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The Continuous Classical Optimal Control Problems for Triple Nonlinear Elliptic Boundary Value Problem

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Abstract

In this research, our aim is to study the optimal control problem (OCP) for triple nonlinear elliptic boundary value problem (TNLEBVP). The Mint-Browder theorem is used to prove the existence and uniqueness theorem of the solution of the state vector for fixed control vector. The existence theorem for the triple continuous classical optimal control vector (TCCOCV) related to the TNLEBVP is also proved. After studying the existence of a unique solution for the triple adjoint equations (TAEqs) related to the triple of the state equations, we derive The Fréchet derivative (FD) of the cost function using Hamiltonian function. Then the theorems of necessity conditions and the sufficient condition for optimality of the constraints problem are proved.

Keywords: Triple nonlinear elliptic value problem, continuous classical optimal control vector, Mint-Browder theorem, triple adjoint equations, Fréchet derivative necessity and sufficient theorems.

1. Introduction

The OCP is one of the most important subject not only in mathematics, but in all branches of science, for instance, in engineering such as robotics [1]. And aeronautics [2]. In the medicine and mathematical biology, such as modeling and optimal controlling the infectious diseases [3]. In the life sciences, such as sustainable forest management [4].

In the past few decades, there were many studies and papers published in OCPs for systems that related to nonlinear ordinary differential equations [5]. or systems related to nonlinear partial differential equation (NLPDEqs) either of: a hyperbolic type [6]. Or of a parabolic type [7]. Or of an elliptic type [8].

or OCP are related to couple of NLPDEqs of: a hyperbolic [9]. Or of hyperbolic but include a boundary control [10]. Or of a parabolic type [11].Or of a parabolic type but includes a boundary control [12]. Or of an elliptic type [13]. Of an elliptic type that includes a Numann boundary control [14]. While other papers deals with the optimal control problems that are related to triple linear partial differential equation of : an elliptic type [15]. Or of an parabolic type [16].

In this work, the Minty-Browder theorem is used to prove the existence theorem for a unique solution (continuous state vector) for the TNLEBVP for fixed TCCOCV, and to state and prove the theorem for the existence TCCOCV related to the TNLEBVP, so as the theorem of the existence of a unique solution of the TAEqs related to the TNLEBVP. The FD of the cost function is derived. At the end the theorem of necessity conditions is stated and proved so as is the sufficient condition theorem for optimality of the constrained problem.

2. The Problem Description

Let Λ be an open (bounded) connected subset in $\mathbb{R}\times\mathbb{R}$ with Lipschitz boundary $\partial\Lambda$. Consider the TCCOC of the TNLEBVP

$$-B_1\,\xi_1+\xi_1-\xi_2-\xi_3+a_1(x,\xi_1,v_1)=a_2(x,v_1),\ in\ \Lambda \quad (1)$$

$$-B_2\,\xi_2+\xi_1+\xi_2+\xi_3+p_1(x,\xi_2,v_2)=p_2(x,v_2),\ in\ \Lambda \quad (2)$$

$$-B_3\,\xi_3+\xi_1-\xi_2+\xi_3+k_1(x,\xi_3,v_3)=k_2(x,v_3),\ in\ \Lambda \quad (3)$$

with the Dirchlet boundary condition

$$\xi_1=\xi_2=\xi_3=0\ ,\ in\ \partial\Lambda \quad (4)$$

Where $B_r\xi_r=\sum_{i,j}^{2}\frac{\partial}{\partial x_i}\left(b_{ij}\frac{\partial\xi_r}{\partial x_j}\right)$, $r=1,2,3$, $b_{ij}=b_{ij}(x)\in L^\infty(\Lambda)$, $\forall i,j=1,2$, $x=(x_1,x_2)$

$\vec{\xi}=(\xi_1(x),\xi_2(x),\xi_3(x))\in\left(H_0^2(\Lambda)\right)^3$ is the classical solution of the system (1)-(4), $\vec{v}=(v_1(x),v_2(x),v_3(x))\in\left(L^2(\Lambda)\right)^3$ is the CCV, the functions $a_1(x,\xi_1,v_1)$, $p_1(x,\xi_2,v_2)$ and $k_1(x,\xi_3,v_3)$ are defined on $\Lambda\times\mathbb{R}\times V_1$, $\Lambda\times\mathbb{R}\times V_2$ and $\Lambda\times\mathbb{R}\times V_3$ respectively, and the functions $a_2(x,v_1)$, $p_2(x,v_2)$ and $k_2(x,v_3)$ are defined on $\Lambda\times V_1$, $\Lambda\times V_2$ and $\Lambda\times V_3$ respectively with $V_1,V_2,V_3\subset\mathbb{R}$.

The control constraint is $(v_1,v_2,v_3)\in U_1\times U_2\times U_3=\vec{U}$, $\vec{U}\subset\left(L^2(\Lambda)\right)^3$, where $\vec{U}$ is the control set has the form

$$\vec{U}=\{\vec{u}\in\left(L^2(\Lambda)\right)^3|\vec{u}=(u_1,u_2,u_3)\in V_1\times V_2\times V_3=\vec{V}\ a.e.\ in\ \Lambda\}$$

With $\vec{V}\subset\mathbb{R}^3$ that is convex and compact set.

The cost function is

$$Y_0(\vec{v})=\int_\Lambda y_{01}(x,\xi_1,v_1)dx+\int_\Lambda y_{02}(x,\xi_2,v_2)dx+\int_\Lambda y_{03}(x,\xi_3,v_3)dx \quad (5)$$

The state –control constraints are

$$Y_1(\vec{v})=\int_\Lambda y_{11}(x,\xi_1,v_1)dx+\int_\Lambda y_{12}(x,\xi_2,v_2)dx+\int_\Lambda y_{13}(x,\xi_3,v_3)dx=0 \quad (6)$$

$$Y_2(\vec{v})=\int_\Lambda y_{21}(x,\xi_1,v_1)dx+\int_\Lambda y_{22}(x,\xi_2,v_2)dx+\int_\Lambda y_{23}(x,\xi_3,v_3)dx\le 0 \quad (7)$$

The set of the admissible controls is $\vec{U}_A=\{\vec{v}\in\vec{U}|Y_1(\vec{v})=0,Y_2(\vec{v})\le 0\}$

The TCCOC problem is to minimize the cost function (5) subject to the state constraints of (6) and (7), i.e. to find $\vec{v}$ such that $\vec{v}\in\vec{U}_A$ and $Y_0(\vec{v})=\min_{\vec{u}\in\vec{U}_A}Y_0(\vec{u})$.

Let $\vec{W}=W_1\times W_2\times W_3=H_0^1(\Lambda)\times H_0^1(\Lambda)\times H_0^1(\Lambda)$, y $\|w\|_1$ and $\|\vec{w}\|_1$ are denoted by the norm in $H^1(\Lambda)$ and $(\left(H^1(\Lambda)\right)^3$ respectively, y $\|w\|_0$ ($\|\vec{w}\|_0$) are denoted the norm in $L^2(\Lambda)$

and in $(L^2(\Lambda))^3$ respectively and the inner product in W is denoted by (w, w), with $\|\vec{w}\| = \|w_1\| + \|w_2\| + \|w_3\|$, $\overrightarrow{W^*}$ is dual of $\overrightarrow{W}$.

3. Weak Formulation of the TNLEBVP

The weak form (WF) of (1)-(4) is obtained through multiplying both sides of Equations (1)-(3) by$w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$ respectively, then integrating the obtained equations. Finally, using the generalize Green's theorem for the 1st term in left hand side (L.H.S) of the three obtained equations, once get $\forall w_1, w_2, w_3 \in W_2$

$$b_1(\xi_1, w_1) + (\xi_1, w_1) - (\xi_2, w_1) - (\xi_3, w_1) + (a_1(\xi_1, v_1), w_1) = (a_2(v_1), w_1) \quad (8)$$

$$b_2(\xi_2, w_2) + (\xi_1, w_2) + (\xi_2, w_2) + (\xi_3, w_2) + (p_1(\xi_2, v_2), w_2) = (p_2(v_2), w_2) \quad (9)$$

$$b_3(\xi_3, w_3) + (\xi_1, w_3) - (\xi_2, w_3) + (\xi_3, w_3) + (k_1(\xi_3, v_3), w_3) = (k_2(v_3), w_3) \quad (10)$$

where $b_r(\xi_r, w_r) = \int_\Lambda \sum_{i,j=1}^{2} b_{ij} \frac{\partial \xi_r}{\partial x_i} \cdot \frac{\partial w_r}{\partial x_j} dx$, $(\xi_r, w_p) = \int_\Lambda \xi_r w_p dx$, $(\Theta, w_r) = \int_\Lambda \Theta\, w_r dx$,

with $\Theta = a_\iota$ *or* p_ι *or* k_ι , $r, p = 1,2,3$, $\iota = 1,2$.

By blending to gather equations (8), (9) and (10), once get

$$B(\vec{\xi}, \vec{w}) + (a_1(\xi_1, v_1), w_1) + (p_1(\xi_2, v_2), w_2) + (k_1(\xi_3, v_3), w_3) = (a_2(v_1), w_1) + (p_2(v_2), w_2) + (k_2(v_3), w_3) \quad (11)$$

where $B(\vec{\xi}, \vec{w}) = b_1(\xi_1, w_1) + (\xi_1, w_1) - (\xi_2, w_1) - (\xi_3, w_1) + b_2(\xi_2, w_2) + (\xi_1, w_2) + (\xi_2, w_2) + (\xi_3, w_2) + b_3(\xi_3, w_3) + (\xi_1, w_3) - (\xi_2, w_3) + (\xi_3, w_3)$

Hypotheses A:

a) $B(\vec{\xi}, \vec{w})$ is coercive, i.e. $\frac{B(\vec{\xi}, \vec{\xi})}{\|\vec{\xi}\|_1} \geq \epsilon \|\vec{\xi}\|_1 > 0$, $\vec{\xi} \in \overrightarrow{W}$

b) $|B(\vec{\xi}, \vec{w})| \leq \epsilon_1 \|\vec{\xi}\|_1 \|\vec{w}\|_1$, $\epsilon_1 > 0$

c) the functions $a_1(x, \xi_1, v_1)$, $p_1(x, \xi_2, v_2)$ and $k_1(x, \xi_3, v_3)$ are of Carathéodory type on $\Lambda\times \mathbb{R} \times V_1$, $\Lambda\times \mathbb{R} \times V_2$ and $\Lambda \times \mathbb{R} \times V_3$ respectively and satisfy the following sublinearity conditions with respect to (w.r.t.)(ξ_1, v_1), (ξ_2, v_2) and (ξ_3, v_3) respectively.

$$|a_1(x, \xi_1, v_1)| \leq \vartheta_1(x) + c_1|\xi_1| + \bar{c}_1|v_1| \ , \ |p_1(x, \xi_2, v_2)| \leq \vartheta_2(x) + c_2|\xi_2| + \bar{c}_2|v_2| ,$$
$$|k_1(x, \xi_3, v_3)| \leq \vartheta_3(x) + c_3|\xi_3| + \bar{c}_3|v_3|$$

$\forall (x, \xi_i, v_i) \in \Lambda \times \mathbb{R} \times U_i$ with $\vartheta_i \in L^2(\Lambda)$, $c_i, \bar{c}_i \geq 0$, $i = 1,2,3$.

d) $a_1(x, \xi_1, v_1)$, $p_1(x, \xi_2, v_2)$ and $k_1(x, \xi_3, v_3)$ are monotone w.r.t. ξ_1, ξ_2, ξ_3 respectively for each $x \in \Lambda$, $v_1 \in V_1$, $v_2 \in V_2$, $v_3 \in V_3$.

e) $a_1(x, 0, v_1) = 0$, $\forall x \in \Lambda, v_1 \in V_1$, $p_1(x, 0, v_2) = 0$, $\forall x \in \Lambda, v_2 \in V_2$, $k_1(x, 0, v_3) = 0$, $\forall x \in \Lambda, v_3 \in V_3$.

f) the functions $a_2(x, v_1)$, $p_2(x, v_2)$ and $k_2(x, v_3)$ are of Carathéodory type on $\Lambda\times V_1$, $\Lambda\times V_2$ and $\Lambda \times V_3$ respectively and satisfy the following conditions

$$|a_2(x, v_1)| \leq \vartheta_4(x) + c_4|v_1| , \quad |p_2(x, v_2)| \leq \vartheta_5(x) + c_5|v_2| , \quad |k_2(x, v_3)| \leq \vartheta_6(x) + c_6|v_3|$$

$\forall (x, v_i) \in \Lambda \times U_i$, $i = 1,2,3$ with $\vartheta_r \in L^2(\Lambda)$, $c_r \geq 0$, $r = 4,5,6$.

Theorem 3.1 (The Minty-Browder theorem)[17]. let W be a reflexive Banach space and $D: W \to W^*$ be a nonlinear continuous map such that

$$(Dw_1 - Dw_2, w_1 - w_2) > 0, \ \forall w_1, w_2 \in W, w_1 \neq w_2 \quad \textit{and} \quad \lim_{\|w\| \to \infty} \frac{(Dw, w)}{\|w\|} = \infty$$

Then the equation $D\xi = a$ has a unique (solution) $\xi \in W$ for every $a \in W^*$.

Proposition 3.1 [18]. Let $a: \Lambda \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is of Carathéodory type, and the functional A is defined by $A(\xi) = \int_\Lambda a(x, \xi(x))\, dx$, where Λ is a measurable subset of $\mathbb{R}^n$, and suppose that

$\|a(x,\xi)\| \leq \vartheta(x) + \eta(x)\|\xi\|^\alpha, \forall (x,\xi) \in \Lambda \times \mathbb{R}^n, \xi \in L^P(\Lambda \times \mathbb{R}^n)$

where $\vartheta \in L^1(\Lambda \times \mathbb{R}), \eta \in L^{\frac{P}{P-\alpha}}(\Lambda \times \mathbb{R})$, and $\alpha \in [0, P]$, if $P \in [1, \infty)$, and $\eta \equiv 0$, if $P = \infty$. Then A is continuous on $L^P(\Lambda \times \mathbb{R}^n)$.

Theorem 3.2: In addition to the hypo.(A-a&d), If at least one of the functions a_1, p_1 or k_1 in hypo.(A-d) is strictly monotone. Then for any fixed control$\vec{v} \in \vec{U}_A$, the WF (11) has a unique solution $\vec{\xi} \in \vec{W}$.

Proof: let $\bar{D}: \vec{W} \to \vec{W}^*$, then the WF (11) is rewriting as

$$\left(\bar{D}(\vec{\xi}), \vec{w}\right) = (a_2(v_1), w_1) + (p_2(v_2), w_2) + (k_2(v_3), w_3) \tag{12}$$

where$\left(\bar{D}(\vec{\xi}), \vec{w}\right) = B(\vec{\xi}, \vec{w}) + (a_1(\xi_1, v_1), w_1) + (p_1(\xi_2, v_2), w_2) + (k_1(\xi_3, v_3), w_3)$ (13)

Then $\bar{D}$ satisfies the following:

i) $\bar{D}$ is coercive from hypo. (A-a&e&d)

ii) from hypotheses (A-a&c) and using proposition3.1 the maping $\vec{\xi} \longmapsto \left(\bar{D}(\vec{\xi}), \vec{w}\right)$ is continuous w.r.t. $\vec{\xi}$.

iii) from hypotheses (A-a&b) and (i) $\bar{D}$ is strictly monotone w.r.t. $\vec{\xi}$.

Hence by Theorem3.1, there exists a unique weak solution $\vec{\xi} \in \vec{W}$ of (11).

4. Existence of the TCCOC

Lemma 4.1: If the functions(a_1& a_2), (p_1& p_2) and (k_1&k_2) are Lipschitz w.r.t. v_1, v_2 and v_3 respectively, moreover the hypothesis (A). Then the transformation $\vec{v} \longmapsto \vec{\xi}_{\vec{v}}$ from $\vec{U}$ to $\left(L^2(\Omega)\right)^3$is Lipschitz continuous.

Proof: let $\vec{\check{v}} = (\check{v}_1, \check{v}_2, \check{v}_3) \in \vec{U}$ be a given control of WF(8)-(10) with its corresponding state solution$\left(\check{\xi}_1, \check{\xi}_2, \check{\xi}_3\right)$, then by subtracting (8)-(10) from the equations which are obtained from substituting $\delta\xi_i = \check{\xi}_i - \xi_i$, $\delta v_i = \check{v}_i - v_i$ $(i = 1,2,3)$ in (8)-(10) respectively, setting $w_1 = \delta\xi_1$, $w_2 = \delta\xi_2$ and $w_3 = \delta\xi_3$ and blending together the obtained equation, to give

$$\begin{aligned}
&b_1(\delta\xi_1, \delta\xi_1) + (\delta\xi_1, \delta\xi_1) + b_2(\delta\xi_2, \delta\xi_2) + (\delta\xi_1, \delta\xi_2) + b_3(\delta\xi_3, \delta\xi_3) + (\delta\xi_1, \delta\xi_3)\\
&+(a_1(\xi_1 + \delta\xi_1, v_1 + \delta v_1) - a_1(\xi_1, v_1 + \delta v_1, \delta\xi_1)\\
&+(p_1(\xi_2 + \delta\xi_2, v_2 + \delta v_2) - p_1(\xi_2, v_2 + \delta v_2), \delta\xi_2)\\
&+(k_1(\xi_3 + \delta\xi_3, v_3 + \delta v_3) - k_1(\xi_3, v_3 + \delta v_3), \delta\xi_3)\\
&= -(a_1(\xi_1, v_1 + \delta v_1) - a_1(\xi_1, v_1), \delta\xi_1) - (p_1(\xi_2, v_2 + \delta v_2) - p_1(\xi_2, v_2), \delta\xi_2)\\
&-(k_1(\xi_3, v_3 + \delta v_3) - k_1(\xi_3, v_3), \delta\xi_3) + (a_2(v_1 + \delta v_1), \delta\xi_1) - (a_2(v_1), \delta\xi_1)\\
&+(p_2(v_2 + \delta v_2), \delta\xi_2) - (p_2(v_2), \delta\xi_2) + (k_2(v_3 + \delta v_3), \delta\xi_3) - (k_2(v_3), \delta\xi_3)
\end{aligned} \tag{14}$$

By hypotheses (A-a&d), one has:

$$\begin{aligned}
\epsilon \|\overrightarrow{\delta\xi}\|_1^2 \leq &\left|\int_\Lambda \left(a_1(x, \xi_1, v_1 + \delta v_1) - a_1(x, \xi_1, v_1)\right)\delta\xi_1\, dx\right| + \left|\int_\Lambda (p_1(x, \xi_2, v_2 + \delta v_2) - p_1(x, \xi_2, v_2))\delta\xi_2 dx\right| \\
&+ \left|\int_\Lambda \left(k_1(x, \xi_3, v_3 + \delta v_3) - k_1(x, \xi_3, v_3)\right)\delta\xi_3 dx\right| + \left|\int_\Lambda \left(a_2(x, v_1 + \delta v_1) - a_2(x, v_1)\right)\delta\xi_1 dx\right| \\
&+ \left|\int_\Lambda \left(p_2(x, v_2 + \delta v_2) - p_2(x, v_2)\right)\delta\xi_2 dx\right| + \left|\int_\Lambda \left(k_2(x, v_3 + \delta v_3) - k_2(x, v_3)\right)\delta\xi_3 dx\right|
\end{aligned}$$

By using Lipchitz condition on (a_1& a_2), (p_1& p_2) and (k_1&k_2) w.r.t. v_1, v_2 ,v_3 respectively and Cauch-Schwarz Inequality (C-S-I) of the obtained inequality, to get:

$\|\overrightarrow{\delta\xi}\|_1^2 \leq L_4\|\delta v_1\|_0\|\delta\xi_1\|_0 + L_5\|\delta v_2\|_0\|\delta\xi_2\|_0 + L_6\|\delta v_3\|_0\|\delta\xi_3\|_0 \implies$

$\|\overrightarrow{\delta\xi}\|_0 \leq \breve{L}\|\overrightarrow{\delta v}\|_0$, with $L_4 = max\left(\frac{L_1}{\epsilon}, \frac{\bar{L}_1}{\epsilon}\right), L_5 = max\left(\frac{L_2}{\epsilon}, \frac{\bar{L}_2}{\epsilon}\right),\ L_6 = max\left(\frac{L_3}{\epsilon}, \frac{\bar{L}_3}{\epsilon}\right)$ (15)

Hypotheses B:

Suppose that $y_{\ell i}$ ($\forall\ \ell = 0,1,2$ & $i = 1,2,3$) is of Carathéodory type on $\Lambda \times \mathbb{R} \times V_i$, satisfies the following condition w.r.t.(ξ_i, v_i), i.e.

$|y_{\ell i}(x, \xi_i, v_i)| \leq \vartheta_{\ell i}(x) + c_{\ell i}\xi_i^2 + \breve{c}_{\ell i}v_i^2$, where$(\xi_i, v_i) \in \mathbb{R} \times v_i$,$\vartheta_\ell \in L^1(\Lambda)$ and $c_{\ell i}, \breve{c}_{\ell i} \geq 0$.

Lemma 4.2: With hypotheses (B), the functional $\vec{v} \longmapsto Y_\ell(\vec{v})$,$(\forall \ell = 0,1,2,)$ defines on $\left(L^2(\Lambda)\right)^3$is continuous.

Proof: hypotheses (B) and proposition 3.1, gives that $\int_\Lambda y_{\ell i}(x, \xi_i, v_i)\, dx (\forall \ell = 0,1,2, \& \ i = 1,2,3)$, is continuous on $L^2(\Lambda)$. Hence $Y_\ell(\vec{v})$ is continuous on$\left(L^2(\Lambda)\right)^3$.

Lemma 4.3[18]**.** Let $y : \Lambda \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is of Carathéodory type on $\Lambda \times \mathbb{R}^2$, with $\left| y(x, \xi, v) \right| \leq \eta(x) + \mathbb{C}y^2 + \mathbb{C}'u^2$, where $\eta \in L^1(\Lambda, \mathbb{R}), \mathbb{C}, \mathbb{C}' \geq 0$.

Then $\int_\Lambda y(x, \xi, v)dx$ is continuous on $L^2(\Lambda, \mathbb{R}^2)$, with $v \in V$, $V \subset \mathbb{R}$ is compact.

Theorem 4.1: In addition to hypotheses (A & B), we suppose that the set of controls $\vec{U}$, with $\vec{V}$ is convex and compact, $\vec{U}_A \neq \phi$, where a_1, p_1 and k_1 are independent of v_1, v_2 and v_3 respectively, and a_2, p_2 and k_2 are linear w.r.t. v_1, v_2 and v_3 respectively, i.e.

$a_1(x, \xi_1, v_1) = a_1(x, \xi_1)$, $p_1(x, \xi_2, v_2) = p_1(x, \xi_2)$, $k_1(x, \xi_3, v_3) = k_1(x, \xi_3)$

$a_2(x, v_1) = a_2(x)v_1$, $p_2(x, v_2) = p_2(x)v_2$, $k_2(x, v_3) = k_2(x)v_3$, such that

$|a_1(x, \xi_1)| \leq \vartheta_1(x) + \hat{c}_1|\xi_1|$, $|p_1(x, \xi_2)| \leq \vartheta_2(x) + \hat{c}_2|\xi_2|$, $|k_1(x, \xi_3)| \leq \vartheta_3(x) + \hat{c}_1|\xi_3|$

where $\vartheta_1, \vartheta_2, \vartheta_3 \in L^2(\Lambda)$ and $\hat{c}_1, \hat{c}_2, \hat{c}_3 \geq 0$,$|a_2(x)| \leq n_1$, $|p_2(x)| \leq n_2$, $|k_2(x)| \leq n_3$

y_{1i} is independent ofv_i and $y_{\ell i}$(for $l = 0,2$ and $i = 1,2,3$) is convex w.r.t. v_i for fixed ξ_i, then there exists TCCOCV.

Proof: Since $\vec{V}$ is convex and compact, then $\vec{U}$ is weakly compact.

Since $\vec{U}_A \neq \emptyset$ then there exists $\vec{u} \in \vec{U}_A$ and a minimum sequence$\{\vec{v}_n\} = \{(v_{1n}, v_{2n}, v_{3n})\} \in \vec{U}_A$, such that$\forall\ \vec{v}_n \in \vec{U}_A$,$\forall n$: $\lim\limits_{n\to\infty} Y_0(\vec{v}_n) = \inf\limits_{\vec{u}\in\vec{U}_A} Y_0(\vec{u})$.

Since $\vec{U}$ is weakly compact, then there exists a subsequence of $\{\vec{v}_n\}$, (let it be again $\{\vec{v}_n\}$) which converges weakly to some $\vec{v} \in \vec{U}$, i.e. $\vec{v}_n \longrightarrow \vec{v}$ weakly in$\left(L^2(\Lambda)\right)^3$and $\|\vec{v}_n\|_0 \leq \tilde{c}$, $\forall n$.

Now, by using (12), hypotheses and C-S-I, give

$$\begin{aligned}
\epsilon\|\vec{\xi}_n\|_1^2 &\leq (\bar{D}(\vec{\xi}), \vec{\xi}) = (a_2(x, v_{1n}), \xi_{1n}) + (p_2(x, v_{2n}), \xi_{2n}) + (k_2(x, v_{3n}), \xi_{3n}) \\
&\leq |(a_2(x)v_{1n}, \xi_{1n})| + |p_2(x)v_{2n}, \xi_{2n}| + |(k_2(x)v_{3n}, \xi_{3n})| \\
&\leq n_1c_1\|\xi_{1n}\|_0 + n_2c_2\|\xi_{2n}\|_0 + n_3c_3\|\xi_{3n}\|_0 \\
&\leq (n_1c_1 + n_2c_2 + n_3c_3)\|\vec{\xi}_n\|_1 = \omega\|\vec{\xi}_n\|_1, \text{ where } \omega = \max(n_1c_1, n_2c_2, n_3c_3) > 0
\end{aligned}$$

Then $\|\vec{\xi}_n\|_1 \leq \mu$, for each n with $\mu = \frac{\omega}{\epsilon} > 0$ (i.e. $\vec{\xi}_n$ is bounded $\forall$n)

By Alaoglu theorem(Al.Th.) [19]. there exists a subsequence of $\{\vec{\xi}_n\}$, (let it be again $\{\vec{\xi}_n\}$)such that $\vec{\xi}_n \to \vec{\xi}$ weakly in $\vec{W}$, which mean that $\vec{\xi}_n \longrightarrow \vec{\xi}$ weakly in $(L^2(\Lambda))^3$, then by compactness theorem(Rellich–Kondrachov [20].)$\xi_{in} \longrightarrow \xi_i$ strongly in$(L^2(\Lambda))^3$.Since for each n, $\vec{\xi}_n = (\xi_{1n}, \xi_{2n}, \xi_{3n})$ satisfies (11),i.e.

$$B(\vec{\xi}_n, \vec{w}) + (a_1(\xi_{1n}), w_1) + (p_1(\xi_{2n}), w_2) + (k_1(\xi_{3n}), w_3) = (a_2(x)v_1, w_1) + (p_2(x)v_{2n}, w_2) + (k_2(x)v_{3n}, w_3) \quad (16)$$

Let $(w_1, w_2, w_3) \in (C(\bar{\Lambda}))^3$, to show that (16) converges to (17),such that

$$B(\vec{\xi}, \vec{w}) + (a_1(\xi_1, v_1), w_1) + (p_1(\xi_2, v_2), w_2) + (k_1(\xi_3, v_3), w_3) = (a_2(v_1), w_1) + (p_2(v_2), w_2) + (k_2(v_3), w_3) \quad (17)$$

i. Since $\xi_{in} \longrightarrow \xi_i$ *weakly in* $W_i \xRightarrow{\forall i=1,2,3} \xi_{in} \longrightarrow \xi_i$ weakly in $L^2(\Lambda)$ and $\frac{\partial \xi_{in}}{\partial x_i} \longrightarrow \frac{\partial \xi_i}{\partial x_i}$ weakly in $L^2(\Lambda)$

ii. from the hypotheses on $a_2(x, \xi_{1n}), p_2(x, \xi_{2n})$ *and* $k_2(x, \xi_{3n})$ and by using the result of lemma4.2, give that $\int_\Omega a_1(x, \xi_{1n}) w_1 dx$, $\int_\Lambda p_1(x, \xi_{2n}) w_2\, dx$ and $\int_\Lambda k_1(x, \xi_{3n}) w_3 dx$ are continuous w.r.t. ξ_{1n}, ξ_{2n} and ξ_{3n}respectively since $\xi_{in} \longrightarrow \xi_i$ strongly in $(L^2(\Lambda))^3$, then the L.H.S of (16) →L.H.S of (17).

Also the convergence for the R.H.S of (16) to the R.H.S of (17) is obtained through $(v_{in} \to v_i)$ *weakly in* $L^2(\Lambda)$, $(i = 1,2,3)$.

But $(C(\bar{\Lambda}))^3$ is dense in $\vec{W}$, which gives $\vec{\xi}_n \longrightarrow \vec{\xi} = \vec{\xi}_{\vec{v}}$ is a solution of the state equations in $\vec{W}$.

From lemma4.2, $Y_\ell(\vec{V})$ is continuous on $(L^2(\Lambda))^3$, for each $\ell = 0,1,2$.

From the hypotheses on $y_{\ell i}$($for\ \ell = 0,1,2$ *and* $i = 1,2,3$), and $\xi_{in} \longrightarrow \xi_i$ strongly in $L^2(\Lambda)$, then $Y_1(\vec{v}) = \lim_{n\to\infty} Y_1(\vec{v}_n)$, hence $Y_1(\vec{v}) = 0$.

Now, to prove $Y_\ell(\vec{v})$, $(\ell = 0,2)$ is W.L.Sc. w.r.t. (ξ_i, v_i), $(i = 1,2,3)$.

From hypotheses B, $(v_{1n}, v_{2n}, v_{3n}) \in \vec{V}$ almost everywhere (a.e.) in Λ and $\vec{V}$ is compact, hence $Y_\ell(\vec{v})$ is satisfied the hypotheses of lemma4.3, and gets that

$$\int_\Lambda y_{\ell i}(x, \xi_{in}, v_{in}) dx \longrightarrow \int_\Lambda y_{\ell i}(x, \xi_i, v_{in}) dx$$

Since$y_{\ell i}(x, \xi_i, v_i)$ is convex w.r.t. v_i, then$\int_\Lambda y_{\ell i}(x, \xi_i, v_i) dx$ is W.L.S. w.r.t.v_i, i.e.

$$\int_\Lambda y_{\ell i}(x, \xi_i, v_i) dx \le \underline{\lim}_{n\to\infty} \int_\Lambda y_{\ell i}(x, \xi_i, v_{in}) dx$$
$$= \underline{\lim}_{n\to\infty} \int_\Lambda (y_{\ell i}(x, \xi_{in}, v_{in}) - y_{\ell i}(x, \xi_{in}, v_{in})) dx + \underline{\lim}_{n\to\infty} \int_\Lambda y_{\ell i}(x, \xi_i, v_{in}) dx$$
$$= \underline{\lim}_{n\to\infty} \int_\Lambda (y_{\ell i}(x, \xi_{in}, v_{in}) dx$$

Hence $Y(\vec{v}) \le \underline{\lim}_{n\to\infty} Y_0(\vec{v}_n) = \lim_{n\to\infty} Y_0(\vec{v}_n) = \inf_{\vec{u}\in\vec{U}_A} Y_0(\vec{u}) \Longrightarrow \vec{v}$ is an optimal control

5. The Necessary and the Sufficient Conditions for Optimality Hypotheses C:

a) The functions $a_{1\xi_1}, a_{1v_1}, p_{1\xi_2}, p_{1v_2}, k_{1\xi_3}, k_{1v_3}$ are of the Carathéodory type on $\Lambda\times\mathbb{R}\times\mathbb{R}$ and satisfy for $x \in \Lambda$ and $d_i, j_i \ge 0$, $(i = 1,2,3)$:

$$|a_{1\xi_1}(x, \xi_1, v_1)| \le d_1, \quad |p_{1\xi_2}(x, \xi_2, v_2)| \le d_2, \quad |k_{1\xi_3}(x, \xi_3, v_3)| \le d_3,$$
$$|a_{1v_1}(x, \xi_1, v_1)| \le j_1, \quad |p_{1v_2}(x, \xi_2, v_2)| \le j_2, \quad |k_{1v_3}(x, \xi_3, v_3)| \le j_3$$

b) The functions $a_{2v_1}, p_{2v_2}, k_{2v_3}$ are of the Carathéodory type on $\Lambda\times\mathbb{R}$,with

$$|a_{2v_1}(x, v_1)| \le q_1, \quad |p_{2v_2}(x, v_2)| \le q_2, \quad |k_{2v_3}(x, v_3)| \le q_3$$

where $x \in \Lambda$ and $q_i \ge 0$, $(i = 1,2,3)$.

c) The functions $y_{\ell i \xi_i}$, $y_{\ell i v_i}$($\forall \ell = 0,1,2$ & $i = 1,2,3$) are of the Carathéodory

type on $\Lambda\times\mathbb{R}\times\mathbb{R}$ and satisfy the following conditions for $\eta_{\ell i},\hat{\eta}_{\ell i}\in L^2(\Lambda)$:
$|y_{\ell i\xi_i}|\le\eta_{\ell i}+\Upsilon_{\ell i}|\xi_i|+\Upsilon_{\ell i}|v_i|$ And $|y_{\ell i v_i}|\le\hat{\eta}_{\ell i}+\hat{\Upsilon}_{\ell i}|\xi_i|+\hat{\Upsilon}_{\ell i}|v_i|$, with $\Upsilon_{\ell i},\hat{\Upsilon}_{\ell i}\ge 0$,

Theorem 5.1: With hypotheses A, B and C, the Hamiltonian is:

$$H(x,\vec{\xi},\vec{\zeta},\vec{v})=\zeta_1\big(a_2(x,v_1)-a_1(x,\xi_1,v_1)\big)+y_{01}(x,\xi_1,v_1)+\zeta_2\big(p_2(x,v_2)-p_1(x,\xi_2,v_2)\big)+y_{02}(x,\xi_2,v_2)+\zeta_3\big(k_2(x,v_3)-k_1(x,\xi_3,v_3)\big)+y_{03}(x,\xi_3,v_3)$$

The adjoint vector $(\zeta_1,\zeta_2,\zeta_3)=(\zeta_{1v_1},\zeta_{2v_2},\zeta_{3v_3})$ "equations "of (3.1- 3.4) are:

$$-B_1\zeta_1+\zeta_1+\zeta_2+\zeta_3+\zeta_1 a_{1\xi_1}(x,\xi_1,v_1)=y_{01\xi_1}(x,\xi_1,v_1)\quad, in\ \Lambda \tag{18}$$

$$-B_2\zeta_2-\zeta_1+\zeta_2-\zeta_3+\zeta_2 p_{1\xi_2}(x,\xi_2,v_2)=y_{02\xi_2}(x,\xi_2,v_2)\quad, in\ \Lambda \tag{19}$$

$$-B_3\zeta_3-\zeta_1+\zeta_2+\zeta_3+\zeta_3 k_{1\xi_3}(x,\xi_3,v_3)=y_{03\xi_3}(x,\xi_3,v_3)\quad, in\ \Lambda \tag{20}$$

$$\zeta_1=\zeta_2=\zeta_3=0\quad on\ \partial\Lambda \tag{21}$$

Then the FD of Y_0 is given by:

$$\vec{Y}_0'(\vec{v})\,\overrightarrow{\delta v}=\int_\Lambda H_{\vec{v}}^{\,T}.\overrightarrow{\delta v}\,dx,\qquad H_{\vec{v}}=\begin{pmatrix}H_{v_1}(x,\vec{\xi},\vec{\zeta},\vec{v})\\ H_{v_2}(x,\vec{\xi},\vec{\zeta},\vec{v})\\ H_{v_3}(x,\vec{\xi},\vec{\zeta},\vec{v})\end{pmatrix}=\begin{pmatrix}\zeta_1(a_{2v_1}-a_{1v_1})+y_{1v_1}\\ \zeta_2(p_{2v_2}-p_{1v_2})+y_{2v_2}\\ \zeta_3(k_{2v_3}-k_{1v_3})+y_{3v_3}\end{pmatrix}$$

Proof: Rewriting the TAEqs (18)-(20) by their WF and then blending them together:

$$\bar{\bar{B}}(\vec{\zeta},\vec{w})+\big(\zeta_1 a_{1\xi_1}(\xi_1,v_1),w_1\big)+\big(\zeta_2 p_{1\xi_2}(\xi_2,v_2),w_2\big)+\big(\zeta_3 k_{1\xi_3}(\xi_3,v_3),w_3\big)=\big(y_{01\xi_1}(\xi_1,v_1),w_1\big)+\big(y_{02\xi_2}(\xi_2,v_2),w_2\big)+y_{03\xi_3}(\xi_3,v_3),w_3\big) \tag{22}$$

where $\bar{\bar{B}}(\vec{\zeta},\vec{w})=b_1(\zeta_1,w_1)+(\zeta_1,w_1)+(\zeta_2,w_1)+(\zeta_3,w_1)+b_2(\zeta_2,w_2)-(\zeta_1,w_2)+(\zeta_2,w_2)-(\zeta_3,w_2)+b_3(\zeta_3,w_3)-(\zeta_1,w_3)+(\zeta_2,w_3)+(\zeta_3,w_3)$

The WF of the TAEqs (22) has a unique solution; this can be proved using the same way which is used to prove the WF of the state equation (11).

Now by substituting $\vec{w}=\overrightarrow{\delta\zeta}$ in (22), once has:

$$\bar{\bar{B}}(\vec{\zeta},\overrightarrow{\delta\zeta})+\big(\zeta_1 a_{1\xi_1}(\xi_1,v_1),\delta\zeta_1\big)+\big(\zeta_2 p_{1\xi_2}(\xi_2,v_2),\delta\zeta_2\big)+\big(\zeta_3 k_{1\xi_3}(\xi_3,v_3),\delta\zeta_3\big)$$
$$(23)=\big(y_{01\xi_1}(\xi_1,v_1),\delta\zeta_1\big)+\big(y_{02\xi_2}(\xi_2,v_2),\delta\zeta_2\big)+\big(y_{03\xi_3}(\xi_3,v_3),\delta\zeta_3\big)$$

Setting the solution $\vec{\xi}+\overrightarrow{\delta\xi}$ in (8)-(10) then subtracting (8)-(10) from those equations which are obtained by setting($\vec{\xi}+\overrightarrow{\delta\xi}$), then setting $w_1=\zeta_1$, $w_2=\zeta_2$, $w_3=\zeta_3$ and then blending them together, to get:

$$B(\overrightarrow{\delta\xi},\vec{\zeta})+(a_1(\xi_1+\delta\xi_1,v_1+\delta v_1)-a_1(\xi_1,v_1),\zeta_1)+(p_1(\xi_2+\delta\xi_2,v_2+\delta v_2)-p_1(\xi_2,v_2),\zeta_2)+(k_1(\xi_3+\delta\xi_3,v_3+\delta v_3)-k_1(\xi_3,v_3),\zeta_3)=(a_2(v_1+\delta v_1)-a_2(v_1),\zeta_1)+(p_2(v_2+\delta v_2)-p_2(v_2),\zeta_2)+(k_2(v_3+\delta v_3)-k_2(v_3),\zeta_3) \tag{24}$$

Now, from hypo. on $a_1,p_1,k_1,a_2,p_2\ and\ k_2$, using proposition 3.1 and the Mean value theorem, the FD of $a_1,p_1,k_1,a_2,p_2\ and\ k_2$ are exist, once get that:

$$B(\overrightarrow{\delta\xi},\vec{\zeta})+\big(a_{1\xi_1}\delta\xi_1+a_{1v_1}\delta v_1,\zeta_1\big)+\big(p_{1\xi_2}\delta\xi_2+p_{1v_2}\delta v_2,\zeta_2\big)+\big(k_{1\xi_3}\delta\xi_3+k_{1v_3}\delta v_3,\zeta_3\big)=\big(a_{2v_1}\delta v_1,\zeta_1\big)+\big(p_{2v_2}\delta v_2,\zeta_2\big)+\big(k_{2v_3}\delta v_3,\zeta_3\big)+\tilde{\varepsilon}(\overrightarrow{\delta\Xi})\|\overrightarrow{\delta\Xi}\|_0 \tag{25a}$$

where $\tilde{\varepsilon}(\overrightarrow{\delta\Xi})\|\overrightarrow{\delta\Xi}\|_0=\tilde{\varepsilon}(\overrightarrow{\delta\xi},\overrightarrow{\delta v})\left\|\begin{matrix}\overrightarrow{\delta\xi}\\ \overrightarrow{\delta v}\end{matrix}\right\|$,

From the Minkowiski inequality and lemma 4.1, once obtain that:

$\tilde{\varepsilon}(\overrightarrow{\delta\Xi}) = \tilde{\varepsilon}(\overrightarrow{\delta\xi}, \overrightarrow{\delta v}) = \bar{\varepsilon}_1(\overrightarrow{\delta v}), \quad \|\overrightarrow{\delta\Xi}\|_0 = \left\|\begin{matrix}\delta\xi_1 \\ \delta v_1\end{matrix}\right\| \le c\|\overrightarrow{\delta v}\| \quad \Longrightarrow$

$\tilde{\varepsilon}(\overrightarrow{\delta\Xi})\|\overrightarrow{\delta\Xi}\|_0 = \tilde{\varepsilon}_1(\overrightarrow{\delta v})\|\overrightarrow{\delta v}\|_0$, where $\tilde{\varepsilon}_1(\overrightarrow{\delta v}) \longrightarrow 0$, $and\ \|\overrightarrow{\delta v}\|_0 \longrightarrow 0\ as\ \overrightarrow{\delta v} \longrightarrow 0$

Hence

$$B(\overrightarrow{\delta\xi}, \vec{\zeta}) + (a_{1\xi_1}\delta\xi_1 + a_{1v_1}\delta v_1, \zeta_1) + (p_{1\xi_2}\delta\xi_2 + p_{1v_2}\delta v_2, \zeta_2) + (k_{1\xi_3}\delta\xi_3 + k_{1v_3}\delta v_3, \zeta_3) = (a_{2v_1}\delta v_1, \zeta_1) + (p_{2v_2}\delta v_2, \zeta_2) + (k_{2v_3}\delta v_3, \zeta_3) + \tilde{\varepsilon}_1(\overrightarrow{\delta v})\|\overrightarrow{\delta v}\|_0 \qquad (25b)$$

Now, from definition of the FD, hypotheses on $y_{\ell i}(\ell = 0,2\ , i = 1,2,3)$ and by using the result of lemma 4.1, once obtain that:

$$Y_0(\vec{v} + \overrightarrow{\delta v}) - Y_0(\vec{v}) = \int_\Lambda (y_{01\xi_1}(\xi_1, v_1)\delta\xi_1 + y_{01v_1}(\xi_1, v_1)\delta v_1)dx + \int_\Lambda (y_{02\xi_2}(\xi_2, v_2)\delta\xi_2 + y_{02v_2}(\xi_2, v_2)\delta v_2)dx + \int_\Lambda (y_{03\xi_3}(\xi_3, v_3)\delta\xi_3 + y_{03v_3}(\xi_3, v_3)\delta v_3)dx + \tilde{\varepsilon}(\overrightarrow{\delta v})\|\overrightarrow{\delta v}\|_0 \qquad (26)$$

where $\tilde{\varepsilon}(\overrightarrow{\delta v}) \to 0$, $and\ \|\overrightarrow{\delta v}\|_0 \longrightarrow 0\ as\ \overrightarrow{\delta v} \longrightarrow 0$

By subtracting (23) from (25b), and substituting the rustle in (26), once get

$$Y_0(\vec{v} + \overrightarrow{\delta v}) - Y_0(\vec{v}) = \int_\Lambda (\zeta_1(a_{2v_1} - a_{1v_1}) + y_{01v_2})\delta v_1\, dx + \int_\Lambda (\zeta_2(p_{2v_2} - p_{1v_2}) + y_{02v_2})\delta v_2 + \int_\Lambda (\zeta_3(k_{2v_3} - k_{1v_3}) + y_{03v_3})\delta v_3\, dx + \tilde{\varepsilon}(\overrightarrow{\delta v})\|\overrightarrow{\delta v}\|_0 \qquad (27)$$

Then from FD, we have that $\acute{\vec{Y}}_0(\vec{v})\,\overrightarrow{\delta v} = \int_\Lambda H_{\vec{v}}{}^T . \overrightarrow{\delta v}\, dx.$

Note: In the proof of the theorem 5.1, we find the FD for the functional Y_0, so the same technique is used to find the FD for Y_1 and Y_2.

Theorem 5.2: Optimality Necessary Conditions

(a) With hypotheses A, B and C, assume $\vec{U}$ is convex, if $\vec{v} \in \vec{U}_A$ is optimal, then there exist multipliers $\lambda_\ell \in \mathbb{R}$,(ℓ =0,1,2 with λ_0, $\lambda_2 \ge 0$, $\sum_{\ell=0}^{2}|\lambda_\ell|= 1$), such that the following The Kuhn- Tucker- Lagrange's Multipliers (K.T.L) are satisfied:

$$\int_\Lambda H_{\vec{v}}{}^T \cdot \overrightarrow{\delta v}\, dx \ge 0, \forall \vec{u} \in \vec{U}, \overrightarrow{\delta v} = \vec{u} - \vec{v} \qquad (28a)$$

where $y_i = \sum_{\ell=0}^{2} \lambda_\ell y_{\ell i}$ and $\zeta_i = \sum_{\ell=0}^{2} \lambda_\ell \zeta_{\ell i}$, $(i = 1,2,3)$ in the definition of H, and also

$\lambda_2 Y_2(\vec{v}) = 0$, (Transversality condition) (28b)

(b) If $\vec{U}$ is of the form

$$\vec{U} = \{\vec{u} \in (L^2(\Lambda, \mathbb{R}))^3 \mid u_i(x) \in V_i, \text{a.e. on } \Lambda\}, \text{with } V_i \subset \mathbb{R}, i = 1,2,3.$$

Then (28a) is equivalent to the minimum element wise (29), where:

$$H_{\vec{v}}{}^T . \vec{v} = \min_{\vec{u} \in \vec{V}} H_{\vec{v}}{}^T . \vec{u} \quad \text{a.e. on } \Lambda \qquad (29)$$

Proof : (a) From theorem 4.2, the functional $Y_\ell(\vec{v})$ has a continuous FD at each $\vec{v} \in \vec{U}$, since the control $\vec{v} \in \vec{U}_A$ is optimal, then by K.T.L theorem there exist multipliers $\lambda_\ell \in \mathbb{R}$, ℓ =0,1,2, with λ_0,$\lambda_2 \ge 0$, $\sum_{\ell=0}^{2}|\lambda_\ell|= 1$, such that $(\lambda_0 \acute{\vec{Y}}_{0\vec{v}}(\vec{v}) + \lambda_1 \acute{\vec{Y}}_{1\vec{v}}(\vec{v}) + \lambda_2 \acute{\vec{Y}}_{2\vec{v}}(\vec{v})).(\vec{u} - \vec{v}) \ge 0$, $\forall \vec{u} \in \vec{U}$ and $\lambda_2 Y_2(\vec{v}) = 0$, substituting the FD of $Y_\ell(\vec{v})$ $(\forall \ell = 0,1,2)$ in the above inequality, to get

$$\int_\Lambda ((\zeta_1(a_{2v_1} - a_{1v_1}) + y_{1v_1})\,\delta v_1 + (\zeta_2(p_{2v_2} - p_{1v_2}) + y_{2v_2})\delta v_2 + (\zeta_3(k_{2v_3} - k_{1v_3}) + y_{3v_3})\delta v_3)dx \ge 0$$

where $\zeta_i = \sum_{\ell=0}^{2} \lambda_\ell \zeta_{i\ell}$, $y_{iv_i} = \sum_{\ell=0}^{2} \lambda_\ell y_{i\ell v_i}$, for $i = 1,2,3$,
$\Longrightarrow \int_\Lambda H_{\vec{v}}{}^T . \overrightarrow{\delta v}\, dx \geq 0, \forall \vec{u} \in \vec{U}\ \ \overrightarrow{\delta v} = \vec{u} - \vec{v}$.

(b) Let $\vec{U} = \{u \in \left(L^2(\Lambda, \mathbb{R})\right)^3 \,|\, u_i(x) \in V_i, \text{a.e. on } \Lambda\}$, with $V_i \subset \mathbb{R}$, $i = 1,2,3$,μ is a "Lebesgue" measure on Λ, $\{v_n\}$ be a sequence in $\vec{U}_{\vec{V}}$ and assume $S \subset \Lambda$ be a measurable set such that $\vec{u}(x) = \begin{cases} \vec{u}_n(x), \text{if } x \in S \\ \vec{v}(x), \text{if } x \notin S \end{cases}$. Hence (28a), becomes
$\int_S H_{\vec{v}}{}^T . (\vec{u}_n - \vec{v}) dx \geq 0$, for each such set $S \Longrightarrow H_{\vec{v}}{}^T . (\vec{u}_n - \vec{v}) \geq 0$, a.e. on Λ
That is it satisfies in φ with $\varphi = \bigcap_n \varphi_n$, *where* $\varphi_n = \Lambda - \Lambda_n$, with $\mu(\Lambda_n) = 0$, but φ is independent of n,with $\mu\left(\Lambda/\varphi\right) = 0$ and since$\{\vec{v}_n\}$ is dense in $\vec{U}_{\vec{V}}$, then
$H_{\vec{u}}{}^T . (\vec{u} - \vec{v}) \geq 0$, a.e. on $\Lambda \Longrightarrow H_{\vec{v}}{}^T . \vec{v} = \min_{\vec{u} \in \vec{V}} H_{\vec{v}}{}^T . \vec{u}$ a.e. on Λ.

Theorem 5.3: Optimality Sufficient Conditions:

In addition to the hypotheses A,B&C, with $\vec{U}$ is convex ,$(a_1 \& y_{11})$, $(p_1 \& y_{12})$, , $(k_1 \& y_{13})$ are affine w.r.t (ξ_1, v_1),(ξ_2, v_2), (ξ_3, v_3), resp a_2, p_2, k_2 are affine w.r.t v_1, v_2, v_3 resp for each x ,and $y_{\ell i}, (\ell = 0,2, i = 1,2,3)$ is convex w.r.t. (ξ_i, v_i) for each x. Then the necessary conditions in theorem5.2, with $\lambda_0 > 0$, are also sufficient.

Proof: suppose

$$a_1(x, \xi_1, v_1) = a_{11}(x)\xi_1 + a_{12}(x)v_1 + a_{13}(x), \quad a_2(x, , v_1) = a_{21}(x)v_1 + a_{22}(x),$$
$$p_1(x, \xi_2, v_2) = p_{11}(x)\xi_2 + p_{12}(x)v_2 + p_{13}(x) , \quad p_2(x, , v_2) = p_{21}(x)v_2 + p_{22}(x),$$
$$k_1(x, \xi_3, v_3) = k_{11}(x)\xi_3 + k_{12}(x)v_3 + k_{13}(x) , \quad k_2(x, , v_3) = k_{21}(x)v_3 + k_{22}(x) ,$$

And that$\vec{v} \in \vec{U}_A$, $\vec{v}$ is satisfied the K.T.L. and the Transversality condition i.e.
$\int_\Lambda H_{\vec{v}}\left(x, \vec{\xi}, \vec{\zeta}, \vec{v}\right) \cdot \overrightarrow{\delta v} dx \geq 0, \forall \vec{u} \in \vec{U}$ and $\lambda_2 y_2(\vec{v}) = 0$
Let $Y(\vec{v}) = \sum_{\ell=0}^{2} \lambda_\ell y_\ell(\vec{v})$, then$\vec{Y}'(\vec{v})\overrightarrow{\delta v} = \sum_{\ell=0}^{2} \lambda_\ell \vec{Y}_\ell'(\vec{v})\overrightarrow{\delta v}$

$$= \sum_{\ell=0}^{2} \lambda_\ell \int_\Lambda [\left(\zeta_{1\ell}\left(a_{2v_1} - a_{1v_1}\right) + y_{1\ell v_1}\right)\delta v_1 + \left(\zeta_{2\ell}\left(p_{2v_2} - p_{1v_2}\right) + y_{2\ell v_2}\right)\delta v_2$$
$$+ (\zeta_{3\ell}\left(k_{2v_3} - k_{1v_3}\right) + y_{3\ell v_3})\delta v_3) dx = \int_\Lambda H_{\vec{v}}\left(x, \vec{\xi}, \vec{\zeta}, \vec{v}\right) \cdot \overrightarrow{\delta v} dx \geq 0$$

Let (v_1, v_2, v_3) and $(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ are two given controls, then $\left(\xi_1 = \xi_{1v_1}, \xi_2 = \xi_{2v_2}, \xi_3 = \xi_{3v_3}\right)$ and $(\bar{\xi}_1 = \bar{\xi}_{1v_1}, \bar{\xi}_2 = \bar{\xi}_{2v_2}, \bar{\xi}_3 = \bar{\xi}_{3v_3})$ are their corresponding solutions, substituting the pair $(\vec{v}, \vec{\xi})$ in (1)-(4) and multiplying the obtained equation by $\kappa \in [0,1]$ once and once again the pair $(\vec{\bar{v}}, \vec{\bar{\xi}})$ in (1)-(4) multiplying the obtained equation by $(1 - \kappa)$, finally then blending together the obtained equations from each corresponding equations once get:

$$-B_1\left(\kappa\xi_1 + (1-\kappa)\bar{\xi}_1\right) + \left(\kappa\xi_1 + (1-\kappa)\bar{\xi}_1\right) - \left(\kappa\xi_2 + (1-\kappa)\bar{\xi}_2\right) - \left(\kappa\xi_3 + (1-\kappa)\bar{\xi}_3\right)$$
$$+a_{11}(x)\left(\kappa\xi_1 + (1-\kappa)\bar{\xi}_1\right) + a_{12}(x)(\kappa v_1 + (1-\kappa)\bar{v}_1) + a_{13}(x) = a_{21}(x)(\kappa v_1 +$$
$$(1-\kappa)\bar{v}_1) + a_{22}(x) \tag{30a}$$
$$\kappa\xi_1 + (1-\kappa)\bar{\xi}_1 = 0 \tag{30b}$$
$$-B_2\left(\kappa\xi_2 + (1-\kappa)\bar{\xi}_2\right) + \left(\kappa\xi_1 + (1-\kappa)\bar{\xi}_1\right) + \left(\kappa\xi_2 + (1-\kappa)\bar{\xi}_2\right) + (\kappa\xi_3 + (1-\kappa)\bar{\xi}_3) +$$
$$p_{11}(x)\left(\kappa\xi_2 + (1-\kappa)\bar{\xi}_2\right) + p_{12}(x)(\kappa v_2 + (1-\kappa)\bar{v}_2) + p_{13}(x) = p_{21}(x)(\kappa v_2 +$$
$$(1-\kappa)\bar{v}_2) + p_{22}(x) \tag{31a}$$
$$\kappa\xi_2 + (1-\kappa)\bar{\xi}_2 = 0 \tag{31b}$$

$-B_3\big(\kappa\xi_3 + (1-\kappa)\,\bar{\xi}_3\big) + \big(\kappa\xi_1 + (1-\kappa)\bar{\xi}_1\big) - \big(\kappa\xi_2 + (1-\kappa)\bar{\xi}_2\big) + \big(\kappa\xi_3 + (1-\kappa)\bar{\xi}_3\big) +$
$k_{11}(x)\big(\kappa\xi_3 + (1-\kappa)\bar{\xi}_3\big) + k_{12}(x)(\kappa v_3 + (1-\kappa)\bar{v}_3) + k_{13}(x) = k_{21}(x)(\kappa v_3 +$
$(1-\kappa)\bar{v}_3) + k_{22}(x)$ (32a)
$\kappa\xi_3 + (1-\kappa)\bar{\xi}_3 = 0$ (32b)

Now, if we have the control vector $\vec{\bar{\bar{v}}} = (\bar{\bar{v}}_1, \bar{\bar{v}}_2, \bar{\bar{v}}_3)$, with $\bar{\bar{v}}_1 = \kappa v_1 + (1-\kappa)\bar{v}_1$, $\bar{\bar{v}}_2 = \kappa v_2 + (1-\kappa)\bar{v}_2$, $\bar{\bar{v}}_3 = \kappa v_3 + (1-\kappa)\bar{v}_3$. Then from (30 a&b), (31 a&b), (32 a&b), once get that

$\bar{\bar{\xi}}_1 = \xi_{1\bar{\bar{v}}_1} = \xi_{1(\kappa v_1 + (1-\kappa)\bar{v}_1)} = \kappa\xi_1 + (1-\kappa)\bar{\xi}_1$,
$\bar{\bar{\xi}}_2 = \xi_{2\bar{\bar{v}}_2} = \xi_{2(\kappa v_2 + (1-\kappa)\bar{v}_2)} = \kappa\xi_2 + (1-\kappa)\bar{\xi}_2$,
$\bar{\bar{\xi}}_3 = \xi_{3\bar{\bar{v}}_3} = \xi_{3(\kappa v_3 + (1-\kappa)\bar{v}_3)} = \kappa\xi_3 + (1-\kappa)\bar{\xi}_3$

are their corresponding solutions, i.e. $(\bar{\bar{\xi}}_1, \bar{\bar{\xi}}_2, \bar{\bar{\xi}}_3)$ is satisfied (1-4). So, the operator $v_i \longmapsto \xi_{iv_i}$ is convex- linear w.r.t $(\xi_i, v_i)(i = 1,2,3)$, for each $x \in \Lambda$.

Now, since $y_{1i}(x, \xi_i, v_i)$ is affine w.r.t. (ξ_i, v_i), for each $x \in \Lambda$ and from the convex –linear property of operators $v_i \longmapsto \xi_{iv_i}$, once gets that $Y_1(\vec{v})$ is convex-linear w.r.t $(\vec{\xi}, \vec{v})$, $\forall x \in \Lambda$.

The convexity of $Y_\ell(\vec{v})$ (for ℓ= 0,2) w.r.t.$(\vec{\xi}, \vec{v})$, for each $x \in \Lambda$ is obtained from the hypotheses at each of $y_{\ell i}$ is convex w.r.t. $(\xi_i, v_i)\forall x \in \Lambda$, $(\forall \ell = 0,2\,, \& i = 1,2,3)$. Hence $Y(\vec{v})$ is convex w.r.t $(\vec{\xi}, \vec{v})$ in the convex set $\vec{U} = \vec{U}_{\vec{V}}$ and it has a continuous FD satisfied

$\acute{\vec{Y}}(\vec{v})\overrightarrow{\delta v} \geq 0 \Rightarrow Y(\vec{v})$ has a minimum at $\vec{v} \Rightarrow Y(\vec{v}) \leq Y(\vec{u}), \forall \vec{u} \in \vec{U} \Rightarrow$

$$\lambda_0 Y_0(\vec{v}) + \lambda_1 Y_1(\vec{v}) + \lambda_2 Y_2(\vec{v}) \leq \lambda_0 Y_0(\vec{u}) + \lambda_1 Y_1(\vec{u}) + \lambda_2 Y_2(\vec{u}) \qquad (33)$$

Now, let $\vec{u}$ be an admissible control and since $\vec{v}$ is also admissible and satisfies the Transversality condition, then (33) becomes$Y_0(\vec{v}) \leq Y_0(\vec{u})$, $\forall \vec{u} \in \vec{U}$ i.e. $\vec{v}$ is an optimal control for the problem.

6. Conclusion

The existence and uniqueness theorem for the solution (continuous state vector) of the TNLEBVP is stated and proved successfully using the Mint-Browder theorem when the TCCOCV is given. Also, the existence theorem of a TCCOCV governing by the TNLEBVP is proved. The existence and uniqueness solution of the TAEqs related with the TNLEBVP is studied. The derivation of the FD of the Hamiltonian is obtained. Finally, the theorem of necessary conditions so as the sufficient condition theorem for optimality of the constrained problem are stated and proved.

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