Estimation of the Reliability Function of Basic Gompertz Distribution under Different Priors

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Abstract

In this paper, some estimators for the reliability function R(t) of Basic Gompertz (BG) distribution have been obtained, such as Maximum likelihood estimator, and Bayesian estimators under General Entropy loss function by assuming non-informative prior by using Jefferys prior and informative prior represented by Gamma and inverted Levy priors. Monte-Carlo simulation is conducted to compare the performance of all estimates of the R(t), based on integrated mean squared.

Keywords: Basic Gompertz distribution, General Entropy loss function, integrated mean squared errors, Maximum likelihood estimator, Reliability function.

1. Introduction

The British Benjamin Gompertz (1825) reached to the law of geometrical progression pervades large portions of different tables of mortality for humans. The formula he derived was commonly called the Gompertz equation, which is a valuable tool in demography, reliability analysis, and life testing. It is widely used in Bayesian estimation as a conjugate prior also in demonstrating individuals' mortality and actuarial chart and different scientific disciplines fields such as biological, Marketing Science, also in network theory.

Therefore, the main objective of this paper is to obtain the best estimator for the reliability function of BG distribution under General Entropy error loss function (GELF) with assuming different priors.

The Gompertz distribution has the following p.d.f [1].

$$f(t\,;\zeta,\varphi) = \varphi \exp\left[\zeta t + \frac{\varphi}{\zeta}\left(1 - e^{\zeta t}\right)\right] \; ; \quad t \geq 0 \quad \zeta, \varphi > 0$$

Where ζ is the scale parameter and φ is shape parameter of the Gompertz distribution.

In this paper, a special case of Gompertz distribution knows as BG distribution will be assumed by letting that $\zeta = 1$ which is given by the following probability density function [2].

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$$f(t;\varphi) = \varphi \exp[t + \varphi(1 - e^t)] \quad ; \quad t \geq 0, \quad \varphi > 0 \tag{1}$$

The corresponding cumulative distribution function F(t) and reliability or survival function R(t) of BG distribution are given by:

$$F(t) = 1 - \exp[\varphi(1 - e^t)] \qquad ; \quad t \geq 0$$
$$\widehat{R}(t) = \overline{F(t)} = \exp[\varphi(1 - e^t)] \quad ; \quad t \geq 0$$

2. Maximum likelihood Estimator of the Shape Parameter (φ)

Assume that, $t_1, t_2,..., t_n$ is the set of n random lifetimes from the BG distribution defined by equation (1), the likelihood function for the sample observation will be as follows [3].

$$L(t_1, t_2, \ldots, t_n; \varphi) = \prod_{i=1}^{n} f(t_i; \varphi)$$

$$L(t_1, t_2, \ldots, t_n; \varphi) = \varphi^n \exp[\sum_{i=1}^{n} t_i + \varphi \sum_{i=1}^{n}(1 - e^{t_i})] \tag{2}$$

Assume that

$$\frac{\partial}{\partial \varphi} \ln L(t_i; \varphi) = 0$$

The MLE of φ becomes

$$\widehat{\varphi}_{ML} = \frac{-n}{T} \tag{3}$$

Where $T = \sum_{i=1}^{n}(1 - e^{t_i})$

Based on the invariant property of the MLE, the MLE for R(t) will be as follows

$$\widehat{R}(t) = \exp[\widehat{\varphi}_{MLE}(1 - e^t)]$$

$$\tag{4}$$

3. Bayesian Estimation

We provide a Bayesian estimation method for R(t) of BG distribution, including non-informative and informative priors.

3.1 Posterior Density Functions Using Jeffreys Prior

In this subsection, φ will be assumed has non-informative prior density defined as using Jeffreys prior information as follows [4].

$$g_1(\varphi) \propto \sqrt{I(\varphi)}$$

Where $I(\varphi)$ represents Fisher information [5]. That is given by:

$$I(\varphi) = -nE\left(\frac{\partial^2 \ln f}{\partial \varphi^2}\right)$$

Therefore,

$$g_1(\varphi) = k\sqrt{-nE\left(\frac{\partial^2 \ln f(t; \varphi)}{\partial \varphi^2}\right)} \quad , \quad k \text{ is a constant} \tag{5}$$

Taking the natural logarithm for p.d.f. of BG distribution and taking the second partial derivative with respect to φ, gives:

$$E\left(\frac{\partial^2 \ln f(t; \varphi)}{\partial \varphi^2}\right) = -\frac{1}{\varphi^2}$$

After substitution into eq. (5) yields,

$$g_1(\varphi) = \frac{k}{\varphi}\sqrt{n} \qquad , \qquad \varphi > 0$$

In general, the posterior probability density function of unknown parameter φ with prior $g(\varphi)$ can be expressed as form:

$$\pi(\varphi|\underline{t}) = \frac{L(t_1, t_2, \ldots, t_n; \varphi)\, g(\varphi)}{\int_{\forall \varphi} L(t_1, t_2, \ldots, t_n; \varphi)\, g(\varphi) d\varphi} \tag{6}$$

After substituting into eq.(6), the posterior density function based on Jeffreys prior to become

$$\pi_1(\varphi|\underline{t}) = \frac{\varphi^{n-1}e^{-\varphi\sum(e^t-1)}}{\int_0^\infty \varphi^{n-1}e^{-\varphi\sum(e^t-1)}\, d\varphi} = \frac{P^n \varphi^{n-1}e^{-\varphi P}}{\Gamma(n)}$$

Where,

$$P = \sum(e^t - 1) = -T$$

The posterior density $\pi_1(\varphi|\underline{t})$ is a Gamma distribution, i.e.

$$(\varphi|t_1, \ldots, t_n) \sim \text{Gamma}(n, P), \text{ with } E(\varphi|t_1, \ldots, t_n) = \frac{n}{P}\ ; \ ver(\varphi|t_1, \ldots, t_n) = \frac{n}{P^2}$$

3.2 Posterior Density Functions Using Gamma Distribution [6].

Suppose that φ is distributed Gamma as a prior distribution with the following p.d.f

$$g_2(\varphi) = \frac{\beta^\alpha}{\Gamma(\alpha)}\varphi^{\alpha-1}e^{-\beta\varphi} \ ; \qquad \varphi > 0, \quad \alpha, \beta > 0 \tag{7}$$

Now, the posterior density functions using Gamma distribution can be obtained by combining eq. (2) with eq. (7) in eq. (6), as follows:

$$\pi_2(\varphi|\underline{t}) = \frac{\varphi^{n+\alpha-1}e^{-\varphi[\beta-\sum_{i=1}^n(1-e^{t_i})]}}{\int_0^\infty \varphi^{n+\alpha-1}e^{-\varphi[\beta-\sum_{i=1}^n(1-e^{t_i})]}\, d\varphi}$$

After simplification, we get:

$$\pi_2(\varphi|\underline{t}) = \frac{(\beta-T)^{n+\alpha}\varphi^{n+\alpha-1}e^{-\varphi(\beta-T)}}{\Gamma(n+\alpha)}$$

Notice that the posterior p.d.f. of the parameter φ is a Gamma distribution, i.e.

$$\varphi|\underline{t} \sim \text{Gamma}(n+\alpha, \beta-T), \text{ with } E(\varphi|\underline{t}) = \frac{n+\alpha}{\beta-T} \quad , \quad Var(\varphi|\underline{t}) = \frac{n+\alpha}{(\beta-T)^2}.$$

3. 3 Posterior Density Functions Using Inverted Levy prior

Assume that φ has inverted Levy prior with hyper-parameter b with the following p.d.f. [7].

$$g_3(\varphi) = \sqrt{\frac{b}{2\pi}}\ \varphi^{\frac{-1}{2}}e^{\frac{-b\varphi}{2}} \qquad , \qquad \varphi > 0, \ b > 0 \tag{8}$$

Combining eq. (2) with eq. (8) into eq. (6), yields the posterior probability density function of the shape parameter φ as the following

$$\pi_3(\varphi|\underline{t}) = \frac{\varphi^{n-\frac{1}{2}}e^{-\varphi(\frac{b}{2}-T)}}{\int_0^\infty \varphi^{n-\frac{1}{2}}e^{-\varphi(\frac{b}{2}-T)}\, d\varphi}$$

After simplification, it yields

$$\pi_3(\varphi|\underline{t}) = \frac{(\frac{b}{2}-T)^{n+\frac{1}{2}}\varphi^{n-\frac{1}{2}}e^{-\varphi(\frac{b}{2}-T)}}{\Gamma(n+\frac{1}{2})}$$

The posterior p.d.f. of the parameter φ is Gamma distribution, i.e.

$\varphi|t \sim Gamma\left(n + \frac{1}{2}, \frac{b}{2} - T\right)$, with $E(\varphi|\underline{t}) = \frac{n+\frac{1}{2}}{\frac{b}{2}-T}$, $Var\left(\varphi|\underline{t} \sim \varphi\right) = \frac{n+\frac{1}{2}}{(\frac{b}{2}-T)^2}$.

Where $T = \sum_{i=1}^{n}(1 - e^{t_i})$

4.1 Bayes Estimation under General Entropy Error Loss Function (GELF)

In many practical situations, it appears more realistic to express the loss function in terms of the ratio $\widehat{\varphi}/\varphi$. In this case, a useful asymmetric loss function is the general entropy proposed by Calabria and Pulcini (1996) [8].

$$L(\widehat{\varphi}, \varphi) = w\left[\left(\frac{\widehat{\varphi}}{\varphi}\right)^s - s\ln\left(\frac{\widehat{\varphi}}{\varphi}\right) - 1\right]; \quad w > 0, \quad s \neq 0$$

Where minimum value occurs when $\widehat{\varphi} = \varphi$.

When (s > 0), a positive error ($\widehat{\varphi} > \varphi$) causes more serious consequences than a negative error. Without any loss of generality, it can be assumed that, $w = 1$. Then, the risk function under the General Entropy loss function is denoted by $R_{BGE}(\widehat{\varphi}, \varphi)$.

$\qquad R_{BGE}(\widehat{\varphi}, \varphi) = E(L(\widehat{\varphi}, \varphi))$

Let $w = 1$, then

$$R_{BGE}(\widehat{\varphi}, \varphi) = \int_0^{\infty}\left[\left(\frac{\widehat{\varphi}}{\varphi}\right)^s - s\ln\left(\frac{\widehat{\varphi}}{\varphi}\right) - 1\right]\pi(\varphi|\underline{t})d\varphi$$

$$\frac{\partial R_{BGE}(\widehat{\varphi}, \varphi)}{\partial \widehat{\varphi}} = \int_0^{\infty}\left[\frac{s}{\varphi}\left(\frac{\widehat{\varphi}}{\varphi}\right)^{s-1} - \frac{s}{\widehat{\varphi}}\right]\pi(\varphi|\underline{t})d\varphi$$

The value of $\widehat{\varphi}$ minimizes the risk function under General Entropy loss function which satisfies the following condition:

$$\frac{\partial R_{BGE}(\widehat{\varphi}, \varphi)}{\partial \widehat{\varphi}} = 0$$

$$s\,\widehat{\varphi}^{s-1}E\left(\frac{1}{\varphi^s}|\underline{t}\right) - \frac{s}{\widehat{\varphi}} = 0$$

$$\widehat{\varphi}^s = \left[\frac{1}{E\left(\frac{1}{\varphi^s}|\underline{t}\right)}\right]^{\frac{1}{s}}$$

Accordingly,

$$\widehat{R}(t) = \left[\frac{1}{E\left(\frac{1}{(R(t))^s}|\underline{t}\right)}\right]^{\frac{1}{s}} \tag{9}$$

The Bayes estimator for Reliability function under Jefferys prior can be derived as follows

$$E\left(\frac{1}{(R(t))^s}|\underline{t}\right) = P^n\int_0^{\infty}\frac{e^{-s\varphi(1-e^t)}\varphi^{n-1}e^{-\varphi P}d\varphi}{\Gamma(n)}$$

$$= \frac{P^n}{[P+s(1-e^t)]^n}\int_0^{\infty}\frac{[P+s(1-e^t)]^n\,\varphi^{n-1}e^{-\varphi(P+s(1-e^t))}}{\Gamma(n)}d\varphi$$

$$E\left(\frac{1}{(R(t))^s}|\underline{t}\right) = \left[\frac{P}{P+s(1-e^t)}\right]^n \tag{10}$$

After substituting eq. (10) into eq. (9), we will get the Bayes estimator for the R(t) of BG distribution under the General Entropy loss function with Jefferys prior denoted by $\widehat{R}_J$ as follow:

$$\widehat{R}(t)_J = \left[\frac{P+s\ (1-e^t)}{P}\right]^{\frac{n}{s}} \tag{11}$$

Similarly, the Bayesian estimation of Reliability function under Gamma prior distribution information can be derived as follows

$$E\left(\frac{1}{(R(t))^s}|\underline{t}\right) = (\beta - T)^{n+\alpha} \int_0^\infty \frac{e^{-s\varphi(1-e^t)}\varphi^{n+\alpha-1}e^{-\varphi(\beta-T)}d\varphi}{\Gamma(n+\alpha)}$$

$$= \frac{(\beta - T)^{n+\alpha}}{[(\beta - T) + s\ ((1-e^t)]^{n+\alpha}} \int_0^\infty \frac{[(\beta - T) + s\ ((1 - e^t)]^{n+\alpha}\ \varphi^{n+\alpha-1}e^{-\varphi((\beta-T)+s\ (1-e^t))}}{\Gamma(n+\alpha)}d\varphi$$

Therefore,

$$E\left(\frac{1}{(R(t))^s}|\underline{t}\right) = \left[\frac{(\beta-T)}{(\beta-T)+s\ (1-e^t)}\right]^{n+\alpha} \tag{12}$$

After substituting eq.(12) into eq. (9), it will yield the Bayesian estimation for R(t) of BG distribution under General Entropy loss function with assuming Gamma prior

$$\widehat{R}(t)_G = \left[\frac{(\beta-T)+s\ (1-e^t)}{(\beta-T)}\right]^{\frac{n+\alpha}{s}} \tag{13}$$

Now, $\widehat{R}(t)$ under Inverted Levy prior Information can be obtained as follows:

$$E\left(\frac{1}{(R(t))^s}|\underline{t}\right) = \int_0^\infty \frac{1}{(R(t))^s}\pi_3\left(\varphi|\underline{t}\right)d\varphi \tag{14}$$

$$= \frac{(\frac{b}{2}-T)^{n+\frac{1}{2}}}{\left[(\frac{b}{2}-T)+s\ ((1-e^t)\right]^{n+\frac{1}{2}}} \int_0^\infty \frac{\left[(\frac{b}{2}-T)+s\ ((1-e^t)\right]^{n+\frac{1}{2}}\varphi^{n-\frac{1}{2}}e^{-\varphi((\frac{b}{2}-T)+s\ (1-e^t))}}{\Gamma\left(n+\frac{1}{2}\right)}d\varphi$$

$$= \left[\frac{(\frac{b}{2}-T)}{(\frac{b}{2}-T)+s\ (1-e^t)}\right]^{n+\frac{1}{2}} \tag{15}$$

After substituting eq. (15) into eq.(9), the Bayesian estimation for the R(t) of BG distribution using the General Entropy loss function with Inverted Levy prior was obtained as:

$$\widehat{R}(t)_{IL} = \left[\frac{(\frac{b}{2}-T)+s\ (1-e^t)}{(\frac{b}{2}-T)}\right]^{\frac{n+\frac{1}{2}}{s}} \tag{16}$$

5. Simulation Study

In this section, the Monte-Carlo simulation was done to compare the accuracy of the different estimators of the Reliability function R(t) for BG distribution. The process (L) has been repeated 5000 times with different sample sizes (n = 15, 50, and100).

The default values of shape parameter φ were chosen to be less and greater than 1 as φ= 0.5, 3. Different values of the Gamma prior parameters were chosen as $\alpha = 0.8, 3$ and $\beta = 0.5,\ 3$. Two different values for the parameter of Inverted Levy prior were chosen as (b=0.5, 5).

The integrated mean squared error (IMSE) was employed to compare the accuracy of the different estimates for R(t). IMSE is an important global measure and it more accurate than

MSE which is defined as the distance between the estimated value and actual value of reliability function given by

$$\text{IMSE}(\hat{R}(t)) = \frac{1}{L} \sum_{i=1}^{L} \left[\frac{1}{n_t}\sum_{j=1}^{n_t} (\hat{R}_i(t_j) - R(t_j))\right]^2 = \frac{1}{n_t} \sum_{j=1}^{n_t} \text{MSE}(\hat{R}_i(t_j))$$

Where $i=1, 2,\ldots, L$, n_t the random limits of t_i.

In this paper, we chose t = 0.1, 0.2, 0.3, 0.4, 0.5.

The results were summarized and tabulated in the following tables for each estimator and for all sample sizes.

Table 1. IMSE's of the different estimates for R(t) of BG distribution under MLE and Jefferys prior where $\varphi = 0.5$.

Estimator		N		
		15	50	100
$\hat{\lambda}_{ML}$		0.0014595	0.0003847	0.0001869
$\hat{R}(t)_J$	s=1	0.0015166	0.0003889	0.0001879
	s=3	0.0016452	0.0003981	0.0001901

Table 2. IMSE's of the different estimates for R(t) of Gompertz distribution under Gamma Prior where $\varphi = 0.5$.

Estimator			N					
			15		50		100	
			$\beta = 0.5$	$\beta = 3$	$\beta = 0.5$	$\beta = 3$	$\beta = 0.5$	$\beta = 3$
$\hat{R}(t)_G$	$\alpha = 0.8$	s=1	0.0013158	0.0011254	0.0003748	0.0003485	0.0001846	0.0001774
		s=3	0.0014162	0.0010857	0.0003832	0.0003442	0.0001867	0.0001763
	$\alpha = 3$	s=1	0.0022689	**0.0007777**	0.0004638	**0.0003114**	0.0002071	**0.0001679**
		s=3	0.0024683	**0.0007776**	0.0004792	**0.0003123**	0.0002108	**0.0001682**

Table 3. IMSE's of the different estimates for R(t) of Gompertz distribution under Inverted Levy Prior where $\varphi = 0.5$.

Estimator		N					
		15		50		100	
		$b = 0.5$	$b = 5$	$b = 0.5$	$b = 5$	$b = 0.5$	$b = 5$
$\hat{R}(t)_{IL}$	s=1	0.0015545	0.0010174	0.0003934	0.0003379	0.0001891	0.0001748
	s=3	0.0083362	0.0105278	0.0088036	0.0095093	0.0089058	0.0092641

Table 4. IMSE's of the different estimates for R(t) of Gompertz distribution under MLE and Jefferys Prior where $\varphi = 3$.

Estimator		N		
		15	50	100
$\hat{\lambda}_{ML}$		0.0069471	0.0021114	0.0010669
$\hat{R}(t)_J$	s=1	0.0076524	0.0021757	0.0010831
	s=3	0.0101046	0.0023806	0.0011335

Table 5. IMSE's of the different estimates for R(t) of Gompertz distribution
under Gamma Prior where $\varphi = 3$.

Estimator			N					
			15		50		100	
			$\beta = 0.5$	$\beta = 3$	$\beta = 0.5$	$\beta = 3$	$\beta = 0.5$	$\beta = 3$
$\widehat{R}(t)_G$	$\alpha = 0.8$	s=1	0.0072419	0.0780281	0.0021427	0.0170420	0.0010744	0.0057196
		s=3	0.0061791	0.0745552	0.0020318	0.0159772	0.0010458	0.0053534
	$\alpha = 3$	s=1	**0.0054667**	0.0613839	**0.0019455**	0.0140180	**0.0010227**	0.0048371
		s=3	0.0059374	0.0581128	0.0019979	0.0130569	0.0010367	0.0045065

Table 6. IMSE's of the different estimates for R(t) of Gompertz distribution
under Inverted Levy Prior where $\varphi = 3$.

Estimator			N					
			15		50		100	
			$b = 0.5$	$b = 5$	$b = 0.5$	$b = 5$	$b = 0.5$	$b = 5$
$\widehat{R}(t)_{IL}$		s=1	0.0064792	0.0619624	0.0020660	0.0121698	0.0010553	0.0040611
		s=3	0.0935634	0.2011713	0.0951978	0.1419205	0.0955598	0.1212603

6. Results Discussion and Analysis
The discussion of the results obtained from applying the simulation study can be summarized as follows:
- When shape parameter φ=0.5, the Bayes estimator under General Entropy loss function based on Gamma prior, with ($\alpha = 3, \beta = 3$, s=1 and 2) is the best estimates for R(t) in comparing to other estimates for all sample sizes see **Tables 1, 2, 3.**
- When shape parameter φ=3, from **Tables 4, 5, 6.** Notice that the performance of Bayes estimator under General Entropy loss function based on Gamma prior, is the best with ($\alpha = 3, \beta = 0.5$, s=1) for all sample sizes.

1. Conclusion
The simulation study has shown that:
1. In general, Bayesian estimation for Reliability function of Basic Gompertz distribution with Gamma prior is the best compared with the corresponding estimates based on Jeffreys prior and Inverted Levy prior by using the same loss function (General Entropy loss function) in addition to MLE.
2. To increase the accuracy of Bayesian estimation of R(t) under General Entropy loss function using Gamma prior, the value of scale parameter (β) of Gamma prior should be chosen to be inversely proportional to the value of the shape parameter of Basic Gompertz distribution (φ).

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