , \ \ c > 0 \quad (20)$$

Then, the sequence $\{\vec{v}_n\}$ has a subsequence for simplicity say again$\{\vec{v}_n\}$ such that $\vec{v}_n \longrightarrow \vec{v}$ weakly in $(L^2(\Lambda))^3$, (by using the Aloglu theorem). But theorem 3.1, tell us that the sequence of problems (9) has the sequence of solutions$\{\vec{\xi}_n\}$. To prove$\{\vec{\xi}_n\}, \forall n$, is bounded in $\overrightarrow{W}$, the hypotheses (a and c), and the C-S-I, are used to get that:

$$\epsilon\left\|\vec{\xi}_n\right\|_1^2 \le B\left(\vec{\xi}_n,\vec{\xi}_n\right) = \breve{A}\left(\vec{\xi}_n\right) \quad (21)$$

$\le\|a_1\|_0\|\xi_{1n}\|_0 + \|v_{1n}\|_0\|\xi_{1n}\|_0 + \|a_2\|_0\|\xi_{2n}\|_0 + \|v_{2n}\|_0\|\xi_{2n}\|_0 + \|a_3\|_0\|\xi_{3n}\|_0 + \|v_{3n}\|_0\|\xi_{3n}\|_0 \le h_1\|\xi_{1n}\|_0 + \varepsilon_1\|\xi_{1n}\|_0 + h_2\|\xi_{2n}\|_0 + \varepsilon_2\|\xi_{2n}\|_0 + h_3\|\xi_{3n}\|_0 + \varepsilon_3\|\xi_{3n}\|_0$
$\le \varpi\left\|\vec{\xi}_n\right\|_1$

Where $\varpi = \max(\gamma_1,\gamma_2,\gamma_3), \gamma_1 = max(h_1,\varepsilon_1),\ \gamma_2 = max(h_2,\varepsilon_2)\ and\ \gamma_3 = max(h_3,\varepsilon_3)$ then $\left\|\vec{\xi}_n\right\|_1 \le \mu$,for each n, with $\mu = \frac{\varpi}{\epsilon} > 0$.

By Alaoglu theorem there exists a subsequence of $\{\vec{\xi}_n\}$ (for simplicity say again $\{\vec{\xi}_n\}$) such that $\vec{\xi}_n \to \vec{\xi}$ weakly in $\vec{U}$.
Since for each n, $\vec{\xi}_n$ satisfies the weak form (9),then

$$\mathrm{B}\left(\vec{\xi}_n,\vec{\mathrm{w}}\right) = \breve{\mathrm{A}}_n(\vec{\mathrm{w}}) = (a_1,w_1) + (v_{1n},w_1) + (a_2,w_2) + (v_{2n},w_2) + (a_3,w_3) + (v_{3n},w_3) \quad (22)$$

To show that (22) converges to

$$\mathrm{B}\left(\vec{\xi},\vec{\mathrm{w}}\right) = \breve{\mathrm{A}}(\vec{\mathrm{w}}) \quad (23)$$

First, since $\forall i,\ \xi_{in} \longrightarrow \xi_i$weakly in $L^2(\Lambda)$. Then by using the C-S-I and hypothesis (b), once gets:

$$|b_1(\xi_{1n},w_1) + (\xi_{1n},w_1) - (\xi_{2n},w_1) - (\xi_{3n},w_1) + b_2(\xi_{2n},w_2) + (\xi_{1n},w_2) + (\xi_{2n},w_2) + (\xi_{3n},w_2)$$
$$+b_3(\xi_{3n},w_3) + (\xi_{1n},w_3) - (\xi_{2n},w_3) + (\xi_{3n},w_3) - b_1(\xi_1,w_1) - (\xi_1,w_1) + (\xi_2,w_1) + (\xi_3,w_1)$$
$$-b_2(\xi_2,w_2) - (\xi_1,w_2) - (\xi_2,w_2) - (\xi_3,w_2) - b_3(\xi_3,w_3) - (\xi_1,w_3) + (\xi_2,w_3) - (\xi_3,w_3)|$$

$$\le \epsilon_1\|\xi_{1n} - \xi_1\|_1\|w_1\|_1 + \|\xi_{1n} - \xi_1\|_0\|w_1\|_0 + \|\xi_2 - \xi_{2n}\|_0\|w_1\|_0 + \|\xi_3 - \xi_{3n}\|_0\|w_1\|_0$$
$$+\epsilon_1\|\xi_{2n} - \xi_2\|_1\|w_2\|_1 + \|\xi_{1n} - \xi_1\|_0\|w_2\|_0 + \|\xi_{2n} - \xi_2\|_0\|w_2\|_0 + \|\xi_{3n} - \xi_3\|_0\|w_2\|_0$$
$$+\epsilon_1\|\xi_{3n} - \xi_3\|_1\|w_3\|_1 + \|\xi_{1n} - \xi_1\|_0\|w_3\|_0 + \|\xi_2 - \xi_{2n}\|_0\|w_3\|_0 + \|\xi_{3n} - \xi_3\|_0\|w_3\|_0 \longrightarrow 0$$

Second, the right hand side (R.H.S)of (22) converges to the R..H.S of (23), since $\vec{v}_n \to \vec{v}\ weakly\ in\ (L^2(\Lambda))^3$, which gives (22) converges to (23).
But $Y_0(\vec{v})$ is W.L.S., with $\vec{v}_n \to \vec{v}\ weakly\ in\ \left(L^2(\Lambda)\right)^3$, then

$$Y_0(\vec{v}) \le \lim_{n\to\infty} Y_0(\vec{v}_n) = \inf_{\vec{u}\in\vec{U}} Y_0(\vec{u}),\text{ which gives}$$

$$Y_0(\vec{v}) = \inf_{\vec{u}\in\vec{U}} Y_0(\vec{u})$$

i.e., $\vec{v}$ is a ccocv. One can easily applies remark 4.1, to get the uniqueness of $\vec{v}$.

5. The Necessary Conditions for Optimality

Theorem 5.1: Consider the cost function (5), and the TAEqs $(\zeta_1,\zeta_2,\zeta_3)$ equations of the state Equations (1-4) are given by :

$$-B_1\zeta_1 + \zeta_1 + \zeta_2 + \zeta_3 = \xi_1 - \xi_{1d} \quad (24)$$

$$-B_2\zeta_2 - \zeta_1 + \zeta_2 - \zeta_3 = \xi_2 - \xi_{2d} \quad (25)$$

$$-B_3\zeta_3 - \zeta_1 + \zeta_2 + \zeta_3 = \xi_3 - \xi_{3d} \quad (26)$$

$\zeta_1 = \zeta_2 = \zeta_3 = 0 \ on \ \partial\Lambda$ (27)

Then the Fréchet derivative of Y_0 is

$(Y'_0(\vec{v}), \overrightarrow{\delta v}) = (\vec{\zeta} + \alpha\vec{v}, \overrightarrow{\delta v})$

Proof: Writing the TAEqs (19-22) by their WF, then adding them and then substituting $\vec{w} = \overrightarrow{\delta\zeta}$ in the resulting equation to get the following WF (the proof of the existences of a unique solution $\vec{\zeta}$ for this WF is simpler than the proof of theorem (3.1)):

$$b_1(\zeta_1, \delta\xi_1) + (\zeta_1, \delta\xi_1) + (\zeta_2, \delta\xi_1) + (\zeta_3, \delta\xi_1) + b_2(\zeta_2, \delta\xi_2) - (\zeta_1, \delta\xi_2) + (\zeta_2, \delta\xi_2) - (\zeta_3, \delta\xi_2) + b_3(\zeta_3, \delta\xi_3) - (\zeta_1, \delta\xi_3) + (\zeta_2, \delta\xi_3) + (\zeta_3, \delta\xi_3) = (\xi_1 - \xi_{1d}, \delta\xi_1) + (\xi_2 - \xi_{2d}, \delta\xi_2) + (\xi_3 - \xi_{3d}, \delta\xi_3) \quad (28)$$

Now, substituting the solutions ξ_1 and $\xi_1 + \delta\xi_1$ in (6) separately, then subtracting the obtained 1st equation from the 2nd one, finally setting $w_1 = \zeta_1$, to obtain

$$b_1(\delta\xi_1, \zeta_1) + (\delta\xi_1, \zeta_1) - (\delta\xi_2, \zeta_1) - (\delta\xi_3, \zeta_1) = (\delta v_1, \zeta_1) \quad (29)$$

Same steps can be used in Equation (7)for the solutions ξ_2 and $\xi_2 + \delta\xi_2$ with $w_2 = \zeta_2$, (in Equation (8) for the solutionξ_3 and $\xi_3 + \delta\xi_3$ with $w_3 = \zeta_3$), to get respectively

$$b_2(\delta\xi_2, \zeta_2) + (\delta\xi_1, \zeta_2) + (\delta\xi_2, \zeta_2) + (\delta 3, \zeta_2) = (\delta v_2, \zeta_2) \quad (30)$$

$$b_3(\delta\xi_3, \zeta_3) + (\delta\xi_1, \zeta_3) - (\delta\xi_2, \zeta_3) + (\delta\xi_3, \zeta_3) = (\delta v_3, \zeta_3) \quad (31)$$

Blending together the above triple equations, then subtracting the obtained equation from (28), to get

$$(\delta v_1, \zeta_1) + (\delta v_2, \zeta_2) + (\delta v_3, \zeta_3) = (\xi_1 - \xi_{1d}, \delta\xi_1) + (\xi_2 - \xi_{2d}, \delta\xi_2) + (\xi_3 - \xi_{3d}, \delta\xi_3) \quad (32)$$

Now, (5) , once get

$$Y_0(\vec{v} + \overrightarrow{\delta v}) - Y_0(\vec{v}) = (\xi_1 - \xi_{1d}, \delta\xi_1) + (\xi_2 - \xi_{2d}, \delta\xi_2) + (\xi_3 - \xi_{3d}, \delta\xi_3) + (v_1, \delta v_1) + (v_2, \delta v_2) + (v_3, \delta v_3) + \frac{1}{2}\|\overrightarrow{\delta\xi}\|_0^2 + \frac{\alpha}{2}\|\overrightarrow{\delta v}\|_0^2 \quad (33)$$

From (32)and (33), once get

$$Y_0(\vec{v} + \overrightarrow{\delta v}) - Y_0(\vec{v}) = (\vec{\zeta} + \alpha\vec{v}, \overrightarrow{\delta v}) + \frac{1}{2}\|\overrightarrow{\delta\xi}\|_0^2 + \frac{\alpha}{2}\|\overrightarrow{\delta v}\|_0^2 \quad (34)$$

from lemma (4.1), once obtain

$$\frac{1}{2}\|\overrightarrow{\delta\xi}\|_0^2 + \frac{\alpha}{2}\|\overrightarrow{\delta v}\|_0^2 = \epsilon(\overrightarrow{\delta v}) \parallel \overrightarrow{\delta v} \parallel_0 \quad (35)$$

where $\epsilon(\overrightarrow{\delta v}) = \epsilon_1(\overrightarrow{\delta v}) + \epsilon_2(\overrightarrow{\delta v}) \longrightarrow 0$, as $\parallel \overrightarrow{\delta v} \parallel_0 \longrightarrow 0$

Then from the definition of FD ofY_0, and (34-35), once get

$$(Y_0'(\vec{v}), \overrightarrow{\delta v}) = (\vec{\zeta} + \alpha\vec{v}, \overrightarrow{\delta v}).$$

Theorem 5.2 : The CCOCV of (1- 4) is:

$Y'(\vec{v}) = \vec{\zeta} + \alpha\vec{v} = 0$ with $\vec{\xi} = \vec{\xi}_{\vec{v}}$ and $\vec{\zeta} = \vec{\zeta}_{\vec{v}}$.

Proof: If $\vec{v}$ is CCOCV of (1-4), then

$Y_0(\vec{v}) = \min_{\vec{u} \in \vec{U}} Y_0(\vec{u}), \forall \vec{u} \in (L^2(\Lambda))^3$, i.e., $Y_0'(\vec{v}) = 0 \Longrightarrow \vec{\zeta} = -\frac{\vec{v}}{\alpha}$

$\overrightarrow{\delta v} = \vec{u} - \vec{v}$

Thus NCO is

$(\vec{\zeta} + \alpha\vec{v}, \vec{v}) \leq (\vec{\zeta} + \alpha\vec{v}, \vec{u}),\ \forall\, \vec{u} \in (L^2(\Omega))^3.$

6. Conclusion:

The existence and uniqueness theorem for the solution (CSV) of the TLEPDEqs is stated and proved successfully by using the GM when the CCCV is given. Also, the existence theorem of a CCOCV governing by the TLEPDEqs is proved. The existence and uniqueness solution of the TAEqs related with the triple of the state equations is sated and studied.The derivation of the FD is given. Finally the NCO of this problem is proved.

References

1. Nguyen, D.B.; Scherpen, J.M.A.; Bliek, F. Distributed Optimal Control of Smart Electricity Grids With Congestion Management. *IEEE Transactions on Automation Science and Engineering*.**2017**, *14*, *2*, 494-504, doi: 10.1109/TASE.2017.2664061.
2. Braun, D.J.; Petit, F.; Huber, F.; Haddadin, S.; Smaga, V.D.P.; Albu-Schaffer, A.; Vijayakumar, S. Robots Driven by Compliant Actuators: Optimal Control Under Actuation Constraints. *IEEE Transactions on Robotics*.**2013**, *29*, *5*, 1085-1101, doi: 10.1109 /TRO.2013.2271099.
3. Chalak, M. Optimal Control for a Dispersing Biological Agent. *Journal of Agricultural and Resource Economics*.**2014**, *39*, *2*, 271–289.
4. Grüne, L.; Semmler, W.; Stieler, M. Using Nonlinear Model Predictive Control for Dynamic Decision Problems in Economics. *Journal of Economic Dynamics and Control*.**2015**, *60*, 112-133, https://hal.inria.fr/hal-01068831.
5. Tilahun, G.T.; Makinde, O. D.; Malonza, D. Modelling and optimal control of typhoid fever disease with cost-effective strategies. *Computational and Mathematical Methods in Medicine*, **2017**.
6. Yilmaz, A.; Mahariq, I.; Yilmaz, F. Numerical Solutions of Optimal Control Problems for Microwave Heating.*International Journal of Advances in Science Engineering and Technology*.**2016**, *4*, *3*, 64-66: ISSN: 2321-9009.
7. Orpel, A. Optimal Control Problems with Higher Order Constraints. *Folia Mathematica*.**2009**, *16*, *1*, 31-44.
8. Al-Hawasy, J. The Continuous Classical Optimal Control of a Nonlinear Hyperbolic Equation (CCOCP). *Al-Mustansiriyah Journal of Science*.**2008**, *19*, *8*, 96-110.
9. Chryssoverghi, I.; Al-Hawasy, J. The Continuous Classical Optimal Control Problem of a Semi linear Parabolic Equations (CCOCP). *Journal of Karbala University*.**2010**, *8*, *3*, 57-70.
10. Bors, D.; Walczak, S. Optimal control elliptic system with distributed and boundary controls. Nonlinear Analysis.**2005**, *63*, 5-7, e1367-e1376. https://doi.org/10.1016/j.na.2005.02.009.
11. Al-Hawasy, J. The Continuous Classical Optimal Control of a Couple Nonlinear Hyperbolic Partial Differential Equations with Equality and Inequality Constraints. *Iraqi Journal of Science*.**2016**, *57*, *2C*, 1528-1538.

12. Al-Hawasy, J.; Kadhem, G.M. The Continuous Classical Optimal Control for a Coupled Nonlinear Parabolic Partial Differential Equations with Equality and Inequality Constraints, *Journal of Al-Nahrain University Science*.**2016**, *19*, *1*, 173-186.
13. Al-Hawasy, J.; Naeif, A.A. The Continuous Classical Boundary Optimal Control of a Couple Linear of Parabolic Partial Differential Equations. *Al-Mustansiriyah Journal of Science*.**2018**, *29*, *1*, 118-126, doi: http://doi.org/10.23851/mjs.v29i1.159.
14. Al-Rawdanee, E.H.M. The Continuous Classical Optimal Control Problem of a *Non-Linear Partial Differential Equations of Elliptic Type*. M.Sc. Thesis, Al-Mustansiriyah University, **2015**.
15. Vexler, B. Finite element approximation of elliptic Dirichlet optimal control problems. *Numer. Funct. Anal. Optim*.**2007**, *28*, *7-8*, 957-973, doi: 10.1080/01630560701493305.
16. Al-Hawasy, J.A.A.; Al-Qaisi, S.J.M. The Continuous Classical Optimal Boundary Control of a Couple Linear Elliptic Partial Differential Equations. *Al-Nahrain Journal of Science*.**2018**, *1*, 137-142.
17. Bacopoulos, A.; Chryssoverghi, I. Numerical Solutions of Partial Differential Equations by Finite Elements Methods. *Symeon Publishing Co Athens*, **2003**.