On the Space of Primary La-submodules

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Article history: Received 23 July 2019, Accepted 22 September 2019, Published in July 2020.

Doi: 10.30526/33.3.2476

Abstract
Suppose that F is a reciprocal ring which has a unity and suppose that H is an F-module. We topologize La-Prim(H), the set of all primary La-submodules of H, similar to that for FPrim(F), the spectrum of fuzzy primary ideals of F, and examine the characteristics of this topological space. Particularly, we will research the relation between La-Prim(H) and La-Prim(F/Ann(H)) and get some results.

Keywords primary La-submodules, Fuzzy primary spectrum, La-top modules.

1. Introduction
Suppose that F is a reciprocal ring with a unity and H is a unitary F-module. The primary spectrum Prim(F) and the topological space acquired by inserting Zariski topology on the collection of primary ideals of a reciprocal ring with unity play an significant role in the fields of reciprocal algebra, algebraic geometry and lattice theory. As well, lately the concept of primary submodules and Zariski topology on Prim(H), the collection of all primary submodules of a module H on a reciprocal ring together identity F, were studied in a previous article [1]. As it is famous [2], inserted the concept of a fuzzy subset ϑ of a nonempty collection L as a mapping from L to [0,1]. Goguen JA [3]. Changed [0,1] by an entire lattice La in the definition of fuzzy collections while inserted the concept of La-fuzzy sets. Rosenfeld inserted the concept of fuzzy groups [4]. While fuzzy submodules of H over F were first inserted by [5], Pan F-Z [6]. Elaborate fuzzy finitely created modules while fuzzy quotient modules (look at [7]). In previous years a large saucepan of labor has been completed on fuzzy ideals in common and primary fuzzy ideals in special, while several motivating topological features of the spectrum of fuzzy primary ideals of a ring were acquired (look at [8-15]).

Suppose that H is an F-module. By G ≤ H, we mean that G is a submodule of H. For any G ≤ H, we indicate the residual of G by H by [G:H], and define[G:H]={ r ∈ F \ r'H⊆G}. In special, [(0):H] is called the annihilator of H and is indicated by Ann(H), that is
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Ann(H) = {r ∈ F \ r'H = 0}. A primary submodule (or a q-primary submodule) of H is a proper submodule Q with Q:H=q, such that r'h ∈ Q for r ∈ F and h ∈ H, either h ∈ Q or r ∈ √q.

The collection of all primary submodules of H is called the primary spectrum of H or, artlessly the p spectra of H and is indicated by Prim(H). Note that the Prim(H) may be empty for some module H. Such a module is said to be primary less (cf. [1]). Clearly, zero module is primary less, but in [1]. Some nontrivial examples are shown.

For example, the Prüfer group ℤ(p∞) as a ℤ-module has no primary submodule for any prime integer p. When Prim(H) ≠ ∅, the map φ:La-Prim(H) → La-Prim(F/Ann(H)) defined by φ(ϑ) = (ϑ: 1H) for ϑ ∈ La-Prim(H), φ will be called the standard map.

In [1]. It is shown that for each multiplication module H,(An F-module H is called a multiplication module if every submodule B of H is of the form IH for some ideal I of F) the Prim(H) is non-empty. For any submodule G of H, V(G) indicates the collection of all primary submodules of H including G. Of course V(H) is just the empty set and V(0) is Prim(H). For any family of submodules Gj (j ∈ J) of H, ∩ V(Gj) = V(∪ Gj).

Thus if ω(H) indicates the set of all subsets V(G) of Prim(H), then ω(H) includes the empty set and Prim(H) and is closed beneath arbitrary intersection. If also ω(H) is closed beneath finite union, i.e. for any submodules G and K of H, there occurs a submodule J of H such that V(G)∪ V(K) = V(J), for in this state ω(H) satisfies the axioms of closed subsets of a topological spaces, which is called Zariski topology. In [1]. A module with Zariski topology is called top module and it is shown that each multiplication module is a top module [1]. In [16,17]. Inserted the concept of primary La-submodules of a module H on a commutative ring together unity F, where La is a whole lattice. The collection of all primary La-submodules of H is called the primary La-spectrum of H or, artlessly the P-La-spectrum of H while is indicated by La-Prim(H).

2. Basic concepts

During this article via F, we mean a reciprocal ring together unity, and H is a unital F-module and La indicates a whole lattice. Via an La-subset ϑ of Y ≠ ∅, we mean a mapping ϑ from Y to La while if La = [0,1], then ϑ is a surname of a fuzzy subset of Y. La Y indicates the collection of each La-subsets of Y. Suppose that C is a subset of Y and b ∈ La. Define bc ∈ La Y as follows:

\[ bc(y) = \begin{cases} b & \text{if } y \in C \\ 0 & \text{otherwise} \end{cases} \]

In particular case if C = {c} we indicate b{c} by bc, while its surname is an La-point of Y.

For ϑ ∈ La Y while c ∈ La, locate ϑc as follows:
\[ \vartheta_c = \{ y \in Y \mid \vartheta(y) \geq c \}, \]

\( \vartheta_c \) is called the c-level subset of \( \vartheta \). The image of \( \vartheta \) is indicated via \( \text{Ima}(\vartheta) \) or \( \vartheta(Y) \). In [18]. It was proved that \( \vartheta = \bigcup_{c \in \theta(Y)} \vartheta_c. \) For \( \vartheta, \varpi \in \text{La}^Y \) we say that \( \vartheta \) is included in \( \varpi \) while we write \( \vartheta \subseteq \varpi \) if for every \( y \in Y \), \( \vartheta(y) \leq \varpi(y). \)

For \( \vartheta, \varpi \in \text{La} \), \( \vartheta \cup \varpi, \vartheta \cap \varpi \in \text{La}^Y \), are defined via \( (\vartheta \cup \varpi)(y) = \vartheta(y) \lor \varpi(y) \) and \( (\vartheta \cap \varpi)(y) = \vartheta(y) \land \varpi(y) \), for each \( y \in Y \).

If \( g \) is a function from \( H \) into \( G \), \( \vartheta \in \text{La}_a \) and \( \varpi \in \text{La}_b \), then the \( \text{La} \)-subsets \( g(\vartheta) \in \text{La}^G \) and \( g^{-1}(\varpi) \in \text{La}^H \) are defined as follows:

\[
\forall \ d \in G, \quad g(\vartheta)(d) = \begin{cases} \bigvee \{ \vartheta(y) \mid y \in g^{-1}(d) \} & g^{-1}(d) \neq \emptyset; \\ 0 & \text{otherwise} \end{cases}
\]

and \( g^{-1}(\varpi)(h) = \varpi(g(h)) \) \( \forall h \in H. \)

Suppose that \( H, G \) are two F-modules while \( g: H \to G \) is an F-homomorphism. Then an \( \text{La} \)-subset \( \vartheta \) of \( H \) is surname \( g \)-invariant if \( \vartheta(a) = g(b) \) then \( \vartheta(b) = \vartheta(b) \) for all \( a, b \in H. \)

**Definition 2.1** Suppose that \( \vartheta \in \text{La}^F \). Then \( \vartheta \) is surname an \( \text{La} \)-ideal of \( F \) if for all \( a, b \in F \) the following situations are satisfied:

1) \( \vartheta(a - b) \geq \vartheta(a) \land \vartheta(b); \)
2) \( \vartheta(ab) \geq \vartheta(a) \lor \vartheta(b). \)

The collection of every \( \text{La} \)-ideals of \( F \) is indicated via \( \text{LaI} (F). \)

For \( \vartheta, \varpi \in \text{LaI} (F) \), \( \vartheta \varpi(a) = \bigvee \{ \vartheta(b) \land \varpi(c) \mid b, c \in F, a = bc \} \) \( \forall a \in F \), and in [18]. It was confirmed that \( \vartheta \varpi \in \text{LaI} (F). \)

If \( \text{La} = [0, 1] \), then an \( \text{La} \)-ideal is surname a fuzzy ideal while the collection of every fuzzy ideals of \( F \) is indicated via \( \text{FI}(F) \).

**Definition 2.2** [18]. Suppose that \( \vartheta \) is a \( \text{La} \)-subset of \( F \). The radical of \( \vartheta \) is indicated by \( (\sqrt{\vartheta}) \) and is defined by

\[
\sqrt{\vartheta}(y) = \bigvee_{n \in \mathbb{N}} \vartheta(y^n) \text{ for all } y \in F.
\]

**Definition 2.3** \( \eta \in \text{LaI} (F) \) is surname a primary \( \text{La} \)-ideal of \( F \) if \( \eta \) is non-fixed and for all \( \vartheta, \varpi \in \text{LaI}(F), \) if \( \vartheta \varpi \subseteq \eta \) then \( \vartheta \subseteq \eta \) or \( \varpi \subseteq \sqrt{\eta}. \)

Via \( \text{La-Prim}(F) \), we mean the collection of each primary \( \text{La} \)-ideals of \( F \).

**Proposition 2.4** [18]. Suppose that \( F \) and \( S' \) are two rings while \( g: F \to S' \) is an epimorphism.

1) Suppose that \( \vartheta \in \text{La-Prim}(F) \) and \( g \)-invariant, then \( g(\vartheta) \in \text{La-Prim}(S'). \)
2) If \( \varpi \in \text{La-Prim}(S') \), then \( g^{-1}(\varpi) \in \text{La-Prim}(F). \)
For $\gamma \in \text{LaI}(F)$, $V(\gamma)$ be the collection of all primary La-ideals of $F$ such that includes $\gamma$, i.e.

\[ V(\gamma) = \{ q \in \text{La-Prim}(F) \mid \gamma \subseteq q \} \]

collection $X(\gamma) = \text{La-Prim}(F) \setminus V(\gamma)$, the La-Prim$(F)$ together the collection $\mathcal{T} = \{ X(\gamma) \mid \gamma \in \text{LaI}(F) \}$ is a topological space while the collection $\mathcal{B} = \{ X(\gamma) \mid \gamma \in F, \alpha \in (0, 1] \}$ formation a basis for $\mathcal{T}$. As well, it can be shown that for two elements $X(y_\alpha), X(x_\alpha)$;

\[ X(y_\alpha) \cap X(x_\alpha) = X((xy)_\alpha \alpha) \]

\[ y, x \in F, \alpha, \alpha' \in \text{La}\{0\} \]

**Definition 2.5** An element $z \in \text{La}\{1\}$ is surname a prime element of La if for $c, d \in \text{La}$, $c \leq z$ or $d \leq z$.

**Definition 2.6** Suppose that $\varepsilon \in \text{LaH}$ and $\vartheta \in \text{LaH}$. Define $\varepsilon \cdot \vartheta \in \text{LaH}$ as follows:

\[ (\varepsilon \cdot \vartheta)(y) = \vee \{ \varepsilon(r') \land \vartheta(b) \mid r' \in F, b \in H, r'b = y \} \]

for all $y \in H$.

**Definition 2.7** An La-subset $\vartheta \in \text{LaH}$ is a La-submodule of $H$ if:

1) $\vartheta(0) = 1$;
2) $\vartheta(r'a) \geq \vartheta(a)$ for all $r' \in F$ and $a \in H$;
3) $\vartheta(a + b) \geq \vartheta(a) \land \vartheta(b)$ for all $a, b \in H$.

The collection of all La-submodules of $H$ is indicated by $\text{La}(H)$.

**Definition 2.8** [18]. Suppose that $\{ \vartheta_j \mid j \in J \} \subseteq \text{La}(H)$. Define the La-submodule $\sum_{j \in J} \vartheta_j$ of $H$ by

\[ (\sum_{j \in J} \vartheta_j)(y) = \vee \{ \land_{j \in J} \vartheta_j(y_j) \mid y = \sum_{j \in J} y_j, y_j \in H, \forall j \in J \} \forall y \in H. \]

It is easy to look that $\sum_{j \in J} \vartheta_j \in \text{La}(H)$.

For $\vartheta, \varpi \in \text{LaH}$ and $\varepsilon \in \text{LaF}$, $\vartheta, \varpi \in \text{LaF}$ and $\varepsilon \in \text{LaH}$ are defined as follows:

$\vartheta: \varpi = \{ y \mid y \in \text{LaF}, y \varpi \leq \vartheta \}$.

$\varepsilon: \varpi = \{ y \mid y \in \text{LaH}, e \varpi \leq \vartheta \}$.

In [18]. It was proved that if $\varpi \in \text{LaH}$, $\vartheta \in \text{La}(H)$, and $\varepsilon \in \text{LaI}(F)$, then

$\vartheta: \varpi = \{ y \mid y \in \text{LaF}, y \varpi \leq \vartheta \}$ and $\varepsilon: \varpi = \{ y \mid y \in \text{LaH}, e \varpi \leq \vartheta \}$.

Also it was shown that if $\vartheta \in \text{La}(H)$, $\varpi \in \text{LaH}$, $\varepsilon \in \text{LaI}(F)$, then $\vartheta: \varpi \in \text{LaI}(F)$ and $\varepsilon: \varpi \in \text{LaI}(F)$ and $\vartheta: \varepsilon \in \text{La}(H)$.

**Theorem 2.9** [18]. If $b \in \text{La}$ and $G$ are a submodule of $H$, then $(1_G \cup b_H): 1_H = 1_{[G,H]} \cup b_F$.

**Definition 2.10** [16]. A non-constant La-submodule $\vartheta$ of $H$ is called primary if for $\varepsilon \in \text{LaI}(F)$ and $\varpi \in \text{La}(H)$ such that $\varepsilon. \varpi \leq \vartheta$ then either $\varpi \leq \vartheta$ or $\varepsilon \leq \sqrt{\vartheta: 1_H}$. In the complement $\text{La-Prim}(H)$ indicates the collection of all primary La-submodules of $H$.

**Theorem 2.11** [16]. $\vartheta \in \text{La-Prim}(H)$ if and only if $\vartheta = 1_{\vartheta*} \cup c_H$ such that $\vartheta* = \{ h \in H \mid \vartheta(h) = 1 \}$ be a primary submodule of $H$ while $z$ is a prime element of La.
Theorem 2.12 [16]. If \( \mathcal{G} \in \text{La-Prim}(H) \), then \( \mathcal{G} : 1_H \) is a primary La-ideal of \( F \).

3. Topologies on La-Prim(H)

In the complement via \( H \) we indicate a unitary module on a reciprocal ring together unity \( F \).

For \( \mathcal{G} \in \text{La}H \), we put \( V^*(\mathcal{G}) = \{ Q \in \text{La-Prim}(H) \mid \mathcal{G} \subseteq Q \} \).

Proposition 3.1 For family \( \{ \mathcal{G}_j \}_{j \in J} \) in \( \text{La}(H) \), the following situations are satisfied:

1. \( V^*(1_{\{0\}}) = \text{La-Prim}(H), V^*(1_H) = \emptyset \);
2. \( \bigcap_{j \in J} V^*(\mathcal{G}_j) = V^*(\sum_{j \in J} \mathcal{G}_j) \), for index collection \( J \) and \( \mathcal{G}_j \in \text{La}(H) \);
3. \( V^*(\mathcal{G}) \cup V^*(\mathcal{P}) \subseteq V^*(\mathcal{G} \cap \mathcal{P}) \), for \( \mathcal{G}, \mathcal{P} \in \text{La}(H) \).

Proof (1) clearly.

(2) Let \( Q \in \bigcap_{j \in J} V^*(\mathcal{G}_j) \), then \( Q \in V^*(\mathcal{G}_j), \forall j \in J \), and hence \( Q \subseteq \mathcal{G}_j, \forall j \in J \). Moreover, we have

\[
(\sum_{j \in J} \mathcal{G}_j)(y) = \bigvee \{ \wedge_{j \in J} \mathcal{G}_j(y_j) \mid y = \sum_{j \in J} y_j, y_j \in H, \forall j \in J \}
\]

\[
= \bigvee \{ \wedge_{j \in J} Q(y_j) \mid y = \sum_{j \in J} y_j, y_j \in H, \forall j \in J \} \leq Q(y).
\]

Then \( \sum_{j \in J} \mathcal{G}_j \subseteq Q \) implies that \( Q \in V^*(\sum_{j \in J} \mathcal{G}_j) \), and hence \( \bigcap_{j \in J} V^*(\mathcal{G}_j) \subseteq V^*(\sum_{j \in J} \mathcal{G}_j) \) (i).

For the converse, \( Q \in V^*(\sum_{j \in J} \mathcal{G}_j) \) then \( \sum_{j \in J} \mathcal{G}_j \subseteq Q \), and so \( \mathcal{G}_j \subseteq \sum_{j \in J} \mathcal{G}_j, \forall j \in J \). So \( \mathcal{G}_j \subseteq Q, \forall j \in J \). Therefore, \( Q \in V^*(\mathcal{G}_j) \), \( \forall j \in J \), and hence \( Q \in \bigcap_{j \in J} V^*(\mathcal{G}_j) \) then \( V^*(\sum_{j \in J} \mathcal{G}_j) \subseteq \bigcap_{j \in J} V^*(\mathcal{G}_j) \) (ii). Now (2) instantly follows from (i) and (ii). For (3) let \( \mathcal{G}, \mathcal{P} \in \text{La}(H) \) and \( Q \in V^*(\mathcal{G}) \cup V^*(\mathcal{P}) \). Then \( \mathcal{G} \subseteq Q \), or \( \mathcal{P} \subseteq Q \), and hence \( \mathcal{G} \cap \mathcal{P} \subseteq Q \). Thus \( Q \in V^*(\mathcal{G} \cap \mathcal{P}) \), while so \( V^*(\mathcal{G}) \cup V^*(\mathcal{P}) \subseteq V^*(\mathcal{G} \cap \mathcal{P}) \). Suppose that \( \mathcal{G} \in \text{La}H \). The La-submodule generated by \( \mathcal{G} \), indicated via \( \downarrow \mathcal{G} \), is the smallest La-submodule of \( H \) including \( \mathcal{G} \). In fact, \( \downarrow \mathcal{G} = \bigcap \{ \mathcal{P} \in \text{La}(H) \mid \mathcal{G} \subseteq \mathcal{P} \} \). For \( \mathcal{G} \in \text{La}(H) \), put \( V(\mathcal{G}) = \{ Q \in \text{La-Prim}(H) \mid 1_H \subseteq Q \} \). Then we have the next outcomes.

Proposition 3.2. Suppose that \( \{ \mathcal{G}_j \}_{j \in J} \), \( \mathcal{G}_j \in \text{La}(H) \). Then the following hold:

\[
V(1_H) = \emptyset, V(1_{\{0\}}) = \text{La-Prim}(H); \tag{1}
\]

\[
V(\mathcal{G}) = V(\mathcal{G} \subseteq \mathcal{G}_j, \forall j \in J); \tag{2}
\]

\[
V(\mathcal{G}_j) = V(\sum_{j \in J} \mathcal{G}_j, 1_H), \bigcap_{j \in J}; \tag{3}
\]

\[
V(\mathcal{G}) \cup V(\mathcal{P}) = V(\mathcal{G} \cap \mathcal{P}), \text{ for } \mathcal{G}, \mathcal{P} \in \text{La}(H). \tag{4}
\]

Proof (1) instant.
It is an instant result of definition of $\langle \vartheta \rangle$. For (3) let $Q \in \bigcap_{j \in J} V(\vartheta_j)$, then $\vartheta_j : 1_H \subseteq Q : 1_H$, $\forall j \in J$. Thus for all $j \in J$ we have $(\vartheta_j : 1_H). 1_H \subseteq (Q : 1_H). 1_H \subseteq Q \Rightarrow (\sum_{j \in J} (\vartheta_j : 1_H). 1_H) : 1_H \subseteq Q \\
= (\sum_{j \in J} (\vartheta_j : 1_H). 1_H) : 1_H \subseteq Q : 1_H$

So $Q \in V (\sum_{j \in J} (\vartheta_j : 1_H). 1_H)$, and hence $\bigcap_{j \in J} V(\vartheta_j) \subseteq V (\sum_{j \in J} (\vartheta_j : 1_H). 1_H)$ (a)

Reciprocally, let $Q \in V (\sum_{j \in J} (\vartheta_j : 1_H). 1_H)$, then $(\sum_{j \in J} (\vartheta_j : 1_H). 1_H) : 1_H \subseteq Q : 1_H$. Thus for all $j \in J$ we have $(\vartheta_j : 1_H). 1_H \subseteq (Q : 1_H). 1_H \Rightarrow \sum_{j \in J} (\vartheta_j : 1_H). 1_H \subseteq Q : 1_H$.

Clearly, we have $(\sum_{j \in J} (\vartheta_j : 1_H). 1_H) : 1_H \subseteq Q : 1_H$, $\forall j \in J$. Also for each $j \in J$, we get that $(\sum_{j \in J} (\vartheta_j : 1_H). 1_H) : 1_H \subseteq Q : 1_H$. Thus for any $j \in J$ it deduces that $\vartheta_j : 1_H \subseteq Q : 1_H \Rightarrow (\sum_{j \in J} (\vartheta_j : 1_H). 1_H) : 1_H \subseteq Q : 1_H$.

For (4) Let $\vartheta, \varpi \in La(H)$ and $Q \in V(\vartheta) \cup V(\varpi)$. Then $Q \in V(\vartheta)$ or $Q \in V(\varpi)$. Without loose of commonness, let $Q \in V(\vartheta)$ we have $\vartheta : 1_H \subseteq Q : 1_H \Rightarrow (\sum_{j \in J} (\vartheta_j : 1_H). 1_H) : 1_H \subseteq Q : 1_H$. Thus $V(\vartheta) \cup V(\varpi) \subseteq V(\vartheta \cap \varpi)$ (c)

For the opposite, let $Q \in V(\vartheta \cap \varpi)$ then $(\vartheta \cap \varpi) : 1_H \subseteq Q : 1_H$. But we have $(\vartheta \cap \varpi) : 1_H = (\vartheta : 1_H) \cap (\varpi : 1_H)$, and hence $(\vartheta : 1_H) (\varpi : 1_H) \subseteq (\vartheta : 1_H) \cap (\varpi : 1_H)$.

Thus $(\vartheta : 1_H) (\varpi : 1_H) \subseteq Q : 1_H$. Since $Q : 1_H$ is a primary La-ideal then $\vartheta : 1_H \subseteq Q : 1_H$ or $\varpi : 1_H \subseteq Q : 1_H$. Hence $V(\vartheta \cap \varpi) \subseteq V(\vartheta) \cup V(\varpi)$ and hence $V(\vartheta \cap \varpi) \subseteq V(\vartheta) \cup V(\varpi)$.

Subsequently (2) follows via (c) and (d).

Now, we set

La- $\varepsilon^\ast(H) = \{V(\vartheta) | \vartheta \in La(H)\}$

La- $\varepsilon \cdot (H) = \{V(\gamma, 1_H) | \gamma \in LaI(F)\}$

La- $\varepsilon (H) = \{V(\vartheta) | \vartheta \in La(H)\}$

We consider the topologies of La-Prim(H) produced, respectively, via these three collections. From Proposition 3.1, we can facilely look that there occurs a topology $\tau^\ast$ say, over La-Prim(H) having La- $\varepsilon^\ast(H)$ as the set of every closed collections if and only if La- $\varepsilon^\ast(H)$ is closed beneath finite union.

In this state, we call the topology $\tau^\ast$ the near-Zariski topology on La-Prim(H). Following [17]. A module H is surname an La-P top module, if La- $\varepsilon^\ast(H)$ result the topology $\tau^\ast$ over La-Prim(H) having La- $\varepsilon^\ast(H)$ as the set of every closed collections if and only if La- $\varepsilon^\ast(H)$ is closed beneath finite union.
Also, La-$\varepsilon$· (H) is closed beneath finite union. Obviously, $\tau'$ is coarser than the near-Zariski topology $\tau^*$, when H be an La-P top module. For each F-module H while $\theta_1, \theta_2 \in \text{La}(H)$ we have the next outcome.

**Proposition 3.3** If $\theta_1 : 1_H = \theta_2 : 1_H$, then $V(\theta_1) = V(\theta_2)$. The converse is true if both $\theta_1$ and $\theta_2$ are primary.

**Proof** First let $\theta_1 : 1_H = \theta_2 : 1_H$, and $\varpi \in V(\theta_1)$. Then $\theta_1 : 1_H \subseteq \varpi : 1_H$ and hence $\theta_2 : 1_H \subseteq \varpi : 1_H$, that is $\varpi \in V(\theta_2)$. Therefore $V(\theta_1) \subseteq V(\theta_2)$. Similarly we get that $V(\theta_2) \subseteq V(\theta_1)$. Therefore $V(\theta_1) = V(\theta_2)$. For the opposite, let $\theta_1, \theta_2 \in \text{La}(H)$ are primary while $V(\theta_1) = V(\theta_2)$.

For $q \in \text{La-Prim}(F)$, by $\text{La-Prim}_q(H)$ we mean the collection of all $\varpi \in \text{La}(H)$ such that $\varpi : 1_H = q$. In other words $\text{La-Prim}_q(H) = \{ \theta \in \text{La}(H) \mid \theta : 1_H = q \}$.

**Proposition 3.4**

(a) $V(\varpi) = \bigcup_{q \in V(\varpi : 1_H)} \text{La-Prim}_q(H)$ for $\varpi \in \text{La}(H)$

(b) $V(\gamma : 1_H) = V^*(\gamma : 1_H)$ for every La-ideal $\gamma$ of F.

Further, if $\theta \in \text{La}(H)$, then $V(\theta) = V((\theta : 1_H).1_H) = V^*((\theta : 1_H).1_H)$.

**Proof** (1) : Suppose that $\varpi \in V(\theta)$. Then $\theta : 1_H \subseteq \varpi : 1_H = q$, and hence $q \in V(\theta : 1_H)$.

Also,

$\varpi \in \text{La-Prim}_q(H) \Rightarrow \varpi \in \bigcup_{q \in V(\varpi : 1_H)} \text{La-Prim}_q(H) \Rightarrow V(\varpi) \subseteq \bigcup_{q \in V(\varpi : 1_H)} \text{La-Prim}_q(H)$

(a)

Now suppose that $\varpi \in \bigcup_{q \in V(\varpi : 1_H)} \text{La-Prim}_q(H)$. Then there occurs $q \in \text{La-Prim}_q(H)$ such that $\theta : 1_H \leq q$ and $\varpi \in \text{La-Prim}_q(H).$ Thus

$\varpi : 1_H = q \Rightarrow \theta : 1_H \subseteq \varpi : 1_H \Rightarrow \varpi \in V(\theta) \Rightarrow \bigcup_{q \in V(\varpi : 1_H)} \text{La-Prim}_q(H) \subseteq V(\theta)$

(b)

Now it follows from (a) and (b).

(2) Suppose that $Q \in V^*(\gamma : 1_H)$. Then we have

$\gamma : 1_H \leq Q \Rightarrow \gamma : 1_H : 1_H \leq Q : 1_H \Rightarrow Q \in V(\gamma : 1_H) \Rightarrow V^*(\gamma : 1_H) \subseteq V(\gamma : 1_H)$

(c)

Let $Q \in V(\gamma : 1_H)$, then $\gamma : 1_H \leq Q : 1_H$.

Clearly, $\gamma \subseteq \gamma : 1_H$. Thus

$\gamma \subseteq Q : 1_H \Rightarrow \gamma : 1_H \subseteq Q \Rightarrow Q \in V^*(\gamma : 1_H) \Rightarrow V(\gamma : 1_H) \subseteq V^*(\gamma : 1_H)$

(d)

Then from (c), (d) the outcome satisfying.

As well, via the preceding debate instantly we get that


Now for $Q \in V(\theta)$, it deduce that $\theta : 1_H \leq Q : 1_H$. Then we get that $(\theta : 1_H).1_H \leq \theta$, and so

$((\theta : 1_H).1_H) : 1_H \leq \theta : 1_H \leq Q : 1_H \Rightarrow Q \in V((\theta : 1_H).1_H) \Rightarrow V(\theta) \subseteq V((\theta : 1_H).1_H)$

(e)

Let $Q \in V^*((\theta : 1_H).1_H)$. Then $(\theta : 1_H).1_H \subseteq Q$, so $\theta : 1_H \subseteq Q : 1_H$. Thus $Q \in V(\theta)$ and
hence
\[ V^*(\langle \varnothing : 1_H \rangle, 1_H) \subseteq V(\varnothing) \quad (f) \]

Consequently from (e) and (f), we get that
\[ V^*(\langle \varnothing : 1_H \rangle, 1_H) \subseteq V(\varnothing) \subseteq V((\varnothing : 1_H), 1_H) \]

Thus
\[ V(\varnothing) = V^*(\langle \varnothing : 1_H \rangle, 1_H) = V((\varnothing : 1_H), 1_H) \]

Note that from Proposition 3.4 we get that La-ε(H) = La-ε(H) ⊆ La-ε*(H).

**Example 3.5**

(1) Let \( H = \mathbb{Z} \) as \( \mathbb{Z} \)-module and suppose that \( La \) is an arbitrary lattice. Let \( q \in \mathbb{Z} \) is prime. For each prime element \( s \in La \), define \( T(s) \in La(\mathbb{Z}) \) by

\[ T(s)(y) = \begin{cases} 1 & \text{if } y \in \langle q \rangle; \\ s & \text{otherwise} \end{cases} \]

Then by Theorem 2.10, \( T(s) \) is a primary \( La \)-submodule of \( H \). Therefore \( La \)-Prim(H) = \( \{ T(s) \mid s \text{ is a prime element of } La \} \), and for \( La = [0, 1] \), then \( La \)-Prim(H) = \( \{ T(s) \mid s \in [0, 1] \text{ while } q \text{ is prime element of } \mathbb{Z} \} \).

(2) Let \( H = \mathbb{R}[x] \) as \( \mathbb{R}[x] \)-module, where \( \mathbb{R} \) is the field of real numbers. For each \( T \in \mathbb{R}[x] \) while each \( s \in La \), defined the fuzzy subset \( T(s) \) of \( \mathbb{R}[x] \) via

\[ T(s)(y) = \begin{cases} 1 & \text{if } y \in \langle q \rangle; \\ s & \text{otherwise} \end{cases} \]

Then by Theorem 2.10, \( T(s) \) primary \( La \)-submodule of \( H \) if and only if \( T \) is irreducible and \( s \) is a prime element of \( La \). Further, for \( La = [0,1] \), we have \( La \)-Prim(H) = \( \{ T(s) \mid q \text{ is irreducible in } \mathbb{R}[y], s \in [0, 1] \} \).

(3) Let \( H \) be an arbitrary \( F \)-module and \( T \) is a prime submodule of \( H \). For each \( s \in La \), define

\[ T(s)(y) = \begin{cases} 1 & \text{if } y \in T; \\ s & \text{otherwise} \end{cases} \]

Then via Theorem 2.10, \( T(s) \) is a primary \( La \)-submodule of \( H \) if and only if \( s \) is a prime element of \( La \). If \( \text{Spec}(La) \) indicate the collection of all prime elements of \( La \), then \( La \)-Prim(H) = \( \{ T(s) \mid s \in \text{Spec}(La) \text{ and } T \text{ be a primary submodule of } H \} \).

(4) If we let \( H = \mathbb{R}[y] \) as \( \mathbb{R} \)-module. Then all proper submodules \( T \) of \( H \), are indicated via \( T < H \), is primary. Then by part (3) \( La \)-Prim(H) = \( \{ T(s) \mid s \in \text{Spec}(La) \text{ and } T < H \} \).

(5) Let \( La = \{0, x, y, 1\} \) is a lattice which is not a chain, that is \( x \) and \( y \) are not similar. Then \( La \)-Prim(H) = \( \emptyset \), for each \( F \)-module \( H \), since \( La \) has not any prime element. This example display that \( La \)-Prim(H) = \( \emptyset \), but \( \text{Prim}(H) \) may be non-empty.

4. The relation between \( La \)-Prim(H) and \( La \)-Prim(F / Ann(H))

Suppose that \( \varnothing \) is a primary \( La \)-submodule of \( H \). Then by Corollary 2.11 we have \( \langle \varnothing : 1_H \rangle \) be a primary \( La \)-ideal of \( F \). Let the quotient ring \( F / \text{Ann}(H) \). We indicate a typical element of \( F / \text{Ann}(H) \) by \( [y] \), where \( y \in F \). Consider the quotient map \( \rho : F \rightarrow F / \text{Ann}(H) \), is defined via \( \rho(y) = [y] \), we indicate the image of \( \varnothing : 1_H \) beneath \( \rho \) by \( \overline{\langle \varnothing : 1_H \rangle} \). In fact, \( \overline{\langle \varnothing : 1_H \rangle}([y]) \)
Proof Suppose that \( \theta \in \text{La}^1 \). Then \( \overline{\theta : 1_H} \) is a primary La-ideal of \( F/\text{Ann}(H) \).

Proposition 4.4 Suppose that \( \theta \in \text{La}^1 \). Then \( \overline{\theta : 1_H} \) is a primary La-ideal of \( F/\text{Ann}(H) \). For each \( F \)-module \( H \) the following assertions are equivalent:

1. \( \sigma \) be injective;
2. for \( \theta, \varpi \in \text{La-Prim}(H) \), if \( V(\theta) = V(\varpi) \), then \( \theta = \varpi \);
3. for every \( q \in \text{La-Prim}(F) \), \( |\text{La-Prim}_p(H)| \leq 1 \).

Proof (1) \( \Rightarrow \) (2): Suppose that \( \theta, \varpi \in \text{La-Prim}(H) \). If \( V(\theta) = V(\varpi) \) then \( \theta : 1_H = \varpi : 1_H \), by Proposition 3.3 and hence \( \overline{\theta : 1_H} = \overline{\varpi : 1_H} \), which lead to that \( \sigma(\theta) = \sigma(\varpi) \). Thus \( \theta = \varpi \), since \( \sigma \) is injective by (1).

(2) \( \Rightarrow \) (3): Suppose that \( \theta, \varpi \in \text{La-Prim}_p(H) \), then \( \theta : 1_H = \varpi : 1_H = q \). Therefore \( V(\theta) = V(\varpi) \) via Proposition 3.3. Then by (2) we have \( \theta = \varpi \), and hence \( |\text{La-Prim}_p(H)| \leq 1 \).

(3) \( \Rightarrow \) (1): Let \( \theta, \nu \in \text{La-Prim}(H) \) and \( \sigma(\theta) = \sigma(\nu) \). Then

\[
\overline{\theta : 1_H} = \overline{\nu : 1_H} \Rightarrow \theta : 1_H = \varpi : 1_H = q \Rightarrow \theta, \varpi \in \text{La-Prim}_p(H) \Rightarrow \theta = \varpi.
\]

That is \( \sigma \) injective.

In the complements we put \( Y = \text{La-Prim}(H) \) and \( \overline{Y} = \text{La-Prim}(F/\text{Ann}(H)) \).

Theorem 4.5 Suppose that \( \sigma \) is the natural map. If \( \sigma \) is inclusive then \( \sigma \) is both closed while open.

Proof Let \( \sigma : Y \rightarrow \overline{Y} \) is the standard function and \( \theta \in Y \). Then via the proof of Proposition 4.3,
\[\sigma^{-1}(V(\overline{\theta} : 1_H)) = V(\overline{\theta} : 1_H). \quad 1_H = V(\theta) \Rightarrow \sigma(V(\theta)) = \sigma \circ \sigma^{-1}(V(\overline{\theta} : 1_H)) = V(\overline{\theta} : 1_H),\]

That is \(\sigma\) is closed. Also we have
\[
\sigma(Y - V(\theta)) = \sigma(\sigma^{-1}(Y)) - \sigma^{-1}(V(\overline{\theta} : 1_H)) = \sigma(\sigma^{-1}(Y - V(\overline{\theta} : 1_H))) = Y - V(\overline{\theta} : 1_H),
\]

That is \(\sigma\) be open.

**Proposition 4.6** Suppose that \(\sigma\) is the standard function from \(Y\) into \(\overline{Y}\) and it is inclusive. Then \(Y\) is linked if and only if \(\overline{Y}\) is linked.

Proof Suppose that \(Y\) is linked. Then \(\overline{Y} = \sigma(Y)\) is linked, since \(\sigma\) be persistent while inclusive. Conversely, let \(\overline{Y}\) is linked but \(Y\) is non-linked. Then \(Y\) includes a non-empty proper subset \(A\) such that \(\sigma(A)\) is both open and closed. We prove that \(\sigma(A)\) is a non-empty proper subset of \(\overline{Y}\). Since \(A\) is open then there occurs \(\overline{\theta} \in \sigma(La(H))\) such that
\[
A = Y \setminus V(\theta).
\]
Thus \(\sigma(A) = \overline{Y} \setminus V(\overline{\theta} : 1_H).\) If \(\sigma(A) = \overline{Y}\) then \(V(\overline{\theta} : 1_H) = \emptyset\), and hence \(\overline{\theta} : 1_H = \chi F / \text{Ann}(H) \Rightarrow \emptyset = 1_H \Rightarrow A = Y \setminus V(\theta) = Y \setminus V(1_H) = Y,
\]
A discrepancy, if \(\sigma(A) = \emptyset\), then we must have \(V(\overline{\theta} : 1_H) = \overline{Y}\), and hence \(\overline{\theta} : 1_H = \chi \emptyset \Rightarrow \emptyset = \chi 0 \Rightarrow A = Y \setminus V(\emptyset) = Y \setminus Y = \emptyset\), which is a discrepancy. Therefore \(\sigma(A)\) is a proper non-empty subset of \(\overline{Y}\) such that it is both open and closed, a discrepancy. Thus \(Y\) is linked.

**Proposition 4.7:** Suppose that \(H\) while \(H'\) is \(F\)-modules. If \(Y = La-\text{Prim}(H), Y' = La-\text{Prim}(H')\) and \(f : H \rightarrow H'\) be an epimorphism, then the function \(g : Y' \rightarrow Y\) is defined via \(g(\theta') = f^{-1}(\theta')\) be persistent.

Proof Let \(\theta \in La(H)\) while \(V(\theta)\) be a closed set in \(Y\). For \(Q \in g^{-1}(V(\theta))\) by Proposition 3.4 (b), we have \(V(\theta) = V^*((\theta; 1_H). 1_H)\).
Thus
\[
Q \in g^{-1}(V^*((\theta; 1_H). 1_H) \Leftrightarrow g(Q) \in V^*((\theta; 1_H). 1_H) \Leftrightarrow (\theta; 1_H). 1_H \subseteq g(Q) = f^{-1}(Q) \Leftrightarrow f((\theta; 1_H). 1_H) \subseteq Q' \Leftrightarrow ((\theta; 1_H). 1_H' \subseteq Q' \Leftrightarrow Q' \in V^*((\theta; 1_H). 1_H) = V((\theta; 1_H). 1_H).
\]
Therefore \(g^{-1}(V(\theta)) = V((\theta; 1_H). 1_H)\), and hence \(g\) is persistent.

5 A basis for the Zariski topology over \(La-\text{Prim}(H)\)

Proposition 5.1 [12] If \(g\) is a homomorphism from \(F\) onto \(F'\), then for each \(y \in F\) and \(\alpha \in \La \setminus \{0\}; g(y \alpha) = (g(y))_\alpha\).

Corollary 5.2 Suppose that \(y \in F\), then for all ideal \(B\) of \(F\), and for all \(\alpha \in \La \setminus \{0\}; \overline{y}_\alpha = (\overline{y})_\alpha\),
where \(\overline{y}_\alpha\) be an \(La\)-point of \(F / B\).

For each \(F\)-module \(H\), we suppose the collection \(\mathcal{C} = \{D(y_\alpha. 1_H) | y \in F, \alpha \in \La \setminus \{0\}\}\) such that
\[
D(y_\alpha. 1_H) = Y \setminus V(y_\alpha. 1_H).
\]
We assumption that if the lattice \(La\) is a chain then \(C\) formation a basis for Zariski topology on \(Y = La-\text{Prim}(H)\).

We suppose the following states:
1) If \(\alpha = 1\) while \(y = 0, D(0_1.1_H) = Y \setminus V(0.1_H) = Y \setminus V(0_H) = \emptyset\).
2) If \(\alpha = 1\) while \(y = 1, D(1_1.1_H) = Y \setminus V(1.1_H) = Y \setminus V(1_H) = Y\).
**Proposition 5.3** If \( \sigma : Y \rightarrow \overline{Y} \) is standard function, then

(a) \( \sigma^{-1}(E(\overline{Y}_\alpha)) = D(\alpha \cdot 1_H) \);

(b) \( \sigma(D(\alpha \cdot 1_H)) \subseteq E(\overline{Y}_\alpha) \). Further if \( \sigma \) is inclusive then the parity satisfies.

**Proof** For (a) we have \( \sigma^{-1}(E(\overline{Y}_\alpha)) = \sigma^{-1}(\overline{Y} \setminus V(\overline{Y}_\alpha)) = Y \setminus \sigma^{-1}(V(\overline{Y}_\alpha)) = Y \setminus V(\alpha \cdot 1_H) = D(\alpha \cdot 1_H) \).

For (b) We have \( \sigma(\sigma^{-1}(E(\overline{Y}_\alpha))) = \sigma(D(\alpha \cdot 1_H)) \) and \( \sigma(\sigma^{-1}(E(\overline{Y}_\alpha))) = E(\overline{Y}_\alpha) \). Then \( \sigma(D(\alpha \cdot 1_H)) \subseteq E(\overline{Y}_\alpha) \).

**Proposition 5.4** If \( a, b \in F \) and \( \alpha, \alpha' \in L_0 \setminus \{0\} \), then \( D(a \cdot 1_H) \cap D(b \cdot 1_H) = \sigma^{-1}(E(\overline{Y}_\alpha)) \cap \sigma^{-1}(E(\overline{Y}_{\alpha'})) = \sigma^{-1}(E(\overline{Y}_\alpha) \cap E(\overline{Y}_{\alpha'})) = \sigma^{-1}(E(\overline{Y}_\alpha \cdot \alpha')) = D((ab) \cdot 1_H) \).

**Theorem 5.5** For each \( F \)-module \( H \), the collection \( C = \{ D(a \cdot 1_H) | x \in F, \alpha \in L_0 \setminus \{0\} \} \) formation a basis for Zariski topology over \( Y = L_0 - \text{Prim}(H) \).

**Proof** Let \( W \) be an arbitrary open set in \( Y \). Then \( W = D(\emptyset) = Y \setminus V(\emptyset) \) for several \( \emptyset \in L_0 \).

Via **Proposition 3.4**, \( V(\emptyset) = V((\emptyset \cdot 1_H) \cdot 1_H) \). By considering \( \gamma = (\emptyset \cdot 1_H) \), then \( V(\emptyset) = V(\gamma \cdot 1_H) \).

As we aforesaid in the basic concepts, we can write \( \gamma = \cup_{c \in Y} c \gamma_c \). Obviously we have \( c \gamma_c = U_{\gamma \in Y} c \gamma_c \). Thus we get that

\[
V(\gamma \cdot 1_H) = V\left((\cup_{c \in Y} c \gamma_c) \cdot 1_H\right) = V\left((\cup_{c \in Y} c \gamma_c) \cdot 1_H\right)
= V\left((\cup_{c \in Y} c \gamma_c) \cdot 1_H\right) \text{ (since } L_0 \text{ is a chain)}
= \cap_{c \in Y} \gamma_c \cdot 1_H
\]

Thus

\[
D(\emptyset) = Y \setminus V(\emptyset) = Y \setminus \cap_{c \in Y} \gamma_c \cdot 1_H
= \cup_{c \in Y} c \gamma_c \cdot 1_H
\]

This proves that \( C \) is a basis for the Zariski topology over \( Y \).

**Proposition 5.6** Suppose that \( H \) is an \( F \)-module. If the standard function \( \sigma \) is inclusive, then \( Y = L_0 - \text{Prim}(H) \) is compact. Proof: Suppose that \( Y = \cup\{D(\gamma \cdot 1_H) | y \in F, \alpha \in L_0 \setminus \{0\} \} \). Then

\[
\bar{Y} = \sigma(Y) = \sigma(\cup\{D(\gamma \cdot 1_H) | y \in F, \alpha \in L_0 \setminus \{0\} \}) = \cup\{\sigma(D(\gamma \cdot 1_H)) | y \in F, \alpha \in L_0 \setminus \{0\} \}
= \cup\{\bar{Y}_\alpha | y \in F, \alpha \in L_0 \setminus \{0\} \} \text{ (since } \sigma \text{ is inclusive)}.
\]
Also, since $\bar{Y}$ is compact, we can write $\bar{Y} = \bigcup_{j=1}^{n} \bar{y}_{aj}$, and hence $\sigma^{-1}(\bar{Y}) = \sigma^{-1}(\bigcup_{j=1}^{n} \bar{(y)}_{aj})$. Thus $Y = \bigcup_{j=1}^{n} \sigma^{-1}(\bar{(y)}_{aj})$, and so $\sigma^{-1}(\bar{(y)}_{aj}) = \bar{(y)}_{aj}$. Therefore $Y$ is compact.

References