Ibn Al-Haitham Jour. for Pure & Appl. Sci. 33 (3) 2020

Ibn Al Haitham Journal for Pure and Applied Science

Journal homepage: http://jih.uobaghdad.edu.iq/index.php/j/index

Solving Some Fractional Partial Differential Equations by Invariant Subspace and Double Sumudu Transform Methods

Hasan Shather Kadhem

Department of Mathematics, College of Education for Pure Sciences (Ibn Al – Haitham), University of Baghdad

hasanshather1968@gmail.com

Sameer Qasim Hasan

Department of Mathematics, College of Science, University of Mustansiriyah, Baghdad, Iraq.

Article history: Received 24 September 2019, Accepted 15 December2019, Published in July 2020.

Doi: 10.30526/33.3.2485

Abstract

In this paper, several types of space-time fractional partial differential equations has been solved by using most of special double linear integral transform" double Sumudu ". Also, we are going to argue the truth of these solutions by another analytically method "invariant subspace method". All results are illustrative numerically and graphically.

Key words fractional calculus, fractional partial differential equation, Sumudu transform, invariant subspace.

1. Introduction

As known, linear integral transformation is used to solve differential equations, by convert the Linear partial differential equation into algebraic equation which can be solved easily. So, the integral transforms such as Mellin, Laplace, Fourier and Sumudu were vastly applied to obtain the solution of differential equations, which are used in astronomy, physics and also in engineering.

Partial differential equations are considered one of the most significant topics in mathematics and others. Motivated by no general methods for solve these equations, integral transform method is one of the most familiar method in order to get the solution of such equations [1-3]. Double Laplace transform method and double Sumudu transform method (DSTM) were used to solve wave and Poisson equations [4, 5].

Moreover, the relation between these linear integral transformations and their applications to differential equations have been determined and studied by [6]. In this study we focus on double Sumudu integral transform in addition to using analyses method [7]. Invariant subspace method (ISM)" to solve some of important time-space and mixed fractional partial differential equations, and we are going to argue the results solution in these two methods graphically and numerically.

2. Some Basic Concepts

1. Important Facts of the Fractional Calculus [8, 9].

In this section, we give some important definitions and notation which are needed in our work.

Definition 1: [8]. The Riemann–Liouville fractional integral of order α for a function f is defined as:

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x)dx \qquad \alpha, t > 0$$

The R-L fractional integral operator has the following properties:

1. J^{α} is linear

2. $J^0 = I$

3. $\lim_{\alpha \to 0} J^{\alpha} = J^0$

Definition 2: [8]. The Caputo fractional derivative of positive order α for a function f is defined as:

$$D_t^{\alpha} f(t) = J^{n-\alpha} D^n f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(x)}{(t-x)^{n-\alpha+1}}\, dx & n \neq \alpha, n \in \mathrm{N} \\ f^{(n)}(x) & n = \alpha \end{cases}$$

Some properties of fractional Caputo derivative and fractional R-L integral are:

1. $J^{\alpha} t^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta+\alpha} t^{\alpha+\beta} \qquad \alpha > 0,\ \beta > -1,\ t > 0$

2. $\frac{d^{\alpha}}{dt^{\alpha}} t^{\beta} = \begin{cases} \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha} t^{\beta-\alpha} & n-1 < \alpha < n,\ \beta > n-1,\ \beta \in R \\ 0 & n-1 < \alpha < n,\ \beta \leq n-1,\ \beta \in N \end{cases}$

3. $\frac{\partial^{\alpha}}{\partial t^{\alpha}} (J^{\alpha}) = I$

4. $J^{\alpha} \left(\frac{d^{\alpha}}{dt^{\alpha}} f(t) \right) = f(t) - \sum_{i=0}^{n-1} \frac{t^i}{i!} f^{(i)}(0)$

5. $\frac{\partial^{\alpha}}{\partial t^{\alpha}} \left(\frac{\partial^{\beta}}{\partial t^{\beta}} f(t) \right) = \frac{\partial^{\beta}}{\partial t^{\beta}} \left(\frac{\partial^{\alpha}}{\partial t^{\alpha}} f(t) \right) = \frac{\partial^{\alpha+\beta}}{\partial t^{\alpha+\beta}} f(t) \quad provided\ that\ f^{(i)} = 0, i = 0,1, \dots, n-1, \alpha + \beta \leq n,\ n \in N$

Definition 3: [8,9]. A two parameters Mittag-Leffler function is defined as:

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)} \qquad \alpha, \beta \in C, \quad R(\alpha), \quad R(\beta) > 0$$

Lemma 1: [8,9]. A Mittag-Leffler function has an interesting properties:

1. $E_{\alpha,1}(x) = E_{\alpha}(x)$ 3. $E_{2,1}(x^2) = cosh(x)$ 5. $E_{2,1}(x^2) = cos(x)$

2. $E_{1,1}(x) = e^{x}$ 4. $xE_{2,1}(x^2) = sinh(x)$ 6. $xE_{2,1}(x^2) = sin(x)$

Lemma 2: [9]. The Caputo derivatives of Mittag-Leffler are given as:

1. $E_{\alpha,\beta}^{(n)}(x) = \sum_{k=0}^{\infty} \frac{(k+n)!x^k}{k!\Gamma(\alpha k+\alpha n+\beta)} \qquad n \in N$
2. $\frac{d^\alpha}{dt^\alpha}\left(E_\alpha(at^\alpha)\right) = aE_\alpha(at^\alpha) \quad \alpha > 0, \quad a \in R$
3. $\frac{d^\gamma}{dt^\gamma}\left(t^{\beta-1}E_{\alpha,\beta}(at^\alpha)\right) = t^{\beta-\gamma-1}E_{\alpha,\beta-\gamma}(at^\alpha) \qquad \gamma > 0$

1. On the Single and Double Sumudu Transform

The Sumudu transform introduced firstly by the Watugala [1,2]. Which can be used to solve the ordinary and partial differential equations with ordinary and fractional order.

The following definitions and properties of single and double Sumudu transform are necessary for our work. For all, we consider Sumudu transform of a function and its inverse is exist, i. e., the considered function is of exponent order.

Definition 4: [10]. A function $f(x)$ is said to be of exponent order $\alpha > 0$ if there exist non-negative constants M, α and T such that $|f(x)| \le Me^{\alpha x} \qquad x \ge T.$

For example, all polynomials e^{ax} where a is a constant, sine and cosine functions and products of these functions are of exponential order.

An example of a function not of exponential order is e^{x^2}. This function grows too rapidly.

Definition 5: [3]. The Sumudu transform for the exponent order function $f\ (x)$ is given by

$$S\{f(x)\} = T(u) = \frac{1}{u}\int_0^\infty e^{\frac{-x}{u}} f(x) \ dx \qquad x>0$$

And the inverse Sumudu transform of T(u) is defined by:

$$S^{-1}\{T(u)\} = f(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{xu}T(\tfrac{1}{u})\frac{du}{u} = \sum_{Residues}\left[e^{xu}\frac{T(u)}{u}\right] \qquad Re(u)<a$$

The Sumudu transform of the important function ”Mittag-Leffler function ” is given by

$$S\{t^{\beta-1}E_{\alpha,\beta}(\lambda t)\} = \frac{u^{\beta-1}}{1-\lambda u^\alpha}$$

Sumudu transform for some famous functions are included in the following table.

Table 1. Sumudu transform for some special functions.

$f(x)$	$S\{f(x)\} = T(u)$	$f(x)$	$S\{f(x)\} = T(u)$
1	**1**	x^α	$\Gamma(1+\alpha)u^\alpha$
e^x	$\frac{1}{1-au}$	$\sin ax$	$\frac{au}{1+a^2u^2}$

$\cos ax$	$\dfrac{1}{1+a^2u^2}$	$\sinh ax$	$\dfrac{au}{1-a^2u^2}$
$\dfrac{x^{\alpha}e^{ax}}{\Gamma(1+\alpha)}$	$\dfrac{u^{\alpha}}{(1-au)^{1+\alpha}}$	$\cosh ax$	$\dfrac{1}{1-a^2u^2}$

Definition 6: [3]. The double Sumudu transform for the function $f\,(t,x)$ is given by:

$$S_2\{f(x,t)\} = T(u,v) = \frac{1}{uv}\int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{x}{u}+\frac{t}{v}\right)} f(x,t)\,dt\,dx$$

We state here some of the important properties of double Sumudu transform which are needed

1. $S_2\{f(ax)g(b\,t)\} = T(au)G(bv)$
2. $S_2\{I_x^{\alpha} f(x,t)\} = u^{\alpha}T(u,v)$

Theorem 1: [11]. If the double Sumudu transform of the function f (x; t) given by $S_2\{\,f\,(x,t) = T(u,v)$, then:

1. $S_2\{x^n f(x,t)\} = u^n \sum_{k=0}^{n} a_k^n\, u^k \frac{\partial^k}{\partial u^k} T(u,v)$
2. $S_2\{x^n t^m f(x,t)\} = u^n v^m \sum_{k=0}^{n} a_k^n\, b_l^m u^k v^l \frac{\partial^{k+l}}{\partial u^k\, v^l} T(u,v)$ where $a_0^n = 1,\ a_n^n = 1,$ $a_1^n = n!\,n,\quad a_k^n = a_{k-1}^{n-1} + (m+k)a_k^{n-1},\quad$ similarly for b_l^m

The double Sumudu transform of the partial Caputo fractional derivatives are given in the following theorem:

Theorem 2: [4]. Let the exponent order function $f\,(x,t)$ has a continuous partial derivative on $R^+ \times R^+$ and $n-1 < \alpha < n,\ m-1 < \beta < m,$ then:

3. $S_2\{D_x^{\alpha} f(x,t)\} = u^{-\alpha}\left[T(u,v) - \sum_{i=0}^{n-1} u^i\, T_i(0,v)\right]$
4. $S_2\left\{D_t^{\beta} f(x,t)\right\} = v^{-\beta}\left[T(u,v) - \sum_{j=0}^{m-1} v^j\, T_j(u,0)\right]$
5. $S_2\left\{D_t^{\beta} D_x^{\alpha} f(x,t)\right\} = u^{-\alpha}v^{-\beta}\Big[T(u,v) - \sum_{i=0}^{n-1} u^i\, T_i(0,v) - \sum_{j=0}^{m-1} v^j\, T_j(u,0) +$

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} u^i v^j \frac{\partial^{i+j}}{\partial t^j\,\partial x^i} f(0,0)\Big]$$

Where $T_i(0,v) = S_2\left\{\frac{\partial^i}{\partial x^i} f(0,t)\right\}$ and $T_j(u,0) = S_2\left\{\frac{\partial^j}{\partial t^j} f(x,0)\right\}$

1. On Invariant Subspace Method

In 2018, authors, [7, 9]. Developed the invariant subspace method for finding exact solutions to some nonlinear partial differential equations with fractional-order mixed partial derivatives (including both fractional space derivatives and time derivatives). These equations are in the form:

$$\sum_{j=0}^{n} \lambda_i \frac{\partial^{\alpha+j}}{\partial t^{\alpha+j}}\, u(x,t) = N\left(x, u, \frac{\partial^{\beta}}{\partial x^{\beta}} u, \frac{\partial^{\beta+1}}{\partial x^{\beta+1}} u, \ldots, \frac{\partial^{\beta+m}}{\partial x^{\beta+m}} u\right) + \mu \frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(\frac{\partial^{\beta}}{\partial x^{\beta}} u\right) \qquad (1)$$

$$a < \alpha \le a+1,\ \ b < \beta \le b+1,\quad a, b, m, n \in N, \lambda_i, \mu \in R$$

Theorem 3: [9]. Suppose $I_{n+1} = L\{\phi_0(x), \phi_1(x), .., \phi_n(x)\}$ is a finite- dimensional linear space, and it is invariant with respect to the operators $N[u]$ and $\frac{\partial^\beta}{\partial x^\beta}u$, then FPDE (1) has an exact solution as follows:

$$u(x,t) = \sum_{i=0}^{n} k_i(t)\phi_i(x) \tag{2}$$

Where $\{k_i(t)\}$ satisfies the following system of FDEs :

$$\sum_{j=0}^{n} \lambda_i \frac{\partial^{\alpha+j}}{\partial t^{\alpha+j}} k_i(t) = \psi_i\big(k_0(t), k_1(t), k_2(t), \dots, k_n(t)\big) + \mu \frac{d^\alpha \psi_{n+1+i}}{dt^\alpha}, \quad i = 0,1,\dots,n \tag{3}$$

Where $\{\psi_0, \psi_1, \dots, \psi_n\}, \{\psi_{n+1}, \psi_{n+2}, \dots, \psi_{2n+1}\}$ are the expansion coefficients of $N[u]$ and $\frac{\partial^\beta}{\partial x^\beta}u$ respectively with respect to $\{\phi_0(x), \phi_1(x), .., \phi_n(x)\}$

6. Numerical Examples

In this section, we consider that the inverse double Sumudu transform is exist.
We apply this transform and technique of invariant subspace method to solve some of the fractional diffusion heat and wave equations in one dimension with initial and boundary conditions and fractional parabolic-hyperbolic differential equations.

Example 1: Consider the homogeneous fractional wave equation:

$$D_t^\alpha f(x,t) = D_x^\beta f(x,t) \qquad 1 < \alpha,\ \beta \le 2 \tag{4}$$

With the initial and boundary conditions

$$f(x,0) = xE_{\beta,2}\big(-x^\beta\big), \quad f_t(x,0) = 2\ , f(0,t) = 2t, \quad f_x(0,t) = E_\alpha(-t^\alpha)$$

Solution by Double Sumudu Transform Method.

Operating double Sumudu transform of Equation (4), and using Theorem 2, we get:

$$\mathrm{v}^{-\alpha}[\mathrm{T(u,v)} - \mathrm{T_0(u,0)} - \mathrm{vT_1(u,0)}] = \mathrm{u}^{-\beta}[\mathrm{T(u,v)} - \mathrm{T_0(0,v)} - \mathrm{uT_1(0,v)}] \tag{5}$$

By applying the single Sumudu transform for conditions, and using properties of Mittag-Leffler function, we get:

$$\mathrm{T_0(u,0)} = \frac{\mathrm{u}}{1+\mathrm{u}^\beta}, \quad \mathrm{T_1(u,0)} = 2, \quad \mathrm{T_0(0,v)} = 2\mathrm{v}, \quad \mathrm{T_1(0,v)} = \frac{\mathrm{v}}{1+\mathrm{v}^\alpha} \tag{6}$$

By substitute (6) in (5), we obtain

$$\mathrm{v}^{-\alpha}\left[\mathrm{T(u,v)} - (2\mathrm{v} + \frac{\mathrm{u}}{1+\mathrm{u}^\beta})\right] = \mathrm{u}^{-\beta}\left[\mathrm{T(u,v)} - (2\mathrm{v} + \frac{\mathrm{u}}{1+\mathrm{v}^\alpha})\right]$$

$$\big(\mathrm{u}^\beta - \mathrm{v}^\alpha\big)\mathrm{T(u,v)} = \mathrm{u}^\beta\left(2\mathrm{v} + \frac{\mathrm{u}}{1+\mathrm{u}^\beta}\right) - \mathrm{v}^\alpha(2\mathrm{v} + \frac{\mathrm{u}}{1+\mathrm{v}^\alpha})$$

$$= 2\mathrm{v}\big(\mathrm{u}^\beta - \mathrm{v}^\alpha\big) + \frac{\mathrm{u}\big(\mathrm{u}^\beta - \mathrm{v}^\alpha\big)}{(1+\mathrm{u}^\beta)(1+\mathrm{v}^\alpha)}$$

$$\mathrm{T(u,v)} = 2\mathrm{v} + \frac{\mathrm{u}}{1+\mathrm{u}^{\beta}}\frac{1}{1+\mathrm{v}^{\alpha}} \tag{7}$$

The inverse double Sumudu transformation of (7) yields the exact solution of (4) as:

$$\mathrm{f(x,t)} = 2\mathrm{t} + \mathrm{x}\mathrm{E}_{\beta,2}\left(-\mathrm{x}^{\beta}\right)\mathrm{E}_{\alpha}(-\mathrm{t}^{\alpha})$$

When $\alpha = \beta = 2$, the exact solution for standard wave equation $f_{tt} = f_{xx}$ is:

$$\mathrm{f(x,t)} = 2\mathrm{t} + \sin(\mathrm{x})\cos(\mathrm{t}).$$

Solution by Invariant Subspace Method:

Let $I_2 = \{1, x^{\beta}\}$ is invariant subspace under the operator $N[f] = \frac{\partial^{\beta}}{\partial x^{\beta}} f$ as:

For $f \in I_2$, then $N[f] = [k_0 + k_1 x^{\beta}] = \frac{\partial^{\beta}}{\partial x^{\beta}}(k_0 + k_1 x^{\beta}) = \Gamma(1+\beta)k_1 \in I_2$

Then by [4]. We conclude the following FDEs:

$$\frac{d^{\alpha}}{dt^{\alpha}} k_0(t) = \Gamma(1+\beta)k_1(t) \tag{8}$$

$$\frac{d^{\alpha}}{dt^{\alpha}} k_1(t) = 0 \Longrightarrow k_1(t) = a + bt \tag{9}$$

Substitute (9) in (8), we obtain

$k_0(t) = \frac{a\Gamma(1+\beta)}{\Gamma(1+\alpha)} t^{\alpha} + \frac{b\Gamma(1+\beta)}{\Gamma(2+\alpha)} t^{\alpha+1} + c + dt$ for any arbitrary constants , b, c, d

So, the exact solution for equation (4) is:

$$f(x,t) = k_0(t) + k_1(t)x^{\beta} = \frac{a\Gamma(1+\beta)}{\Gamma(1+\alpha)} t^{\alpha} + \frac{b\Gamma(1+\beta)}{\Gamma(2+\alpha)} t^{\alpha+1} + c + dt + (a+bt)x^{\beta}$$

If we change the invariant subspace, we get anew exact solution of the same equation (4) as if we consider the new subspace such as $I_2 = \{1, xE_{\beta,2}(-x^{\beta})\}$,

Also, this subspace is an invariant under the operator N, since

$$N[f] = D_x^{\beta}\left[k_0 + k_1\, xE_{\beta,2}(-x^{\beta})\right] = -k_1 x^{1-\beta} E_{\beta,2-\beta}\left(-x^{\beta}\right) = -k_1 x^{\beta} E_{\beta,2}\left(-x^{\beta}\right) \in I_2$$

and we have the following new FDEs :

$$\frac{d^{\alpha}}{dt^{\alpha}} k_0(t) = a + bt$$

$$\frac{d^{\alpha}}{dt^{\alpha}} k_1(t) = -k_1(t) \Longrightarrow k_1(t) = E_{\alpha}(-t^{\alpha})$$

Hence, the exact solution in this case is:

$$f(x,t) = k_0 + k_1\, xE_{\beta,2}\left(-x^{\beta}\right) = a + bt + E_{\alpha}(-t^{\alpha})xE_{\beta,2}\left(-x^{\beta}\right)$$

From our conditions, we get

$$f(x,0) = 0 \Longrightarrow a = 0, \quad f(0,t) = 2t \Longrightarrow b = 2$$

So, the exact solution is

$$f(x,t) = 2t + E_{\alpha}(-t^{\alpha})xE_{\beta,2}(-x^{\beta})$$

Which agree with that by double Sumudu transform.

Table 2. Showing the absolutely error of some of 10-order approximate solutions of Equation (4), for various values of α, β.

Also, 3-D plotting of these solutions are illustrated in **Figure 1.**

Table 2. Absolutely error of 10-order approach solutions of Equation (4).

(x,t)	u_{exa}	u_{10app}	$\vert Error \vert$
	$\alpha = \beta = 2$		
(0.3,0.1)	0.4940438366	0.4940438366	0
(0.6,0.2)	0.9533872166	0.9533872166	0
(0.4,0.8)	1.8713103718	1.8713103718	0
(0.9,0.9)	2.2869238154	2.2869238154	0
	$\alpha = 1.3 \quad \beta = 1.7$		
(0.3,0.1)	0.478523188	0.478523188	0
(0.6,0.2)	0.8867989478	0.8867989478	0
(0.4,0.8)	1.7865918099	1.7865918099	$7.849359999e-14$
(0.9,0.9)	2.1139517311	2.1139517311	$8.1278659999e-13$

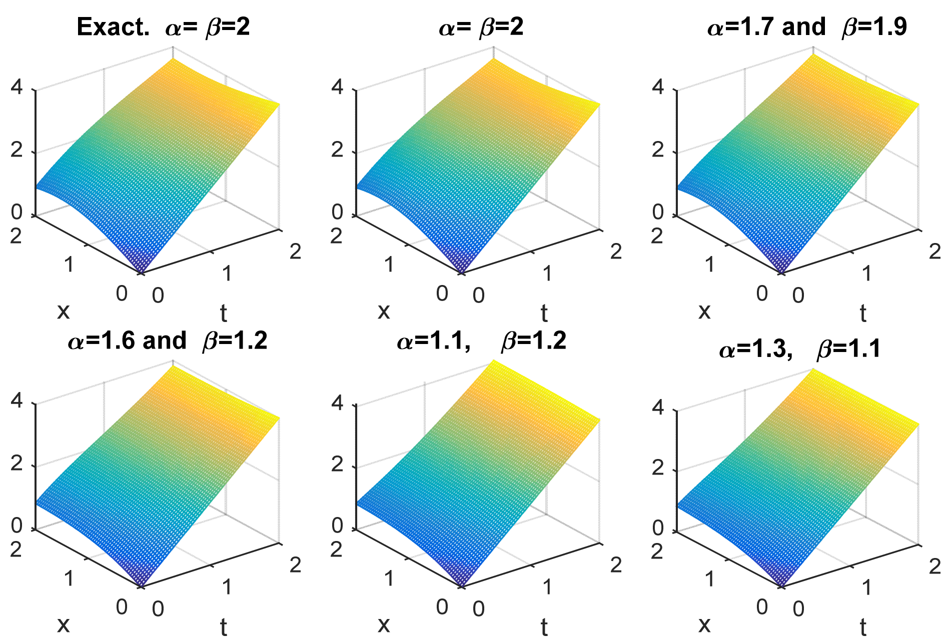

Figure 1. Exact solution and 10-order approximate solutions of Equation (4) for various values of α, β.

Example 2: Consider the following inhomogeneous fractional wave equation:

$$D_t^{\alpha} f(x,t) - D_x^{\beta} f(x,t) = 6t + 2x \qquad 1 < \alpha, \beta \leq 2 \tag{10}$$

With conditions

$$f(x,0) = 0, \qquad f_t(x,0) = xE_{\beta,2}\left(-x^{\beta}\right), \; f(0,t) = t^3, \quad f(0,t) = t^2 + tE_{\alpha,2}(-t^{\alpha})$$

Solution by Double Sumudu Transform Method.
By analogous in previous examples, we get

$$v^{-\alpha}\left(T(u,v) - \frac{uv}{1+u^{\beta}}\right) = u^{-\beta}\left(T(u,v) - 6v^3 - u\left(2v^2 + \frac{v}{1+v^{\alpha}}\right)\right) + 6v + 2u$$

$$\left(u^{\beta} - v^{\alpha}\right)T(u,v) = \frac{u^{1+\beta}v}{1+u^{\beta}} - v^{\alpha}\left(6v^3 + 2uv^2 + \frac{uv}{1+v^{\alpha}}\right) + 2u^{\beta}v^{\alpha}(u+3v)$$

$$= \frac{uv\left(u^{\beta} - v^{\alpha}\right)}{(1+u^{\beta})(1+v^{\alpha})} + 2v^{\alpha}(u+3v)\left(u^{\beta} - v^{\alpha}\right)$$

$$T(u,v) = \frac{u}{1+u^{\beta}}\frac{v}{1+v^{\alpha}} + 2v^{\alpha}(u+3v) \tag{11}$$

The inverse double Sumudu transform of Eq.(11) reads

$$f(x,t) = xE_{\beta,2}\left(-x^{\beta}\right)tE_{\alpha,2}(-t^{\alpha}) + \frac{2xt^{\alpha}}{\Gamma(1+\alpha)} + \frac{6t^{1+\alpha}}{\Gamma(2+\alpha)}$$

Note that, when $= \beta = 2$, we have $f\,(x,t) = xt^2 + t^3 + \sin x \sin t$ which agree with the exact solution of standard non-homogeneous wave equation $f_{tt} - f_{xx} = 2(x+3t)$**.**

Solution by invariant subspace method

The inhomogeneous fractional partial differential equation (10), cannot be solved by invariant subspace method directly, in this case we find the solution of homogeneous equation directly by invariant subspace method and then we can be use any suitable numerical method ”Variation iteration method”, i. e, coupled invariant subspace method with variation iteration method(ISVIM) to find an approximate solution of the original equation.
Table 3. Contains the absolutely error of some of 10-order approximate solutions of eq.(10) for various values of α,β.
3-D plotting of these solutions are illustrated in **Figure 2.**

Table 3. Absolutely error of 10-order approximate solutions of Equation 10.

(x,t)	u_{exa}	u_{10app}	$\vert Error \vert$
		$\alpha=\beta=2$	
$(0.3, 0.1)$	0.0335027919	0.0335027919	0.006
$(0.6, 0.2)$	0.1441771423	0.1441771423	0.048
$(0.4, 0.8)$	0.91935161976	1.0473516198	0.128
$(0.9, 0.9)$	2.0716010473	2.0716010473	0
		$\alpha=1.4\ \beta=1.8$	
$(0.3, 0.1)$	0.0561281872	0.0561281872	0
$(0.6, 0.2)$	0.2500829871	0.2500829871	0
$(0.4, 0.8)$	1.8895767906	1.8895767906	$1.511e-16$
$(0.9, 0.9)$	3.3194135764	3.3194135764	$2.0361999999e-15$

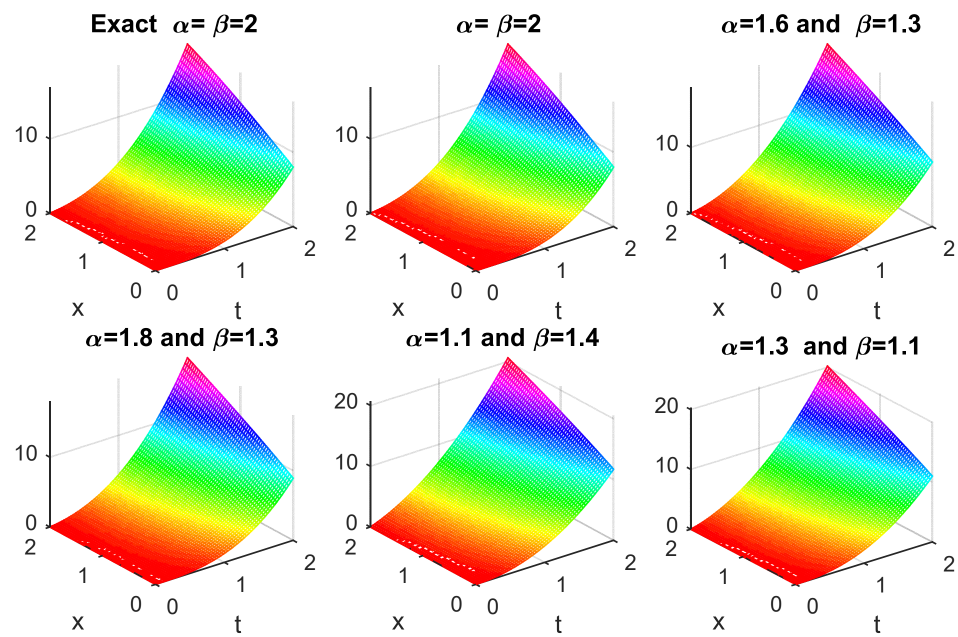

Figure 2. Exact solution and 10-order approximate solutions of Equation 10, for various values of α, β.

Example 3: Consider the following mixed partial fractional differential equation

$$D_t^{\beta} D_x^{\alpha} f(x,t) + f(x,t) = 0, \qquad 0 < \alpha, \beta \leq 1 \tag{12}$$

With conditions

$$f(0,t) = E_{2\beta}(t^{2\beta}), \quad f(x,0) = E_{2\alpha}(x^{2\alpha})$$

Solution by double Sumudu Transform Method:

By Theorem (2) one can obtain

$$\mathrm{u}^{-\alpha}\mathrm{v}^{-\beta}[\mathrm{T}(\mathrm{u},\mathrm{v}) - \mathrm{T}(\mathrm{u},0) - \mathrm{v}\mathrm{T}(\mathrm{u},0)] + \mathrm{T}(\mathrm{u},\mathrm{v}) = 0$$

$$(1 + \mathrm{u}^{-\alpha}\mathrm{v}^{-\beta})T(u,v) = \frac{1}{1-v^{2\beta}} + \frac{1}{1-u^{2\alpha}} - 1 = \frac{1-u^{2\alpha}v^{2\beta}}{(1-v^{2\beta})(1-u^{2\alpha})}$$

$$T(u,v) = \left(\frac{1}{1-v^{2\beta}}\right)\left(\frac{1}{1-u^{2\alpha}}\right) - \left(\frac{v^{\beta}}{1-v^{2\beta}}\right)\left(\frac{u^{\alpha}}{1-u^{2\alpha}}\right) \tag{13}$$

Operating inverse double Sumudu transformation of (13) reads the exact solution of (12) as:

$$f(x,t) = E_{2\beta}\left(t^{2\beta}\right)E_{2\alpha}(x^{2\alpha}) - t^{\beta}E_{2\beta,1+\beta}\left(t^{2\beta}\right)x^{\alpha}E_{2\alpha,1+\alpha}(t^{2\alpha})$$

For $\alpha = \beta = 1$, we obtain the standard differential equation $f_{tx} + f = 0$, which has the exact solution as

$$f(x,t) = \cosh(x-t).$$

Solution by Invariant Subspace Method:

Using Theorem (3), Eq. (12) in the operators form is:

$\mathrm{N[f]} + \frac{\partial^{\beta}}{\partial t^{\beta}}\left(\frac{\partial^{\alpha}}{\partial x^{\alpha}}f\right) = 0$ where $\mathrm{N[f]} = f$

By consider $I_2 = \{1,\ E_{\alpha}(x^{\alpha}), E_{\alpha}(-x^{\alpha})\ \}$ is invariant subspace under the operators $N[f]$ and $\frac{\partial^{\alpha}}{\partial x^{\alpha}}f$ since

$\forall f = a + bE_{\alpha}(x^{\alpha}) + cE_{\alpha}(-x^{\alpha}) \quad \in I_2$ $\qquad a, b, c$ are arbitrary constants

$$N[f] = f \in I_2\ , \qquad \frac{\partial^{\alpha}}{\partial x^{\alpha}}f = bE_{\alpha}(x^{\alpha}) - cE_{\alpha}(-x^{\alpha}), \quad \in I_2$$

According to the invariant subspace method and by considering $f = \sum_{i=0}^{n-1} C_i\,(t)\phi_i(x)$ we have the following FDEs

$$C_0(t) = 0$$

$$\frac{d^{\beta}}{dt^{\beta}}C_1(t) + C_1(t) = 0 \Longrightarrow C_1(t) = AE_{\beta}(-t^{\beta})$$

$$\frac{d^{\beta}}{dt^{\beta}}C_2(t) - C_2(t) = 0 \Longrightarrow C_2(t) = BE_{\beta}(t^{\beta})$$

So, the solution of Eq.(12) is:

$$\boldsymbol{f(x,t)} = C_0(t) + C_1(t)E_{\alpha}(x^{\alpha}) + C_2(t)E_{\alpha}(-x^{\alpha})$$
$$= AE_{\beta}\left(-t^{\beta}\right)E_{\alpha}(x^{\alpha}) + BE_{\beta}(t^{\beta})E_{\alpha}(-x^{\alpha})$$

From our conditions, we have

$$f(x,0) = E_{2\alpha}(x^{2\alpha}) = \frac{1}{2}[E_{\alpha}(x^{\alpha}) + E_{\alpha}(-x^{\alpha})] = AE_{\alpha}(x^{\alpha}) + BE_{\alpha}(-x^{\alpha})$$

Thus, $A = B = \frac{1}{2}$ and $f(x,t) = \frac{1}{2}\left[E_{\beta}\left(-t^{\beta}\right)E_{\alpha}(x^{\alpha}) + E_{\beta}\left(t^{\beta}\right)E_{\alpha}(-x^{\alpha})\right]$

For $\alpha = \beta = 1$

$$f(x,t) = \frac{1}{2}(e^{x-t} + e^{t-x}) = \cosh\,(x-t)$$

which agree with that in double Sumudu transform method.

Exact solution and 10-order approximate solutions of Equation 14, are shown in **Figure 3.** For various values of α, β.

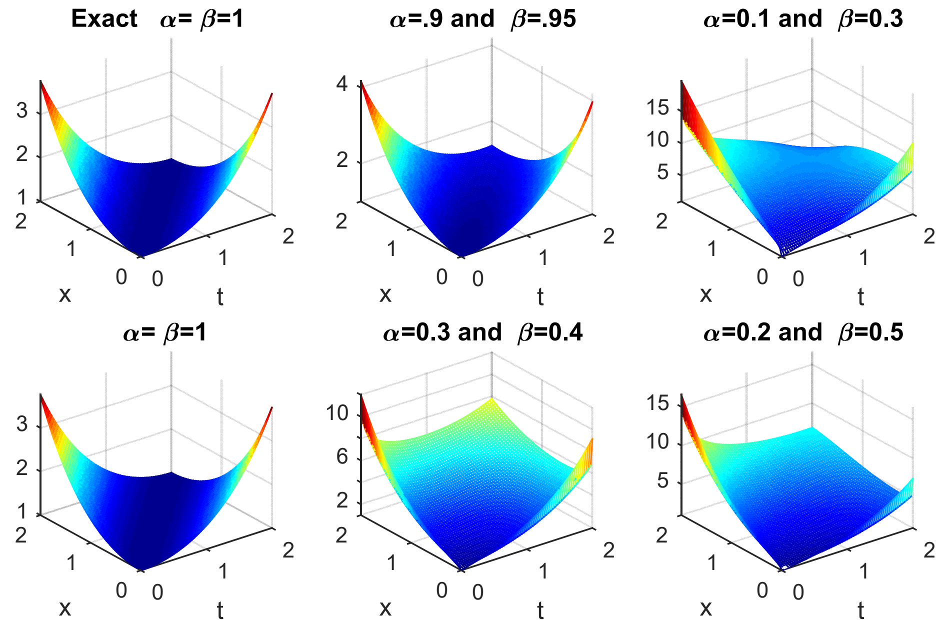

Figure 3. Exact solution and 10-order approximate solutions of Equation 14, for various values of α, β.

Example 4: Consider the following fractional mixed derivatives diffusion equation

$$D_t^{\alpha} f = D_x^{\beta} f + D_t^{\alpha} D_x^{\beta} f \qquad 0 < \alpha \leq 1, \qquad 1 < \beta \leq 2 \tag{14}$$

With conditions

$$f(0,t) = 1 + \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha)} t^{\alpha}, \quad f_x(0,t) = 0, \qquad f(x,0) = 1 + x^{\beta}$$

Solution by Double Sumudu Transform Method:

Operating single Sumudu transform of conditions, we get

$$\boldsymbol{T_0(0,v) = 1} + \Gamma(1+\beta) v^{\alpha}, \qquad \boldsymbol{T_1(0,v) = 0}, \qquad \boldsymbol{T_0(u,0) = 1} + \Gamma(1+\beta) \boldsymbol{u^{\beta}}$$

Using Theorem (2), we get

$v^{-\alpha}[T(u,v) - T_0(u,0)] = u^{-\beta}[T(u,v) - T_0(0,v) - uT_1(0,v)] + v^{-\alpha}u^{-\beta}[T(u,v) -$
$T_0(0,v) - uT_1(0,v) - T_0(u,0) + 1]$

$$\begin{aligned} \left(1 + v^{\alpha} + u^{\beta}\right) T(u,v) \\ &= v^{\alpha}\left(1 + v^{\alpha}\Gamma(1+\beta)\right) + \left(v^{\alpha} + u^{\beta}\right)\Gamma(1+\beta) + 1 - u^{\beta}(1 + u^{\beta}\Gamma(1+\beta)) \\ &= \left(1 + v^{\alpha} + u^{\beta}\right)(1 + \left(v^{\alpha} + u^{\beta}\right)\Gamma(1+\beta)) \\ T(u,v) &= 1 + \left(v^{\alpha} + u^{\beta}\right)\Gamma(1+\beta) \end{aligned}$$

Taking inverse double Sumudu transform of the last equation, we can obtain the exact solution of (14) as

$$\boldsymbol{f(x,t)} = 1 + x^{\beta} + \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha)} t^{\alpha}$$

which agree with that for [9].

3-D plotting of the exact solution and 10-order approximate solutions for Equation (14) for various values of α, β, are contained in **Figure 4.**

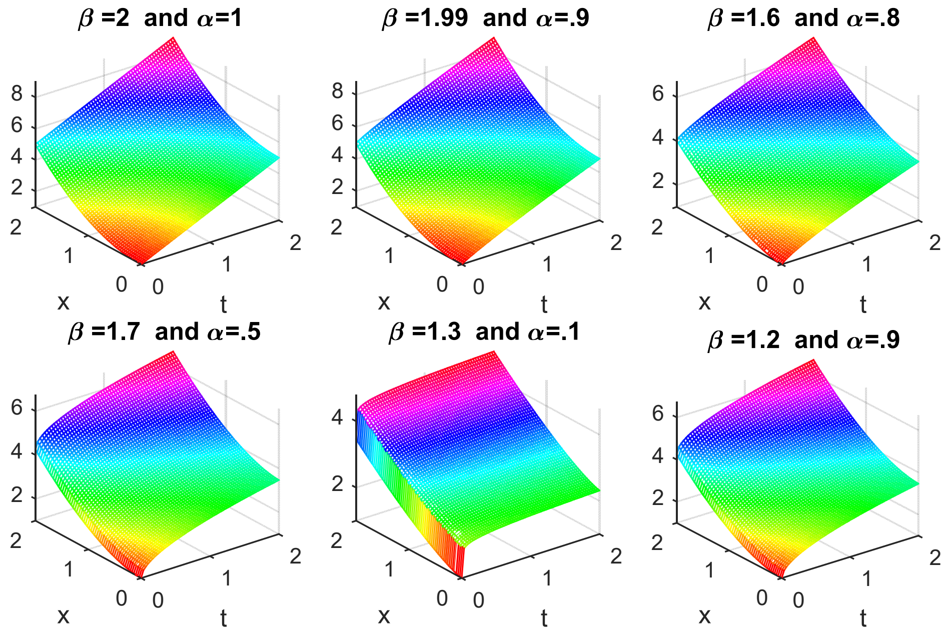

Figure 4. Exact solution and some of approximate solution of Equation 16, for various values of α, β.

7. Conclusion

In this paper, the comparison between the analytical subspace method and double Sumudu transform method has been achieved. From this study, we insure that the double Sumudu transform method is an efficient tool to obtain an exact solutions of many types of linear fractional partial differential equations that have complicated using other numerical methods, however, more of homogeneous type of constant coefficients fractional partial differential equations can be solved analytically with the invariant subspace method than the Sumudu transform.

In both methods, our example illustrated that the solutions which have been achieved in terms of the infinite series Mittag-Leffler function and we have done this by truncated 10-order approximate of this function. This is the reason for explaining the variance of the absolute error for numerical solutions.

Of course, for nonlinear case, we must couple the Sumudu transform with any known numerical method, which need not it with invariant subspace method.

Furthermore, the invariant subspace method is more suitable for solving some of fractional differential equations with variable coefficients than the double Sumudu transform method.

Figures and tabulated results, show how the two methods are have high accuracy. All data type achieved by help of Mathcad 15 and Matlab programs.

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