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λ− Algebra with Some of Their Properties

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Abstract

The objective of this paper is, firstly, we study a new concept noted by λ−algebra and discuss the properties of this concept. Secondly, we introduce a new concept related to the λ−algebra such as smallest λ−algebra. Thirdly, we introduce the notion of the restriction of λ−algebra on a nonempty subset $\mathfrak{D}$ of $\mathfrak{P}$ and investigate some of its basic properties. Furthermore, we present the relationships between α− σ−field, monotone class, β− σ−field and λ−algebra. Finally, we introduce the concept of measure relative to the λ−algebra and prove that every measure relative to the λ− algebra is complete.

Keywords: σ−field, increasing sequence, α− σ−field, monotone class, β− σ−field.

1. Introduction

About forty seven year ago, Robert [1]. Studied the concept of σ−field, where a collection $\mathcal{K}$ is called σ−field of a set $\mathfrak{P}$ if $\mathfrak{P}\epsilon\mathcal{K}$ and $\mathcal{K}$ is closed under complementation and countable union. Many authors studied the concept of σ−field, for example see [2-4]. And [5]. The notion of increasing sequence and decreasing sequence studied by Robert, where $D_1, D_2, \ldots$ are subsets of a set $\mathfrak{P}$, if $D_1 \subset D_2 \subset \cdots$ and $\bigcup_{i=1}^{\infty} D_i = D$. Then we say that D_i increase toD; we write $D_i \uparrow D$. If $D_1 \supset D_2 \supset \cdots$ and $\bigcap_{i=1}^{\infty} D_i = D$, we say that D_i decrease toD; we write $D_i \downarrow D$ [1]. Zhenyuan and George in 2009 studied the concept of monotone class which represents the generalization of σ−field, where a collection $\mathcal{K}$ of subsets of a nonempty set $\mathfrak{P}$ is said to be monotone class iff whenever $D_1, D_2, \ldots \epsilon\mathcal{K}$ such that $D_i \uparrow D$, then $D\epsilon \mathcal{K}$ and if $D_i \downarrow D$, then $D\epsilon \mathcal{K}$ [6]. In 2019, Ibrahim and Hassan introduced some concepts such as α− σ−field and β− σ−field which represent the generalizations of σ−field, where a collection $\mathcal{K}$ is said to be α− σ−field iff $\Phi, \mathfrak{P}\epsilon \mathcal{K}$ and $\mathcal{K}$ is closed under countable union [7]. And a collection $\mathcal{K}$ is said to be β− σ−field if $\Phi, \mathfrak{P}\epsilon \mathcal{K}$ and $\mathcal{K}$ is closed under countable intersection [7]. Ibrahim and Hassan in 2019 also introduced the concept of δ−field as a stronger form of these concepts, where a collection $\mathcal{K}$ is said to δ−field iff $\Phi\epsilon \mathcal{K}$ and if $\Phi \neq A\epsilon\mathcal{K}$ and$A \subset B \subseteq \mathfrak{P}$, then $B\epsilon\mathcal{K}$and $\mathcal{K}$ is closed under countable intersection [8]. The concept of complete measure on σ−field was studied by Robert in 1972, but not necessarily that every measure defined on σ−field is complete. In this work, we prove that every measure defined on λ− algebra is complete.

The main aim of this paper is to introduce and study new concept such as λ− algebra as a stronger from of α− σ−field and monotone class. And we give basic properties and examples of this concept.

2. The main results:

Let $P(\mathfrak{P})$ denoted to the power set of a nonempty set $\mathfrak{P}$ and we start this section by the definition of λ− algebra.

Definition 1

A nonempty collection $\mathcal{K}$ of a set $\mathfrak{P}$, $\mathcal{K} \neq \{\mathfrak{P}\}$ is called λ− algebra or (λ− field) of a set $\mathfrak{P}$ if:

1- $\mathfrak{P} \epsilon \mathcal{K}$.
2- If $D \epsilon \mathcal{K}$ and $E \subset D \subset \mathfrak{P}$, then $E \epsilon \mathcal{K}$.
3- If $D_1, D_2, \dots \epsilon \mathcal{K}$, then $\bigcup_{i=1}^{\infty} D_i \epsilon \mathcal{K}$.

Definition 2

If $\mathcal{K}$ is a λ− algebra of a set $\mathfrak{P}$ .Then a pair $(\mathfrak{P}, \mathcal{K})$ is called measurable space relative to the λ− algebra $\mathcal{K}$ and the elements of $\mathcal{K}$ are called the measurable sets.

Example 3

Let $\mathfrak{P} = \{1,2,3,4\}$ and $\mathcal{K} = \{\Phi, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{1,2,4\}, \mathfrak{P}\}$. Then $(\mathfrak{P}, \mathcal{K})$ is measurable space relative to the λ− algebra $\mathcal{K}$.

Proposition 4

For any λ− algebra $\mathcal{K}$ of a set $\mathfrak{P}$, the following hold:

1- $\Phi \epsilon \mathcal{K}$
2- If $D_1, D_2, \dots, D_n \epsilon \mathcal{K}$, then $\bigcup_{i=1}^{n} D_i \epsilon \mathcal{K}$.
3- If $D_1, D_2, \dots \epsilon \mathcal{K}$, then $\bigcap_{i=1}^{\infty} D_i \epsilon \mathcal{K}$.
4- If $D_1, D_2, \dots, D_n \epsilon \mathcal{K}$, then $\bigcap_{i=1}^{n} D_i \epsilon \mathcal{K}$.

Proof

The proof follows from definition of λ− algebra.

Lemma 5

Let $\{\mathcal{K}_\alpha\}_{\alpha \in I}$ be a collection of λ− algebra on $\mathfrak{P}$. Then $\bigcap_{\alpha \in I} \mathcal{K}_\alpha$ is a λ− algebra on $\mathfrak{P}$.

Proof

Since $\mathcal{K}_\alpha$ is λ− algebra $\forall \alpha \in I$, then $\mathfrak{P} \epsilon \mathcal{K}_\alpha \ \forall \alpha \in I$, hence $\mathcal{K}_\alpha \neq \Phi \ \forall \alpha \in I$ and $\bigcap_{\alpha \in I} \mathcal{K}_\alpha \neq \Phi$, therefore $\mathfrak{P} \epsilon \bigcap_{\alpha \in I} \mathcal{K}_\alpha$. Let $D \epsilon \bigcap_{\alpha \in I} \mathcal{K}_\alpha$ and $E \subset D \subset \mathfrak{P}$, then $D \epsilon \mathcal{K}_\alpha \ \forall \alpha \in I$, but $\mathcal{K}_\alpha$ is λ− algebra $\forall \alpha \in I$ and $E \subset D$. So, we get $E \epsilon \mathcal{K}_\alpha \ \forall \alpha \in I$, hence $E \epsilon \bigcap_{\alpha \in I} \mathcal{K}_\alpha$. Let $D_1, D_2, \dots \epsilon \bigcap_{\alpha \in I} \mathcal{K}_\alpha$.Then, $D_1, D_2, \dots \epsilon \mathcal{K}_\alpha$, $\forall \alpha \in I$, but $\mathcal{K}_\alpha$ is λ− algebra $\forall \alpha \in I$ which implies that $\bigcup_{n=1}^{\infty} D_n \ \epsilon \mathcal{K}_\alpha$, $\forall \alpha \in I$, hence $\bigcup_{n=1}^{\infty} D_n \epsilon \bigcap_{\alpha \in I} \mathcal{K}_\alpha$. Therefore, $\bigcap_{\alpha \in I} \mathcal{K}_\alpha$ is a λ− algebra.

Definition 6

Let $\mathcal{J} \subseteq P(\mathfrak{P})$. Then the intersection of all $\lambda-$ algebra of $\mathfrak{P}$ which includes $\mathcal{J}$ is called the $\lambda-$ algebra generated by $\mathcal{J}$ and denoted by $\lambda(\mathcal{J})$, that is,
$\lambda(\mathcal{J}) = \bigcap\{\mathcal{K}_\alpha : \mathcal{K}_\alpha$ is a $\lambda-$algebra of $\mathfrak{P}$ and $\subseteq \mathcal{K}_\alpha$, $\forall\alpha \in I\}$.

Proposition 7

Let $\mathcal{J} \subseteq P(\mathfrak{P})$. Then $\lambda(\mathcal{J})$ is the smallest $\lambda-$ algebra of $\mathfrak{P}$ which includes $\mathcal{J}$.

Proof

Since $\lambda(\mathcal{J})=\bigcap\{\mathcal{K}_\alpha : \mathcal{K}_\alpha$ is a $\lambda-$algebra of $\mathfrak{P}$ and $\mathcal{J} \subseteq \mathcal{K}_\alpha, \forall\alpha \in I\}$. Then $\lambda(\mathcal{J})$ is $\lambda-$ algebra of $\mathfrak{P}$ by Lemma 5. To prove $\lambda(\mathcal{J}) \supseteq \mathcal{J}$, let each of $\mathcal{K}_\alpha$ is a $\lambda-$ algebra of $\mathfrak{P}$ and $\mathcal{J} \subseteq \mathcal{K}_\alpha, \forall\alpha \in I$. Then $\mathcal{J} \subseteq \bigcap_{\alpha\in I}\mathcal{K}_\alpha$, therefore $\mathcal{J} \subseteq \lambda(\mathcal{J})$. Now, let $\mathcal{K}^*$ is a $\lambda-$ algebra of $\mathfrak{P}$ such that $\mathcal{K}^* \supseteq \mathcal{J}$. Then $\bigcap\{\mathcal{K}_\alpha : \mathcal{K}_\alpha$ is a $\lambda-$algebra of $\mathfrak{P}$ and $\mathcal{J} \subseteq \mathcal{K}_\alpha, \forall\alpha \in I\} \subseteq \mathcal{K}^*$, hence $\lambda(\mathcal{J}) \subseteq \mathcal{K}^*$. Therefore, $\lambda(\mathcal{J})$ is the smallest $\lambda-$ algebra of $\mathfrak{P}$ which includes $\mathcal{J}$.

If we take Example 3 and if we assume $\mathcal{J} =\{\{1\},\{2\}\}$, then $\lambda(\mathcal{J})=\{\Phi,\{1\},\{2\},\{1,2\}, \mathfrak{P}\}$ is the smallest $\lambda-$ algebra of a set $\mathfrak{P}$ which includes $\mathcal{J}$.

Theorem 8

Let $\mathcal{J} \subseteq P(\mathfrak{P})$. Then $(\mathfrak{P}, \mathcal{J})$ is measurable space relative to the $\lambda-$ algebra $\mathcal{J}$.
if and only if $\mathcal{J} = \lambda(\mathcal{J})$.

Proof

Suppose that $(\mathfrak{P}, \mathcal{J})$ is (a) measurable space relative to the $\lambda-$ algebra $\mathcal{J}$. From Proposition 7, we have $\lambda(\mathcal{J})$ is the smallest $\lambda-$algebra of a set $\mathfrak{P}$ which includes $\mathcal{J}$ implies that $\mathcal{J} \subseteq \lambda(\mathcal{J})$. By hypothesis, we have $\mathcal{J}$ is a $\lambda-$ algebra of a set $\mathfrak{P}$, but $\mathcal{J} \subseteq \mathcal{J}$ and $\lambda(\mathcal{J})$ is the smallest $\lambda-$algebra of a set $\mathfrak{P}$ which includes $\mathcal{J}$, then $\lambda(\mathcal{J}) \subseteq \mathcal{J}$, hence $\mathcal{J} = \lambda(\mathcal{J})$.
Conversely) Let $\mathcal{J} \subseteq P(\mathfrak{P})$ and let $\mathcal{J} = \lambda(\mathcal{J})$. Since $\lambda(\mathcal{J})$ is a $\lambda-$ algebra of a set $\mathfrak{P}$, then $\mathcal{J}$ is $\lambda-$ algebra of a set $\mathfrak{P}$.

If we take Example 3 and if we assume $\mathcal{J} =\{\Phi,\{1\},\mathfrak{P}\}$, then we conclude that $\lambda(\mathcal{J}) = \mathcal{J}$.

Now, we introduce the notion of restriction and study the basic properties of this notion.

Definition 9

Let $\mathcal{K} \subseteq P(\mathfrak{P})$ and $\Phi \neq \mathfrak{D} \subseteq \mathfrak{P}$. Then, the restriction of $\mathcal{K}$ over the set $\mathfrak{D}$ is denoted by $\mathcal{K}|_{\mathfrak{D}}$ and defined as follows:
$\mathcal{K}|_{\mathfrak{D}} = \{B: B=E\cap\mathfrak{D}$, for some $E\in \mathcal{K}\}$.

Proposition10

Let $(\mathfrak{P}, \mathcal{K})$ is measurable space relative to the $\lambda-$ algebra $\mathcal{K}$ and $\Phi \neq \mathfrak{D} \subseteq \mathfrak{P}$. Then $\mathcal{K}|_{\mathfrak{D}} = \{E \subseteq \mathfrak{D}: E\in \mathcal{K}\}$.

Proof

Let $B\in \mathcal{K}|_{\mathfrak{D}}$. Then $B=E\cap\mathfrak{D}$, for some $E\in\mathcal{K}$. Since $E\cap\mathfrak{D} \subseteq E$ and $\mathcal{K}$ is $\lambda-$algebra of a set $\mathfrak{P}$, then $E\cap\mathfrak{D}\in\mathcal{K}$, hence $B\in \mathcal{K}$. Since, $E\cap\mathfrak{D} \subseteq \mathfrak{D}$, then $B\subseteq \mathfrak{D}$. Therefore $B\in\{E \subseteq \mathfrak{D}: E\in \mathcal{K}\}$ and $\mathcal{K}|_{\mathfrak{D}} \subseteq\{A \subseteq \mathfrak{D}: A\in \mathcal{K}\}$. Let $C\in\{E \subseteq \mathfrak{D} : E\in \mathcal{K}\}$. Then, $C \subseteq \mathfrak{D}$, and $C \in \mathcal{K}$, hence, C=C$\cap\mathfrak{D}$, but C$\in\mathcal{K}$, then C$\in\mathcal{K}|_{\mathfrak{D}}$ which implies that$\{E\subseteq\mathfrak{D}$:E$\in\mathcal{K}\}\subseteq\mathcal{K}|_{\mathfrak{D}}$, therefore $\mathcal{K}|_{\mathfrak{D}}=\{$A $\subseteq\mathfrak{D}$:A$\in\mathcal{K}\}$.

Corollary 11

Let $(\mathfrak{P},\mathcal{K})$ is measurable space relative to the λ− algebra $\mathcal{K}$ and $\Phi\neq\mathfrak{D}\subseteq\mathfrak{P}$. Then $\mathcal{K}|_{\mathfrak{D}}\subseteq\mathcal{K}$.

Proof

The result follows from Proposition10

Proposition 12

Let $(\mathfrak{P},\mathcal{K})$ is measurable space relative to the λ− algebra $\mathcal{K}$,and $\neq\mathfrak{D}\subseteq\mathfrak{P}$. Then $(\mathfrak{D},\mathcal{K}|_{\mathfrak{D}})$ is measurable space relative to the λ− algebra $\mathcal{K}_{\mathfrak{D}}$

Proof

Since $(\mathfrak{P},\mathcal{K})$ is measurable space relative to theλ− algebra $\mathcal{K}$, then $\mathfrak{P}\in\mathcal{K}$. Since $\subseteq\mathfrak{P}$, then $\mathfrak{D}=\mathfrak{P}\cap\mathfrak{D}$ and $\mathfrak{D}\in\mathcal{K}|_{\mathfrak{D}}$. Let B$\in\mathcal{K}|_{\mathfrak{D}}$and F $\subset$ B $\subset\mathfrak{D}$. Then by Corollary 11, we get B$\in\mathcal{K}$. But F $\subset$ B $\subset\mathfrak{D}\subset\mathfrak{P}$ and $(\mathfrak{P},\mathcal{K})$ is measurable space relative to the λ− algebra $\mathcal{K}$, then F$\in\mathcal{K}$. Now, F $\subset\mathfrak{D}$, and F$\in\mathcal{K}$, then by Proposition 10, we have F$\in\mathcal{K}|_{\mathfrak{D}}$. Let $B_1,B_2,\ldots\in\mathcal{K}|_{\mathfrak{D}}$. Then there exist $E_1,E_2,\ldots\in\mathcal{K}$ such that $B_i=E_i\cap\mathfrak{D}$ where i=1,2,…, hence $\bigcup_{i=1}^{\infty}B_i=\bigcup_{i=1}^{\infty}(E_i\cap\mathfrak{D})=(\bigcup_{i=1}^{\infty}E_i)\cap\mathfrak{D}$. But $(\mathfrak{P},\mathcal{K})$ is measurable space relative to the λ− algebra $\mathcal{K}$ and $E_1,E_2,\ldots\in\mathcal{K}$, then, $\bigcup_{i=1}^{\infty}E_i\in\mathcal{K}$. Hence, $\bigcup_{i=1}^{\infty}B_i\in\mathcal{K}|_{\mathfrak{D}}$. Therefore, $(\mathfrak{D},\mathcal{K}|_{\mathfrak{D}})$ is measurable space relative to the λ− algebra $\mathcal{K}|_{\mathfrak{D}}$**.**

Example 13

Let $\mathfrak{P}=\{1,2,3,4,5\}$ and $\mathcal{K}=\{\Phi,\{1\},\{3\},\{5\},\{1,3\},\{1,5\},\{3,5\},\{1,3,5\},\mathfrak{P}\}$. Then $(\mathfrak{P},\mathcal{K})$ is measurable space relative to the λ− algebra $\mathcal{K}$. If $\mathfrak{D}=\{1,2,4\}$, then $\mathcal{K}|_{\mathfrak{D}}=\{\Phi,\{1\},\mathfrak{D}\}$, hence $(\mathfrak{D},\mathcal{K}|_{\mathfrak{D}})$ is measurable space relative to the λ− algebra $\mathcal{K}|_{\mathfrak{D}}$ and $\mathcal{K}|_{\mathfrak{D}}\subseteq\mathcal{K}$.

Proposition 14

Let $\mathcal{J}\subseteq P(\mathfrak{P})$ and $\Phi\neq\mathfrak{D}\subseteq\mathfrak{P}$. If $\mathcal{K}$ is a λ −algebra of $\mathfrak{P}$ which includes $\mathcal{J}$, then $\lambda(\mathcal{J})|_{\mathfrak{D}}$ is a λ− algebra of a set $\mathfrak{D}$.

Proof

The result follows from Proposition 7 and Proposition 12.

Proposition 15

Let $\mathcal{J}\subseteq P(\mathfrak{P})$ and $\Phi\neq\mathfrak{D}\subseteq\mathfrak{P}$ and $\mathcal{J}|\mathfrak{D}$ is the restriction of $\mathcal{J}$ over the set $\mathfrak{D}$. Then $\lambda(\mathcal{J}|\mathfrak{D})$ is the smallest λ− algebra of a set$\mathfrak{D}$, which includes $\mathcal{J}|\mathfrak{D}$, where $\lambda(\mathcal{J}|_{\mathfrak{D}})=\bigcap\{\mathcal{K}_i|_{\mathfrak{D}}:\mathcal{K}_i|_{\mathfrak{D}}$ is a λ -algebra of $\mathfrak{D}$, and $\mathcal{K}_i|_{\mathfrak{D}}\supseteq\mathcal{J}|_{\mathfrak{D}},\forall i\in I\}$.

Proof

From Lemma 5, we get $\lambda(\mathcal{J}|_{\mathfrak{D}})$ is a λ− algebra of a set$\mathfrak{D}$. To prove that $\lambda(\mathcal{J}|_{\mathfrak{D}})\supseteq\mathcal{J}|_{\mathfrak{D}}$, suppose that each of $\mathcal{K}_i|_{\mathfrak{D}}$ is a λ -algebra of a set $\mathfrak{D}$ and$\mathcal{K}_i|_{\mathfrak{D}}\supseteq\mathcal{J}|_{\mathfrak{D}},\forall i\in I$, then $\mathcal{J}|_{\mathfrak{D}}\subseteq\bigcap_{i\in I}\mathcal{K}_i|_{\mathfrak{D}}$, hence $\mathcal{J}|_{\mathfrak{D}}\subseteq\lambda(\mathcal{J}|_{\mathfrak{D}})$. Now, let $\mathcal{K}^*|_{\mathfrak{D}}$ is a λ− algebra of a set $\mathfrak{D}$ such that $\mathcal{K}^*|_{\mathfrak{D}}\supseteq\mathcal{J}|_{\mathfrak{D}}$. Then $\mathcal{K}^*|_{\mathfrak{D}}\supseteq\lambda(\mathcal{J}|_{\mathfrak{D}})$. Therefore, $\lambda(\mathcal{J}|_{\mathfrak{D}})$ is the smallest λ− algebra of a set $\mathfrak{D}$ includes$\mathcal{J}|_{\mathfrak{D}}$.

Proposition 16

Let $\mathcal{J} \subseteq P(\mathfrak{P})$ and $\Phi \neq \mathfrak{D} \subseteq \mathfrak{P}$, define the collection $\mathcal{K}$ as:
$\mathcal{K} = \{E \subseteq \mathfrak{P}: (E \cap \mathfrak{D}) \in \lambda(\mathcal{J}|_{\mathfrak{D}})\}$. Then $(\mathfrak{P}, \mathcal{K})$ is measurable space relative to the $\lambda-$ algebra $\mathcal{K}$.

Proof

Since $\lambda(\mathcal{J}|_{\mathfrak{D}})$ } is a $\lambda-$ algebra of a set $\mathfrak{D}$, then $\Phi, \mathfrak{D} \in \lambda(\mathcal{J}|_{\mathfrak{D}})$. Since $\mathfrak{D} \subseteq \mathfrak{P}$, then $\mathfrak{D} = \mathfrak{P} \cap \mathfrak{D}$ and $\mathfrak{P} \in \mathcal{K}$. Let $E \in \mathcal{K}$ and $F \subset E \subset \mathfrak{P}$. Then, $(E \cap \mathfrak{D}) \in \lambda(\mathcal{J}|_{\mathfrak{D}})$. Since, $F \subset E$, then $(F \cap \mathfrak{D}) \subset (E \cap \mathfrak{D})$. But $\lambda(\mathcal{J}|_{\mathfrak{D}})$ is a $\lambda-$ algebra of a set $\mathfrak{D}$, which implies that $(F \cap \mathfrak{D}) \in \lambda(\mathcal{J}|_{\mathfrak{D}})$ and $F \in \mathcal{K}$. Let $E_1, E_2, \ldots \in \mathcal{K}$. Then $(E_i \cap \mathfrak{D}) \in \lambda(\mathcal{J}|_{\mathfrak{D}})$, for all i=1,2,…, hence $\bigcup_{i=1}^{\infty}(E_i \cap \mathfrak{D}) \in \lambda(\mathcal{J}|_{\mathfrak{D}})$ and $((\bigcup_{i=1}^{\infty} E_i) \cap \mathfrak{D}) \in \lambda(\mathcal{J}|_{\mathfrak{D}})$ implies that $\bigcup_{i=1}^{\infty} E_i \in \mathcal{K}$. Therefore $\mathcal{K}$ is $\lambda-$algebra of a set $\mathfrak{P}$.

Theorem 17

Let $\mathcal{J} \subseteq P(\mathfrak{P})$ and $\Phi \neq \mathfrak{D} \subseteq \mathfrak{P}$. Then $\lambda(\mathcal{J}|_{\mathfrak{D}}) = \lambda(\mathcal{J})|_{\mathfrak{D}}$.

Proof

Let $B \in \mathcal{J}|_{\mathfrak{D}}$, then $B = E \cap \mathfrak{D}$, for some $E \in \mathcal{J}$. But $\mathcal{J} \subseteq \lambda(\mathcal{J})$, then $E \in \lambda(\mathcal{J})$, thus $B \in \lambda(\mathcal{J})|_{\mathfrak{D}}$, hence $\mathcal{J}|_{\mathfrak{D}} \subseteq \lambda(\mathcal{J})|_{\mathfrak{D}}$, but $\lambda(\mathcal{J}|_{\mathfrak{D}})$ is smallest λ –algebra of a set $\mathfrak{D}$, which include $\mathcal{J}|_{\mathfrak{D}}$ and $\lambda(\mathcal{J})|_{\mathfrak{D}}$ is a $\lambda-$ algebra of a set $\mathfrak{D}$ which include $\mathcal{J}|_{\mathfrak{D}}$, then $\lambda(\mathcal{J}|_{\mathfrak{D}}) \subseteq \lambda(\mathcal{J})|_{\mathfrak{D}}$. Now, define collection $\mathcal{K}$ as: $\mathcal{K} = \{E \subseteq \mathfrak{P} : E \cap \mathfrak{D} \in \lambda(\mathcal{J}|_{\mathfrak{D}})\}$, then from Proposition 16, we obtain $\mathcal{K}$ is a $\lambda-$ algebra of a set $\mathfrak{P}$. Let $C \in \mathcal{J}$, then $(C \cap \mathfrak{D}) \in \mathcal{J}|_{\mathfrak{D}}$, but $\mathcal{J}|_{\mathfrak{D}} \subseteq \lambda(\mathcal{J}|_{\mathfrak{D}})$ implies that $(C \cap \mathfrak{D}) \in \lambda(\mathcal{J}|_{\mathfrak{D}})$, hence $C \in \mathcal{K}$ and $\mathcal{J} \subseteq \mathcal{K}$. Let $B \in \lambda(\mathcal{J})|_{\mathfrak{D}}$, then $B = F \cap \mathfrak{D}$, for some $F \in \lambda(\mathcal{J})$. But $\lambda(\mathcal{J}) \subseteq \mathcal{K}$, then $F \in \mathcal{K}$, hence $B \in \lambda(\mathcal{J}|_{\mathfrak{D}})$ and $\lambda(\mathcal{J})|_{\mathfrak{D}} \subseteq \lambda(\mathcal{J}|_{\mathfrak{D}})$, consequently $\lambda(\mathcal{J}|_{\mathfrak{D}}) = \lambda(\mathcal{J})|_{\mathfrak{D}}$.

We end this section by introduce the relationships between $\alpha-\sigma-$field, monotone class, $\beta-\sigma-$field and $\lambda-$ algebra.

Proposition 18

Every $\lambda-$ algebra is a $\alpha-\sigma-$field.

Proof

Let $\mathcal{K}$ be a $\lambda-$ algebra of a set $\mathfrak{P}$. Then by definition of $\lambda-$ algebra, we have $\Phi, \mathfrak{P} \in \mathcal{K}$. Let $D_1, D_2, \ldots \in \mathcal{K}$. Since $\mathcal{K}$ is a $\lambda-$ algebra, then by definition of $\mathcal{K}$, we have $\bigcup_{i=1}^{\infty} D_i \in \mathcal{K}$. Therefore $\mathcal{K}$ is a $\alpha-\sigma-$field.

In general, the converse of above proposition is not true. For example, if $\mathfrak{P} = \{1,2,3\}$ and $\mathcal{K} = \{\Phi, \{1\}, \{1,3\}, \mathfrak{P}\}$, then $\mathcal{K}$ is $\alpha-\sigma-$ field but not $\lambda-$ algebra, because $\{1,3\} \in \mathcal{K}$ and $\{3\} \subset \{1,3\}$, but $\{3\} \notin \mathcal{K}$.

Proposition 19

Every $\lambda-$ algebra is a $\beta-\sigma-$field.

Proof

The proof follows from Proposition 4 and definition of $\lambda-$ algebra.

In general, the converse of above proposition is not true as shown in following example.

Example 20

Let $\mathfrak{P}=\{1,2,3,4\}$ and $\mathcal{K}=\{\Phi,\{1\},\{1,3,4\},\{3,4\},\mathfrak{P}\}$. Then, $\mathcal{K}$ is $\beta-\sigma-$ field but not $\lambda-$ algebra, because $\{1,3,4\}\in\mathcal{K}$ and $\{3,4\}\subset\{1,3,4\}$, but $\{3,4\}\notin\mathcal{K}$.

Proposition 21

Every $\lambda-$ algebra is a monotone class.

Proof

Let $\mathcal{K}$ be a $\lambda-$ algebra of a set $\mathfrak{P}$ and $D_1, D_2, \ldots \in \mathcal{K}$ such that $D_i \uparrow D$. Then $\bigcup_{i=1}^{\infty} D_i = D$ Since $\mathcal{K}$ is a $\lambda-$ algebra, then by definition of $\mathcal{K}$, we have $\bigcup_{i=1}^{\infty} D_i \in \mathcal{K}$ which implies that $D\in\mathcal{K}$. Let $D_1, D_2, \ldots \in \mathcal{K}$ such that $D_i \downarrow D$. Then, $\bigcap_{i=1}^{\infty} D_i = D$, but $\mathcal{K}$ is a $\lambda-$ algebra, implies that $\bigcap_{i=1}^{\infty} D_i \in\mathcal{K}$ and $D\in\mathcal{K}$. Hence $\mathcal{K}$ is a monotone class.

In general, the converse of above proposition is not true. For example, if $\mathfrak{P}=\{1,2,3\}$ and $\mathbb{M}=\{\Phi,\{1\},\{1,2\}\}$, then $\mathbb{M}$ is a monotone class, but not $\lambda-$ algebra, because $\{1,2\}\in\mathbb{M}$ and $\{2\}\subset\{1,2\}$, but $\{2\}\notin\mathbb{M}$.

Definition 22 [6]

Let $\mathcal{J}\subseteq P(\mathfrak{P})$. Then the intersection of all monotone classes of $\mathfrak{P}$ which include $\mathcal{J}$ is called the monotone class generated by $\mathcal{J}$ and denoted by $\mathbb{M}(\mathcal{J})$, that is, $\mathbb{M}(\mathcal{J}) = \bigcap\{\mathbb{M}_i : \mathbb{M}_i$ is a monotone class of $\mathfrak{P}$ and $\mathcal{J}\subseteq\mathbb{M}_i$, $\forall i \in I\}$.

Lemma 23 [6]

Let $\{\mathbb{M}_i\}_{i\in I}$ be a collection of monotone classes on $\mathfrak{P}$. Then $\bigcap_{i\in I}\mathbb{M}_i$ is a monotone class on $\mathfrak{P}$.

Proposition 24 [6]

Let $\mathcal{J}\subseteq P(\mathfrak{P})$. Then $\mathbb{M}(\mathcal{J})$ is the smallest monotone class of $\mathfrak{P}$ which includes $\mathcal{J}$.

Theorem 25

Let $\mathcal{J}\subseteq P(\mathfrak{P})$. Then $\mathbb{M}(\mathcal{J})\subseteq\lambda(\mathcal{J})$.

Proof

Let $\mathcal{J}\subseteq P(\mathfrak{P})$. Then by Proposition 7, we have $\lambda(\mathcal{J})$ is a $\lambda-$ algebra of $\mathfrak{P}$ which includes $\mathcal{J}$. From Proposition 21, we have, every $\lambda-$ algebra is a monotone class, implies that $\lambda(\mathcal{J})$ is a monotone class which includes $\mathcal{J}$. But $\mathbb{M}(\mathcal{J})$ is the smallest monotone class which includes $\mathcal{J}$ by Proposition 24, then $\mathbb{M}(\mathcal{J})\subseteq\lambda(\mathcal{J})$.

3. Measure Defined on $\boldsymbol{\lambda}-$ algebra

Our aim in this section is to prove that any measure defined on $\lambda-$ algebra is complete. We begin with the notions of measure on $\lambda-$ algebra.

Definition 26

Let $(\mathfrak{P},\mathcal{K})$ is measurable space relative to the $\lambda-$ algebra $\mathcal{K}$. Then, a set function $\mathfrak{M}$, $\mathfrak{M}:\mathcal{K}\to[0,\infty]$ is called measure relative to the $\lambda-$ algebra $\mathcal{K}$ if whenever $D_1, D_2, \ldots$ form a finite or countably infinite collection of disjoint sets in $\mathcal{K}$, we have $\mathfrak{M}(\bigcup_{n=1}^{\infty} D_n) = \sum_{n=1}^{\infty} \mathfrak{M}(D_n)$ and $\mathfrak{M}(\Phi) = 0$.

Example 27

Let $\mathfrak{P}=\{1,2,3\}$ and $\mathcal{K}=\{\Phi,\{1\},\{3\},\{1,3\},\mathfrak{P}\}$. Then $(\mathfrak{P},\mathcal{K})$ is measurable space relative to the λ− algebra $\mathcal{K}$. If we define a set function $\mathfrak{M}:\mathcal{K}\to[0,\infty]$ by

$$\mathfrak{M}(D)=\begin{cases} o & ;\ if\ D=\Phi \\ \frac{1}{2} & ;\ if\ D=\{1\}\ or\ \{3\} \\ 1 & ;other\ wise \end{cases}$$

Then $\mathfrak{M}$ is a measure relative to the λ− algebra $\mathcal{K}$.

Definition 28

A measure space relative to the λ− algebra $\mathcal{K}$ is a triple $(\mathfrak{P},\mathcal{K},\mathfrak{M})$ where $(\mathfrak{P},\mathcal{K})$ is measurable space relative to the λ− algebra $\mathcal{K}$ and $\mathfrak{M}$ is a measure relative to the λ− algebra $\mathcal{K}$.

In the following Theorem, we use mathematical induction to prove that the linear combination of measure relative to the λ− algebra $\mathcal{K}$ is also measure relative to the λ− algebra $\mathcal{K}$.

Theorem 29

Let $(\mathfrak{P},\mathcal{K},\mathfrak{M}_j)$ be a measure space relative to the λ− algebra $\mathcal{K}$ and $c_j\in[0,\infty)$ for all $j=1,2,\dots,k$. If a set function $\sum_{j=1}^{k}c_j\mathfrak{M}_j:\wp\to[0,\infty]$ is defined by:
$(\sum_{j=1}^{k}c_j\mathfrak{M}_j)(D)=\sum_{j=1}^{k}c_j\,.\,\mathfrak{M}_j(D)\ \forall D\epsilon\wp$, then $(\mathfrak{P},\mathcal{K},\sum_{j=1}^{k}c_j\mathfrak{M}_j)$ is measure space relative to the λ− algebra $\mathcal{K}$.

Proof

If $k=2$, then $(c_1\mathfrak{M}_1+c_2\mathfrak{M}_2)(\Phi)=c_1\,.\,\mathfrak{M}_1(\Phi)+c_2\,.\,\mathfrak{M}_2(\Phi)$
$$=c_1\,.0+c_2\,.0=0$$
Let $D_1,D_2,\dots$ are disjoint sets in $\mathcal{K}$. Since $\mathfrak{M}_j$ is measure relative to the λ− algebra $\mathcal{K}$, $j=1,2$

Then, $\mathfrak{M}_j(\bigcup_{n=1}^{\infty}D_n)=\sum_{n=1}^{\infty}\mathfrak{M}_j(D_n)$. So, we have

$$\begin{aligned}(c_1\mathfrak{M}_1+c_2\mathfrak{M}_2)(\textstyle\bigcup_{n=1}^{\infty}D_n) &= c_1\,.\,\mathfrak{M}_1(\textstyle\bigcup_{n=1}^{\infty}D_n)+c_2\,.\,\mathfrak{M}_2(\textstyle\bigcup_{n=1}^{\infty}D_n)\\ &= c_1\,.\,\textstyle\sum_{n=1}^{\infty}\mathfrak{M}_1(D_n)+c_2\,.\,\textstyle\sum_{n=1}^{\infty}\mathfrak{M}_2(D_n)\\ &= \textstyle\sum_{n=1}^{\infty}c_1\,.\,\mathfrak{M}_1(D_n)+\textstyle\sum_{n=1}^{\infty}c_2\,.\,\mathfrak{M}_2(D_n)\\ &= \textstyle\sum_{n=1}^{\infty}[\,c_1\,.\,\mathfrak{M}_1(D_n)+c_2\,.\,\mathfrak{M}_2(D_n)]\\ &= \textstyle\sum_{n=1}^{\infty}(\,c_1\mathfrak{M}_1+c_2\mathfrak{M}_2)(D_n)\end{aligned}$$

Hence, $(\mathfrak{P},\mathcal{K},(c_1\mathfrak{M}_1+c_2\mathfrak{M}_2))$ is measure space relative to the λ− algebra $\mathcal{K}$.

Now, we assume that $(\mathfrak{P},\mathcal{K},\sum_{j=1}^{k}c_j\mathfrak{M}_j)$ is measure space relative to the λ− algebra $\mathcal{K}$, when $k=\mathrm{m}$ and we prove this fact when $k=\mathrm{m}+1$. Let $(\mathfrak{P},\mathcal{K},\mathfrak{M}_j)$ be a measure space relative to the λ− algebra $\mathcal{K}$ and $c_j\in[0,\infty)$ for all $j=1,2,\dots,m,m+1$. Then

$$\begin{aligned}(\textstyle\sum_{j=1}^{m+1}c_j\mathfrak{M}_j)(\Phi) &= (\textstyle\sum_{j=1}^{m}c_j\mathfrak{M}_j+c_{m+1}\mathfrak{M}_{m+1})(\Phi)\\ &= \textstyle\sum_{j=1}^{m}c_j\,.\,\mathfrak{M}_j(\Phi)+c_{m+1}\,.\,\mathfrak{M}_{m+1}(\Phi)\\ &= 0\ \text{ since, } \mathfrak{M}_j \text{ is measure relative to the λ− algebra } \mathcal{K}.\end{aligned}$$

Let $D_1,D_2,\dots$ are disjoint sets in $\mathcal{K}$. Since $(\mathfrak{P},\mathcal{K},\sum_{j=1}^{m}c_j\mathfrak{M}_j)$ is measure space relative to the λ− algebra $\mathcal{K}$, then $\sum_{j=1}^{m}c_j\mathfrak{M}_j(\bigcup_{n=1}^{\infty}D_n)=\sum_{n=1}^{\infty}[\sum_{j=1}^{m}c_j\mathfrak{M}_j](D_n)$. So, we have

$$(\sum_{j=1}^{m+1} c_j\mathfrak{M}_j)(\bigcup_{n=1}^{\infty} D_n) = (\sum_{j=1}^{m} c_j\mathfrak{M}_j + c_{m+1}\mathfrak{M}_{m+1})(\bigcup_{n=1}^{\infty} D_n)$$
$$= \sum_{j=1}^{m} c_j \,.\, \mathfrak{M}_j(\bigcup_{n=1}^{\infty} D_n) + c_{m+1} \,.\, \mathfrak{M}_{m+1}(\bigcup_{n=1}^{\infty} D_n)$$
$$= (\sum_{j=1}^{m} c_j\mathfrak{M}_j)(\bigcup_{n=1}^{\infty} D_n) + c_{m+1} \,.\, \mathfrak{M}_{m+1}(\bigcup_{n=1}^{\infty} D_n)$$
$$= \sum_{n=1}^{\infty} (\sum_{j=1}^{m} c_j\mathfrak{M}_j)(D_n) + c_{m+1} \,.\, \sum_{n=1}^{\infty} \mathfrak{M}_{m+1}(D_n)$$
$$= \sum_{n=1}^{\infty} [\sum_{j=1}^{m} c_j \,.\, \mathfrak{M}_j(D_n)] + \sum_{n=1}^{\infty} c_{m+1} \,.\, \mathfrak{M}_{m+1}(D_n)$$
$$= \sum_{n=1}^{\infty} [\sum_{j=1}^{m} c_j \,.\, \mathfrak{M}_j(D_n) + c_{m+1} \,.\, \mathfrak{M}_{m+1}(D_n)]$$
$$= \sum_{n=1}^{\infty} [\sum_{j=1}^{m} c_j\mathfrak{M}_j + c_{m+1}\mathfrak{M}_{m+1}](D_n)$$
$$= \sum_{n=1}^{\infty} [\sum_{j=1}^{m+1} c_j\mathfrak{M}_j](D_n).$$

Hence, $\sum_{j=1}^{m+1} c_j\mathfrak{M}_j$ is measure relative to $\mathcal{K}$, therefore $(\mathfrak{P}, \mathcal{K}, \sum_{j=1}^{k} c_j\mathfrak{M}_j)$ is measure space relative to the λ− algebra $\mathcal{K}$.

Definition 30 [1]

A measure on a σ− field $\mathcal{K}$ is a nonnegative, extended real-valued set function $\mathfrak{M}$ on $\mathcal{K}$ such that whenever $A_1, A_2, \ldots$ form a finite or countably infinite collection of disjoint sets in $\mathcal{K}$, we have, $\mathfrak{M}(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathfrak{M}(A_n)$.

Definition 31 [1, 3]

A measure $\mathfrak{M}$ on a σ−field $\mathcal{K}$ is said to be complete iff whenever A $\epsilon\mathcal{K}$ and $\mathfrak{M}(A) = 0$, we have $B \,\epsilon\mathcal{K}$ for all $B \subset A$.

The following example shows that, if $\mathfrak{M}$ is a measure on σ−field $\mathcal{K}$, then not necessarily that $\mathfrak{M}$ is complete.

Example 32

Let $\mathfrak{P}$ ={1,2,3} and $\mathcal{K}$ ={ Φ,{1},{2,3},$\mathfrak{P}$ }. Then $\mathcal{K}$ is σ−field of a set $\mathfrak{P}$. If we define a set function $\mathfrak{M}: \mathcal{K} \to [0, \infty]$ by

$$\mathfrak{M}(D) = \begin{cases} o & ;\ if\ D = \Phi\ or\ D = \{2,3\} \\ 1 & ;\ other\ wise \end{cases}$$

Then $\mathfrak{M}$ is a measure on σ−field $\mathcal{K}$, it is clear that $\mathfrak{M}$ is not complete, because {2,3}∈ $\mathcal{K}$ and $\mathfrak{M}(\{2,3\}) = 0$, now {2},{3}⊂{2,3}, but {2},{3}∉ $\mathcal{K}$.

Theorem 33

Every measure relative to the λ− algebra is complete.

Proof

Let $\mathfrak{M}$ be a measure relative to the λ− algebra $\mathcal{K}$. Assume that A $\epsilon\mathcal{K}$ such that $\mathfrak{M}(A) = 0$, since $\mathcal{K}$ is a λ− algebra, then $B \,\epsilon\mathcal{K}$ for all $B \subset A$. Therefore $\mathfrak{M}$ is complete measure.

Example 34

Let $\mathfrak{P}$ ={a,b,c,d} and $\mathcal{K}$ ={ Φ,{a},{c},{d},{a,c},{c,d},{a,d},{a,c,d},$\mathfrak{P}$ }. Then $\mathcal{K}$ is λ− algebra of a set $\mathfrak{P}$. If we define a set function $\mathfrak{M}: \mathcal{K} \to [0, \infty]$ by

$$\mathfrak{M}(D) = \begin{cases} o & ;\ if\ D \neq \mathfrak{P} \\ 1 & ;\ if\ D = \mathfrak{P} \end{cases}$$

Then $\mathfrak{M}$ is a measure on $\lambda-$ algebra $\mathcal{K}$. Now, for any $A\epsilon\mathcal{K}$ such that $\mathfrak{M}(A) = 0$, then $B\epsilon\mathcal{K}$ for all $B \subset A$. Therefore $\mathfrak{M}$ is complete measure.

4. Conclusions

The main results of this paper are the following:

(1) Let $\{\mathcal{K}_i\}_{i\in I}$be a collection of $\lambda-$ algebra on $\mathfrak{P}$. Then $\bigcap_{i\in I}\mathcal{K}_i$ is a $\lambda-$ algebra on $\mathfrak{P}$.

(2) Let $\mathcal{J} \subseteq P(\mathfrak{P})$. Then $\lambda(\mathcal{J})$ is the smallest $\lambda-$ algebra of $\mathfrak{P}$ which includes $\mathcal{J}$.

(3) Let $\mathcal{J} \subseteq P(\mathfrak{P})$. Then $\mathcal{J}$ is a λ –algebra of a set $\mathfrak{P}$ if and only if $\mathcal{J} = \lambda(\mathcal{J})$.

(4) Let $\mathcal{J} \subseteq P(\mathfrak{P})$ and $\Phi \neq \mathfrak{D} \subseteq \mathfrak{P}$. If $\mathcal{K}$ is a λ –algebra of $\mathfrak{P}$ which includes $\mathcal{J}$, then $\lambda(\mathcal{J})|_{\mathfrak{D}}$
is a $\lambda-$ algebra of a set $\mathfrak{D}$.

(5) Let $\mathcal{J} \subseteq P(\mathfrak{P})$ and $\Phi \neq \mathfrak{D} \subseteq \mathfrak{P}$. Then $\lambda(\mathcal{J}|_{\mathfrak{D}})$=$\lambda(\mathcal{J})|\mathfrak{D}$.

(6) Every $\lambda-$ algebra is a $\alpha-\sigma-$field.

(7) Every $\lambda-$ algebra is a $\beta-\sigma-$field.

(8) Every $\lambda-$ algebra is a monotone class.

(9) Let $\mathcal{J}$ be a collection of subsets of a nonempty set $\mathfrak{P}$. Then $\mathbb{M}(\mathcal{J}) \subseteq \lambda(\mathcal{J})$.

(10) Let $(\mathfrak{P}, \mathcal{K}, \mathfrak{M}_j)$ be a measure space relative to the $\lambda-$ algebra $\mathcal{K}$ and $c_j \in [0,\infty)$ for all $j = 1,2,\dots,k$. If a set function $\sum_{j=1}^{k} c_j\mathfrak{M}_j\colon \wp \to [0,\infty]$ is defined by:
$(\sum_{j=1}^{k} c_j\mathfrak{M}_j)\,(D) = \sum_{j=1}^{k} c_j\,.\,\mathfrak{M}_j\,(D)\ \forall D\epsilon\wp$, then $(\mathfrak{P}, \mathcal{K}, \sum_{j=1}^{k} c_j\mathfrak{M}_j)$ is measure space relative to the $\lambda-$ algebra $\mathcal{K}$.

(11) Every measure relative to the $\lambda-$ algebra is complete.

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