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Separation Axioms via Ĭ - Semi- g- Open Sets

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Abstract

The notions *Ĭ-semi-g-closedness* and *Ĭ-semi-g-openness* were used to generalize and introduced new classes of separation axioms in ideal spaces. Many relations among several sorts of these classes are summarized, also.

Keyword: Ideal, separation axioms, Ĭsg-closed set, Ĭsg-open set, Ĭsgo-function, Ĭsg-continuous-function.

1. Introduction

Abdel Karim and Nasir [1]. Introduced the notion Ĭ-*semi-g-closed* set in ideal spaces. A set Ạ of $(\mathcal{X},\mathcal{T},Ĭ)$ is Ĭ-*semi-g-closed* set (artlessly,Ĭ*sg-closed*), when prerequisite Ạ-$\mathcal{U}$ ∈ Ĭ and $\mathcal{U}$ is semi-open, leads to cl(Ạ)-$\mathcal{U}$ ∈ Ĭ. So, the set Ạ⊊$\mathcal{X}$ is nameable Ĭ-*semi-g-open* (artlessly,Ĭ*sg-open*) whenever $\mathcal{X}$ -Ạ is Ĭ*sg-closed* set. The collection of all Ĭ*sg-closed* (respectively, Ĭ*sg-open*) sets in $(\mathcal{X},\mathcal{T},Ĭ)$ notate as Ĭ*sg*-C(Ẋ) (respectively, Ĭ*sg*-$O(\mathcal{X})$) . Every open set in $(\mathcal{X},\mathcal{T})$ is Ĭ*sg-open* in $(\mathcal{X},\mathcal{T},Ĭ)$[1]. And the reverse of this statement is incorrect [1].

The idea of ideals on a nonempty set was started by Kuratowski in 1933[2]. The ideal Ĭ on $\mathcal{X}$ where Ĭ = $\mathbb{P}(\mathcal{X})$ that is valid the preconditions *finite additivity* (Ạ *and* Ḅ ∈ Ĭ *implies* Ạ ∪ Ḅ ∈ Ĭ) and *heredity* (Ạ ⊆ Ḅ *and* Ḅ ∈ Ĭ *implies* Ạ ∈ Ĭ) [2].

In 1945, Vaidynathaswamy [3]. Was the first mathematician who uses the idea of ideal in science of topology, which used to create a new generalization of topological spaces, which is called ideal topological space and notate as $(\mathcal{X},\mathcal{T},Ĭ)$[4].

Since then, more authors are interested in this scope of the study. Many results have been achieved [5-8].

The concept of "*semi-open set*" was firstly offered by N.Levine [9]. A set Ạ in $(\mathcal{X},\mathcal{C})$ is called *semi-open* if Ạ⊊$cl(int$(Ạ)) [9].

Recently, the main goal of our work is to construct generalizations of the notion separation axioms in topological space by using the notion of Ĭ*sg-open* set.

2. Separation Axioms via Ǐsg-open Sets.

Here, we will present new type of separation axioms in ideal spaces via Ǐsg-$open$ set. Also, relations between these sorts are discussed. The following definition is for the research who did this research.

Definition 2.1. The triple $(\mathcal{X}, \mathcal{T}, $Ǐ$)$ is called Ǐ-$semi$-g-T_0-$space$ (shortly,Ǐsg-T_0-$space$), if for any element $ç_1 \neq ç_2$, there is an Ǐsg-$open$ set $\mathcal{U}$ containing only one of them.

Remark 2.2. If $(\mathcal{X}, \mathcal{T})$ is T_0-$space$ then $(\mathcal{X}, \mathcal{T}, $Ǐ$)$ Ǐsg-T_0-$space$.

Remark 2.3. The below phrases are rewards;

i. $\mathcal{X}$ is Ǐsg-T_0-$space$.

ii. For each element $ç_1 \neq ç_2$, there is an Ǐsg-$closed$ set $\mathcal{V}$ containing only one of them.

Definition 2.4. $(\mathcal{X}, \mathcal{C}, $Ǐ$)$ is called Ǐ-$semi$-g-T_1-$space$ (artlessly, Ǐsg-T_1-$space$), if for any elements $ç_1 \neq ç_2$, there are Ǐsg-$open$ sets, $\mathcal{V}$ whenever $(ç_1 \in \mathcal{U}$-$\mathcal{V})$, $(ç_2 \in \mathcal{V}$-$\mathcal{U})$.

Remark 2.5. If $(\mathcal{X}, \mathcal{T})$ is T_1-$space$ then $(\mathcal{X}, \mathcal{T}, $Ǐ$)$ Ǐsg-T_1-$space$.

Remark 2.6. $(\mathcal{X}, \mathcal{T}, $Ǐ$)$ is Ǐsg-T_0-$space$ whenever it is an Ǐsg-T_1-$space$.

The conclusions in Remark 2.6, is not reversible by example below.

Example 2.7. The Ǐsg-T_0-$space$ $(\mathcal{X}, \mathcal{T}, $Ǐ$)$ (where $\mathcal{X} = \{ç_1, ç_2, ç_3\}$, $\mathcal{C} = \{\mathcal{X}, \phi, \{ç_1\}, \{ç_2\}, \{ç_1, ç_2\}\}$ and Ǐ $= \{\phi\}$), is not Ǐsg-T_1-$space$.

Remark 2.8. The below phrases are rewards;

i. $\mathcal{X}$ is Ǐsg-T_1-$space$.

ii. For each elements $ç_1 \neq ç_2$, there are two Ǐsg-$closed$ sets Ạ and $\underset{.}{B}$ satisfy $(ç_1 \in$ Ạ $\cap\, \underset{.}{B}^c)$ and $(ç_2 \in \underset{.}{B} \cap$ Ạ$^c)$.

Remark 2.9. For $(\mathcal{X}, \mathcal{C}, $Ǐ$)$, if $\{ç\}$ is Ǐsg-$closed$ set $\forall ç \in \mathcal{X}$, then $\mathcal{X}$ is Ǐsg-T_1-$space$.

The conclusions in Remark 2.9, is not reversible by example below.

Example 2.10. Given $(\mathcal{X}, \mathcal{T}, $Ǐ$)$; $(\mathcal{X} = \{ç_1, ç_2, ç_3\}$, $\mathcal{C} = \{\phi, \mathcal{X}\}$ and Ǐ $= \{\mathcal{X}, \phi, \{ç_1\}\})$ is Ǐsg-T_1-$space$, while $\{ç_1\}$ is not Ǐsg-$closed\ set$.

Definition 2.11. $(\mathcal{X}, \mathcal{T}, $Ǐ$)$ is called Ǐ-$semi$-g-T_2-$space$ (artlessly, Ǐsg-T_2-$space$), if for any elements $ç_1 \neq ç_2$, there are Ǐsg-$open$ sets $\mathcal{V}$ and $\mathcal{U}$ satisfy $ç_1 \in \mathcal{U}$, $ç_2 \in \mathcal{V}$ and $\mathcal{V} \cap \mathcal{U} = \emptyset$.

Remark 2.12. If $(\mathcal{X}, \mathcal{T})$ is T_2-$space$, then $(\mathcal{X}, \mathcal{T}, $Ǐ$)$ is Ǐsg-T_2-$space$.

Remark 2.13. A space $(\mathcal{X}, \mathcal{T}, $Ǐ$)$ is Ǐsg-T_1-$space$ whenever it is an Ǐsg-T_2-$space$.

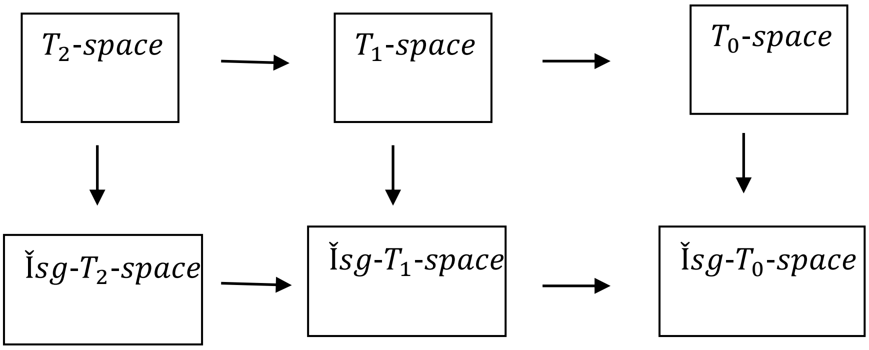
The conclusions in Remark 2.13, is not reversible by example below.

Example 2.14. A space $(\mathcal{X}, \mathcal{T}, \check{I})$ when $\mathcal{X} = \mathcal{N}$ the set of all natural numbers, $\mathcal{T} = \mathcal{T}_{cof}$ is the cofinite topology on $\mathcal{X}$ and Ǐ $= \{\phi\}$. Clearly, that $(\mathcal{X}, \mathcal{T}, $Ǐ$)$ is Ǐsg-T_1-$space$ which is not Ǐsg-T_2-$space$.

We have previously noted that $\mathcal{X}$ is Ǐsg-T_i-$space$ whenever it is T_i-$space$ $(\forall i = 0{,}1\ and\ 2)$. The opposite is not generally achieved by example below.

Example 2.15. $(\mathcal{X}, \mathcal{T}, $Ǐ$)$ is Ǐsg-T_i-$space$ $(\forall i = 0{,}1$ and 2), where $\mathcal{X} = \{ç_1, ç_2, ç_3\}$, $\mathcal{T} = \{\mathcal{X}, \phi\}$ and Ǐ is the class of all sets in Ẋ. But the space $(\mathcal{X}, \mathcal{T})$ is not T_i-$space$.

The following chart shows the relationships among the various types of notions of our previously mentioned.



3. Separation Axioms via Some Types of Functions

In this portion we will review some sorts of functions which presented by Abdel Karim and Nasir [1]. And study their impact on the notions of separated axioms.

Definition 3.1:[1]. $\wp$: (Ẋ, $\mathcal{T}$, Ĭ) → ($\mathcal{Y}$, $\mathfrak{J}$, ǰ) is;

i. *I-semi-g-open function*, notate as *Isgo-function*, if the precondition $\mathcal{U}$ is Ĭ*sg-open* in Ẋ, leads to $\wp(\mathcal{U})$ is ǰ*sg-open* in $\mathcal{Y}$.

ii. I^**-semi-g-open function*, notate as I^**sgo-function*, if the precondition $\mathcal{U}$ is open in Ẋ, leads to $\wp(\mathcal{U})$ is ǰ*sg-open* in $\mathcal{Y}$

iii. I^{**}*-semi-g-open function*, notate as I^{**}*sgo-function*, if the precondition $\mathcal{U}$ is Ĭ*sg-open* in Ẋ, leads to $\wp(\mathcal{U})$ is open in $\mathcal{Y}$.

Proposition 3.2: $\mathcal{X}$ is Ĭ*sg*-T_i-*space* ($\forall i = 0, 1$ and 2) and $\wp$ is a surjective *Isgo-function* from ($\mathcal{X}$, $\mathcal{T}$, Ĭ) to (Ẏ, $\mathfrak{J}$, ǰ) implies Ẏ is ǰ*sg*-T_i-*space*.

Proof: Follows from $\wp(\mathcal{U})$ is ǰ*sg-open* in Ẏ for all Ĭ*sg-open* set $\mathcal{U}$ in $\mathcal{X}$.

Proposition 3.3: $\mathcal{X}$ is T_i-*space* ($\forall i = 0,1$ and 2) and $\wp$ is a surjective I^**sgo-function* from ($\mathcal{X}$, $\mathcal{T}$, Ĭ) to ($\mathcal{Y}$, $\mathfrak{J}$, ǰ) implies $\mathcal{Y}$ is ǰ*sg*-T_i-*space*.

Proof: Follows from $\wp(\mathcal{U})$ is ǰ*sg-open* in Ẏ for all *open* set $\mathcal{U}$ in $\mathcal{X}$.

Proposition 3.4: $\mathcal{X}$ is Ĭ*sg*-T_i-*space* ($\forall i = 0,1$ and 2) and $\wp$ is a surjective I^{**}*sgo-function* from ($\mathcal{X}$, $\mathcal{T}$, Ĭ) to ($\mathcal{Y}$, $\mathfrak{J}$, ǰ) implies $\mathcal{Y}$ is T_i-*space*.

Proof: Follows from $\wp(\mathcal{U})$ is *open* in Ẏ for all Ĭ*sg-open* set $\mathcal{U}$ in $\mathcal{X}$.

Remark 3.5: If $\wp$ is bijective open function from ($\mathcal{X}$, $\mathcal{T}$) to ($\mathcal{Y}$, $\mathfrak{J}$) and Ẋ is T_i-*space* ($\forall i = 0,1$ and 2), then ($\mathcal{Y}$, $\mathfrak{J}$, ǰ) is ǰ*sg*-T_i-*space*, for any ideal ǰ on Ẏ.

Definition 3.6:[1]. The function $\wp$ from ($\mathcal{X}$, $\mathcal{T}$, Ĭ) to ($\mathcal{Y}$, $\mathfrak{J}$, ǰ) is;

i. *I-semi-g-continuous function*, notate as *Isg-continuous function*, if $\wp^{-1}(\mathcal{U}) \in$ Ĭ*sg*-$O(\mathcal{X})$ for every $\mathcal{U} \in \mathfrak{J}$.

ii. *Strongly-I-semi-g-continuous function*, notate as *strongly-Isg-continuous function*, if $\wp^{-1}(\mathcal{U}) \in \mathcal{T}$ for every $\mathcal{U} \in$ ǰ*sg*-$O(\mathcal{Y})$.

iii. *I-semi-g-irresolute function*, notate as *Isg-irresolute function*, if $\wp^{-1}(\mathcal{U}) \in$ Ĭ*sg*-$O(\mathcal{X})$ for every $\mathcal{U} \in$ ǰ*sg*-$O(\mathcal{Y})$.

Proposition 3.7: If $\mathcal{Y}$ is T_i-*space* ($\forall i = 0,1$ *and* 2) and $\wp$: ($\mathcal{X}$, $\mathcal{T}$, Ĭ) → ($\mathcal{Y}$, $\mathfrak{J}$, ǰ) is an injective *Isg-continuous function* then Ẋ is Ĭ*sg*-T_i-*space*.

Proof: Since $\wp^{-1}(\mathcal{U}) \in$ Ĭ*sg*-$O(\mathcal{X})$ for all $\mathcal{U} \in \mathfrak{J}$.

Corollary 3.8: If $\mathcal{Y}$ is T_i-$space$ ($\forall i = 0,1$ and 2) and $\mathcal{F}: (\dot{X}, \mathcal{T}, \check{I}) \rightarrow (\mathcal{Y}, \mathfrak{I}, \check{j})$ is an injective $continuous\ function$ then Ẏ is $\check{j}sg$-T_i-$space$.

Proof: Since, every continuous function is Isg-continuous [1]. And then Proposition 3.7 is applicable.

Proposition 3.9: If $\mathcal{Y}$ is $\check{j}sg$-T_i-$space$ ($\forall\ i = 0,1$ and 2) and $\mathcal{F}: (\mathcal{X}, \mathcal{T}, \check{I}) \rightarrow (\mathcal{Y}, \mathfrak{I}, \check{j})$ is an injective $strongly$-Isg-$continuous\ function$ then Ẋ is T_i-$space$.

Proof: Follows from, $\mathcal{F}^{-1}(\mathcal{U}) \in \mathcal{T}$ for all $\mathcal{U} \in \check{j}sg$-$O(\mathcal{Y})$.

Proposition 3.10: If $\mathcal{Y}$ is $\check{j}sg$-T_i-$space$ ($\forall\ i = 0,1,2$) and $\mathcal{F}: (\mathcal{X}, \mathcal{T}, \check{I}) \rightarrow (\mathcal{Y}, \mathfrak{I}, \check{j})$ is an injective Isg-$irresolute\ function$ then Ẋ is $\check{I}sg$-T_i-$space$.

Proof: Since $\mathcal{F}^{-1}(\mathcal{U}) \in \check{I}sg$-$O$(Ẋ) for all $\mathcal{U} \in \check{j}sg$-$O(\mathcal{Y})$.

4. I- Semi- g- convergence sequence

Sequences are one of important topics in all branches of mathematics, especially mathematical analysis and topology. So the convergence is the important property for the sequence [10, 11].

In this paragraph, we will use the notion of $\check{I}sg$-$open$ set to create new class of convergence.

Definition 4.1. Given a space $(\mathcal{X}, \mathcal{T}, \check{I})$ where $\varsigma_0 \in \mathcal{X}$ and the sequence $(S_n)_{n \in \mathcal{N}}$ in $\mathcal{X}$. Then $(S_n)_{n \in \mathcal{N}}$ is called Ǐ-$semi$-g-$convergence$ to ς_0 (artlessly, $S_n \rightsquigarrow \varsigma_0$) if for every Ǐsg-open set $\mathcal{U}$ contained ς_0, $\exists \kappa \in N$ where $S_n \in \mathcal{U}\ \forall\ n \geq \kappa$.

A sequence $(S_n)_{n \in \mathcal{N}}$ is nameable Ǐ-$semi$-g-$divergence$ if it is not Ǐ-$semi$-g-$convergence$.

Proposition 4.2. If $(\mathcal{X}, \mathcal{T}, \check{I})$ is Ǐsg-T_2-$space$ then every Ǐ-$semi$-g-$convergence$ sequence in Ẋ has only one limit point.

Proof: If we consider $(S_n)_{n \in N}$ is a seq. in $\mathcal{X}$ and $S_n \rightsquigarrow \varsigma_1$ so $S_n \rightsquigarrow \varsigma_2$; $\varsigma_1 \neq \varsigma_2$ where $\varsigma_1, \varsigma_2 \in \mathcal{X}$. Since $\mathcal{X}$ is Ǐsg-T_2-$space$, then there are disjoint Ǐsg-open sets $\mathcal{U}$ and $\mathcal{V}$ such that $\varsigma_1 \in \mathcal{U}$ and $\varsigma_2 \in \mathcal{V}$. Now, since $S_n \rightsquigarrow \varsigma_1$ and $\varsigma_1 \in \mathcal{U}$ implies $\exists \kappa_1 \in \mathcal{N}$; $S_n \in \mathcal{U}\ \forall\ n \geq \kappa_1$. So, $S_n \rightsquigarrow \varsigma_2$ and $\varsigma_2 \in \mathcal{V}$ implies $\exists \kappa_2 \in \mathcal{N}$; $S_n \in \mathcal{V}\ \ \forall\ n \geq \kappa_2$. Leads to $\mathcal{U} \cap \mathcal{V} \neq \emptyset$, where that contradiction.

The precondition that a space $\mathcal{X}$ is Ǐsg-T_2-$space$ is very requisite to make Proposition 4.2, is valid.

Example 4.3. Consider $(\mathcal{X}, \mathcal{T}, \check{I})$ where $\mathcal{X} = \{\varsigma_1, \varsigma_2, \varsigma_3\}$, $\mathcal{T} = \{\phi, \mathcal{X}\}$ and $\check{I} = \{\phi\}$.

Obviously; the sequence $(S_n)_{n \in \mathcal{N}}$ in $\mathcal{X}$, where $S_n = \varsigma_1$ for all n, has more than one limit point; that $S_n \rightsquigarrow \varsigma_1$, $S_n \rightsquigarrow \varsigma_2$ and $S_n \rightsquigarrow \varsigma_3$.

Proposition 4.4. If a sequence $(S_n)_{n \in \mathcal{N}}$ is Ǐ-$semi$-g-$convergent$ to ς_0 in an ideal space Ẋ , then it is convergent to ς_0.

Proof. Since every open set in $(\mathcal{X}, \mathcal{T}, \check{I})$is Ǐsg-open, then the proof is over.

The meaningfulness in proposition 4.4, is not reversible, in general.

Example 4.5. For an ideal space $(\mathcal{X}, \mathcal{T}, \check{I})$, where $\mathcal{X} = \mathcal{N}$, $\mathcal{T} = \{\mathcal{X}, \phi\}$ and $\check{I} = \mathbb{P}(\mathcal{X})$. The sequence$(S_n)_{n \in N}$, where$S_n = n\ \forall n \in \mathcal{N}$, is convergent to $n = 1$ which is notǏ-$semi$-g-$convergence$.

Proposition 4.6. Let $\mathcal{F}$ be Isg-$irresolute$ function from $\mathcal{X}$ to $\mathcal{Y}$ and $(S_n)_{n \in \mathcal{N}}$ be a sequence in $\mathcal{X}$, $\mathcal{F}(S_n) \rightsquigarrow \mathcal{F}(\varsigma_0)$ in $\mathcal{Y}$ whenever $S_n \rightsquigarrow \varsigma_0$ in $\mathcal{X}$.

Proposition 4.7. Let $\cancel{P}$ be $Isg\text{-}continuous$ function from $\mathcal{X}$ to $\mathcal{Y}$ and $(S_n)_{n\in\mathcal{N}}$ be a sequence in Ẋ. $\cancel{P}(S_n)$ $is\ convergent\ to$ $\cancel{P}(\varsigma_0)$ in Ẏ whenever $S_n \rightsquigarrow \varsigma_0$ in $\mathcal{X}$.

Proposition 4.8. Let $\cancel{P}$ be $strongly\text{-}Isg\text{-}continuous$ function from $\mathcal{X}$ to $\mathcal{Y}$ and $(S_n)_{n\in\mathcal{N}}$ be a sequence in $\mathcal{X}$, $\cancel{P}(S_n) \rightsquigarrow \cancel{P}(\varsigma_0)$ in $\mathcal{Y}$ whenever S_n $is\ convergent\ to$ ς_0 in $\mathcal{X}$.

5. Conclusion

The separation axioms appeared in new types and generalizations by used *Ǐ-semi-g-openness*, in addition a new types of convergence in ideal spaces were studied such as Ǐ-$semi\text{-}g\text{-}convergence$, and the relationships between them was clarified by using several example.

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