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Chromatic Number of Pseudo-Von neuman Regular Graph

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Abstract

Let R be a commutative ring , the pseudo – von neuman regular graph of the ring R is define as a graph whose vertex set consists of all elements of R and any two distinct vertices a and b are adjacent if and only if $a = a^2b$ or $b = b^2a$, this graph denoted by *P-VG(R)* , in this work we got some new results a bout chromatic number of *P-VG(R)*.

Keywords: Graph, Chromatic Number, Commutative Ring.

1. Introduction

Beck [1]. Studied coloring of commutative rings and studied chromatic number of it is graph such that two different elements x and y are adjacent iff $xy = 0$, Bhavanari S. et.al. studied prime graph of a ring with some properties of its graph [2]. Kalita S. [3]. Computed chromatic number of prime graph of some finite ring, patra k. et.al [4]. Studied chromatic number of prime graph of some rings namely Z_n , where n=$\prod_{i=1}^{r} p_i^{\alpha_i}$, Elizabeth R. [5]. Studied colorings of zero divisor graphs of commutative rings , in this paper we define pseudo-von neman regular graph of the ring R with some result of it graph and we study chromatic number of pseudo-von neman regular graph.

2. Primer lay

Definition 1: [6]. A nonempty set R, together with two binary operations (+) and (·) is said to be a ring if the following are satisfied

i- $(R,+)$ is an a belian group

ii- $(R,\cdot)$ is a semi-group

iii-$s\cdot(t+l) = s\cdot t + s\cdot l$ *and* $(s+t)\cdot l = s\cdot l+t\cdot l$ for any $s, t, l \in R$

Definition 2: [7]. let R be a ring and $a \in R$, a is called regular element if there exist $b\in R$ such that $a=aba$, if any element in R is regular then R is regular ring , if R is commutative then $a = a^2b$ and we say that R is Von Neumann regular ring.

Definition 3: [8]. A graph G is defined by an ordered pair ($V(G)$, $E(G)$), when $V(G)$ is a non empty set whose elements are called vertices and $E(G)$ is a set (may be empty) of unordered pairs of distinct vertices of $V(G)$. the element of $E(G)$ are called edges of the graph G . we denote by $\overline{uv}$, an edge between two end vertices u and v .

Definition 4: [8]. An edge whose end-vertices are the same is called loop.

Definition 5: [8]. If there are more than one edges associated with a given pair of vertices , then these edges are called multiple edges or parallel edges.

Definition 6: [8]. A simple graph that has no self-loops or multiple edges .

Definition 7: [8]. A graph H is said to be a subgraph of a graph G if all the edges and all the vertices of H are in G , and each edge of H has the same end vertices in H as in G .

Definition 8: [9]. A path is a graph G that contains a list$v_1, v_2, \dots, v_n$ of vertices of G s.t. for $1 \leq i \leq p - 1$, there is an edge (v_i, v_{i+1}) in G and these are the only edges in G .

Definition 9: [9]. let v_1 and v_2 be two vertices , $d(v_1, v_2)$ is called a distance from v_1 to v_2 if it is the shortest path from v_1 to v_2.

Definition 10: [9]. A close path is called cylce , the degree of each vertex of a cycle graph is two , a cycle with n vertices denoted by C_n .

Definition 11: [10]. Let $G(V, E)$ be a graph and $C \subset G$, is called clique if the induced sub graph of G induced by C is a complete graph .

The clique is called maximal if there is no clique with more vertices .

Definition 12: [11]. A h-coloring of the vertex set of a graph G is a function γ: $V(G) \rightarrow \{1,2, \dots, h\}$ such that $\gamma(v_1) \neq \gamma(v_2)$ whenever v_1 is adjacent to v_2, if a h- coloring of G exists, then G is called h- colorable.

Definition 13: [11]. The chromatic number of G is defined as

$$\mathcal{X}(G) = \min \{ h : G \text{ is } h\text{- colorable} \}$$

Where $\mathcal{X}(G) = h$, G is called h- chromatic.

Theorem 14 [10].

for circular graph C_n one has

$$\mathcal{X}(C_n) = \begin{cases} 2 & when\ n\ is\ even \\ 3 & when\ n\ is\ odd \end{cases}$$

In other words $\mathcal{X}(C_{2i+1}) = 3$, $\mathcal{X}(C_{2i}) = 2$ for $i \in \{1,2,3,\dots\}$.

Theorem 15: [3]. Let R be a ring , $B(R) = \{(a, b) : aRb = 0 \text{ or } bRa = 0, a, b \in R, a \neq b, a \neq 0, b \neq 0\}$. Then $\chi\, PG(R) = \chi\, G(B(R)) + 1$, when $G(B(R))$ is the induced sub graph of $PG(R)$ whose edges are elements of $B(R)$.

3. Main Result

Definition 3.1: let R be a commutative ring. A graph $G(V, E)$ is said to be (pseudo-von neumann regular graph) of R if $V(G) = R$ and $E(G) = \{ \overline{ab} / \ a = a^2 b$ or $b = b^2 a$ and $a \neq b \}$ denoted by *P-VG(R)* , shortly p-von Neumann regular graph .

Example 3.2

Z_2={0,1}

Figure 1: *P-VG*(Z_2)

Z_3={0,1,2}

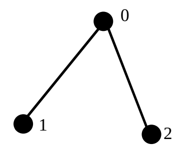

Figure 2:*P-VG*(Z_3)

Z_4={0,1,2,3}

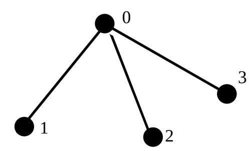

Figure 3:*P-VG*(Z_4)

Z_5={0,1,2,3,4}

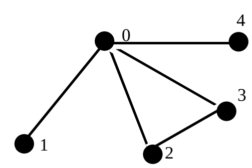

Figure 4:*P-VG*(Z_5)

Note 3.3

$C\,[R] = \{(a,b) : a \neq b\,, a \neq 0\ and\ b \neq 0, a = a^2b$ or $b = b^2a\} \subset R \times R$.

Corollary 3.4

i- The number of element in *C* [*R*] is less than or equal to number of cycle C_3 in *P-VG(R)*.
ii- If *C* [*R*] $\neq \emptyset$, then the longest trail has length ≥ 3.
iii-*C* [*R*] $\neq R \times R$.
Proof

i- Let (*a*,*b*) ∈ *C* [*R*], then { 0, a, b } form a cycle C_3 in *P-VG(R)* ,
if (*a*,*b*) , (*s*,*t*) ∈ *C* [*R*] such that (*a*,*b*) ≠ (*s*,*t*) then *a*≠*s* or *b*≠*t* and so the cycle C_3 { *0*, *a*, *b* } and { *0*, *s*, *t* } in *P-VG(R)* are distinct , this show the number of elements in *C* [*R*] is less than or equal the number of cycle C_3 in *P-VG(R)*.

if *a*=*s* or *b*=*t* then the cycles { *0*, *a*, *b* } , { *0*, *s*, *t*} are adjust.

ii- Let $(a,b) \in C\,[R]$,then $0a$, ab , $0b$ is cycle C_3 , this trail is of length equal to 3 hence , the longest trail has length ≥ 3.

iii-Since $(0,a) \in R \times R$ and $(0,a) \notin C\,[R]$, then we have

$C\,[R] \subset R \times R$ and $C\,[R] \neq R \times R$.

By the same way of (theorem 2.15) we can prove below theorem.

Theorem 3.5

Let R be a ring and let $C\,[R] = \{(a,b) : a \neq b\,, a \neq 0\ and\ b \neq 0, a = a^2b$ or $b = b^2a\}$ then $\mathcal{X}$ (*P-VG(R)*)= $\mathcal{X}$ ($G(C\,[R]$) + 1 , where $G(C\,[R])$ is sub graph of *P-VG(R)* .

Proof

Let R be a ring , let $C\,[R] = \{(a,b) : a \neq b\,, a \neq 0\ and\ b \neq 0, a = a^2b$ or $b = b^2a\}$

If $C\,[R] = \emptyset$, then *P-VG(R)* graph is a star graph and $\mathcal{X}$(*P-VG(R)*) =2

Suppose $C\,[R] \neq \emptyset$, let $|C[R]|$= n

Case i: let $G(C\,[R]) = P_n$ then $\mathcal{X}$ ($G(C\,[R]$) is equal to 2 , since 0 is adjacent to all vertices in *P-VG(R)*, so we have a third color to a vertex 0 , Now the vertices do not belong to P_n will be associated only with vertex 0, this vertices can colored by any color we used it in P_n.

hence $\mathcal{X}$ (*P-VG(R)*) = 3 = $\mathcal{X}$ ($G(C\,[R]$) +1

Case ii: Now if $G(C\,[R]) = C_n$ then $\mathcal{X}$ ($G(C\,[R])$) must equal to 2 when n is even and equal to 3 when n is odd , and the vertex 0 adjacent to all vertices , implies that $\mathcal{X}$ (*P-VG(R)*) equal to 3 if n is even and equal to 4 if n is odd, hence $\mathcal{X}$ (*P-VG(R)*) = $\mathcal{X}$ ($G(C$ [R]) +1.

Case iii : Let $G(C\,[R]$ is an i-partite graph where $G(C\,[R]) = K_{m_1,m_2,\dots,m_i}$, then

$\mathcal{X}$ ($G(C\,[R])$) = i , then we have $\mathcal{X}$ (*P-VG(R)*) = i+1 = $\mathcal{X}$ ($G(C\,[R]$) +1

Case iv: If $G(C\,[R]) = K_h$ then the chromatic number of $G(C\,[R])$) equal to h and therefore $\mathcal{X}$ (*P-VG(R)*)= h+1 = $\mathcal{X}$ ($G(C$ [R]) +1

Case v: Now if $G(C\,[R])$ is a connected graph and it includes a maximal clique K_i , $i > 2$ then the chromatic number of K_i equal to i , and the rest of the vertices of $G(C\,[R])$ can be colored by any colors of the vertices of K_i because they are not associated with it . then $\mathcal{X}$ ($G(C\,[R])$) = i and $\mathcal{X}$ (*P-VG(R)*)= i+1 = $\mathcal{X}$ ($G(C\,[R]$) +1

Case vi : Let $G_1, G_2, \dots, G_k$ be a disjoint components of $G(C\,[R])$ then $\mathcal{X}$ ($G(C\,[R])$) = *max* $\{\mathcal{X}G_1, \mathcal{X}G_2, \dots, \mathcal{X}G_k\}$, suppose $\mathcal{X}$ ($G(C\,[R])$)= s Now if we have a vertices of *P-VG(R)* with degree 1 this can be colored by any color of these s colors used to color $G(C\,[R])$, and the vertex 0 has another color since it adjacent to all vertices , this implies that $\mathcal{X}$ (*P-VG(R)*)= s+1= $\mathcal{X}$ ($G(C$ [R]) +1 .

From all above cases then $\mathcal{X}$ (*P-VG(R)*)= $\mathcal{X}$ ($G(C\,[R]$) +1

Lemma 3.6

Let R=Z_p be a ring , where p ≥ 3 is prime number then $\mathcal{X}$ (*P-VG(R)*) =3.

Proof

Let $\overline{ab} \in VG(R)$ then $a = a^2b$ or $= b^2a$, $a \neq b$ and $a \neq 0 \neq b$,

since p is prime then R is afield implies that $b = a^{-1}$ or $a = b^{-1}$ is unique number satisfy the equation , since $\overline{a0}$, $\overline{b0} \in E(P\text{-}VG(R))$

then P-VG(R) graph has cycle C_3 , then by(throrem 2.14) $\mathcal{X}$ (*P-VG(R)*)=3

Example 3.7

The chromatic number of P –Von Neumann regular graph of Z_{10} , Z_{14} and Z_{15} are 4

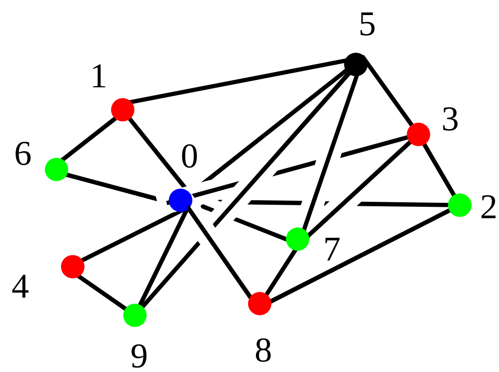

Figure 5:*P-VG*(Z_{10})

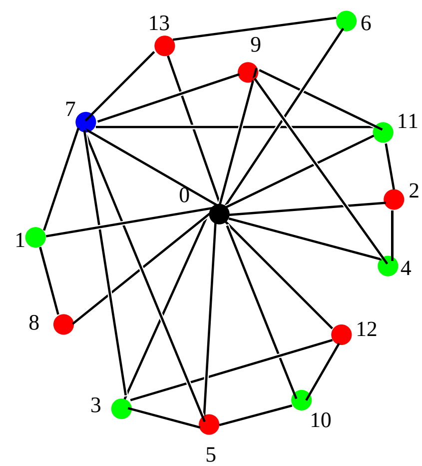

Figure 6: P-*VG*(Z_{14})

Theorem 3.8
The ring Z_{p^2} , $\mathcal{X}$ (*P-VG*(Z_{p^2}))=3 where p ≥ 3 is prime number.

Example 3.9
R= Z_{25} then $\mathcal{X}$ (*P-VG*(Z_{25})) = 3 .

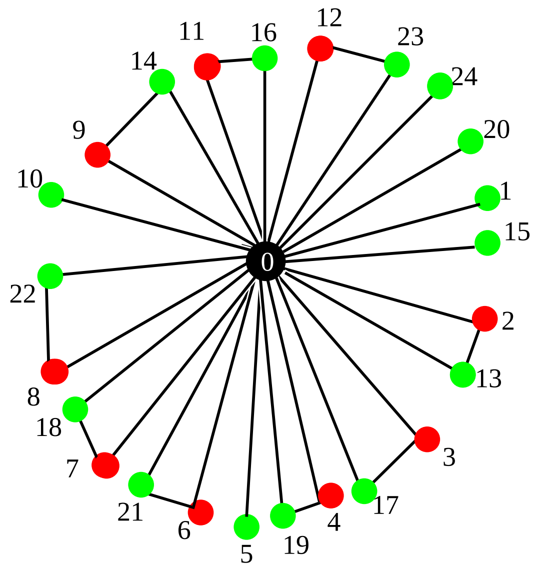

Figure 7: $P\text{-}VG(Z_{25})$

Note 3.10

If $R = Z_p$ and $\overline{ab} \in E\,(P\text{-}VG(Z_p))$ then $a = a^2 b$ and $b = b^2 a$ and ab mod p =1.

Theorem 3.11

The ring Z_{p^n} , $\mathcal{X}\,(P\text{-}VG(Z_{p^n}))$=3, when $p \geq 3$ is a prime number and n a positive integer .

Proof

let a ,$b \in Z_{p^n}$ and $a \neq b$, $a \neq 0 \neq b$ and $\overline{ab} \in E(P\text{-}VG(Z_{p^n}))$

Then $a = a^2 b$ and $b = b^2 a$ and ab=1 and since b is unique (because b is the inverse of a , and it is only element satisfy the equation)

Then $P\text{-}VG(Z_{p^n})$ has only cycle C_3

by theorem (2.14) every cycle has odd vertices has chromatic number equal to 3

and in any $P\text{-}VG(Z_{p^n})$ graph has more than one C_3 , and all C_3 in a graph commend in one vertex (0) , so the vertex has one color

it is clear that the color of other vertices in the same cycle are two another colors , so we have only three colors in any $P\text{-}VG(Z_{p^n})$.

4. Conclusion

In this work we gave a definition of pseudo-von Neumann regular graph $P\text{-}VG(R)$ then proved the chromatic number $\mathcal{X}\,(P\text{-}VG(R)) = \mathcal{X}\,(G(C\,[R]\,) + 1$ when $a \neq b \neq 0$ and $\mathcal{X}\,(P\text{-}VG(Z_p))$, $p \geq 3$ is equal to 3 , and $\mathcal{X}\,(P\text{-}VG(Z_{p^n}))$, $p \geq 3$ equal to 3.

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