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Comparison Among Three Estimation Methods to Estimate Cascade Reliability Model (2+1) Based On Inverted Exponential Distribution

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Abstract

In this paper, we are mainly concerned with estimating cascade reliability model (2+1) based on inverted exponential distribution and comparing among the estimation methods that are used. The maximum likelihood estimator and uniformly minimum variance unbiased estimators are used to get the parameters of the strengths X_k and the stress Y_k;k=1,2,3 respectively then, by using the unbiased estimators, we propose Preliminary test single stage shrinkage (PTSSS) estimator when a prior knowledge is available for the scale parameter as initial value due past experiences . The Mean Squared Error [MSE] for the proposed estimator is derived to compare among the methods. Numerical results about conduct of the considered estimator are discussed including the study of mentioned expressions. The numerical results are exhibited and put it in tables.

Keyword: Inverted exponential distribution, Maximum Likelihood method, Uniformly Minimum Variance Unbiased method, Single Stage Shrinkage Estimator, Mean Squared Error and Cascade Reliability Model (2+1).

1. Introduction

An n - cascade system is defined as a special type of standby system with n components and it could be considered as a kind of stress-strength model. The reliability system of a cascade model can be described by a function of parameters of the identical and independent distributions with strength (X) and stress (Y) and the attenuation factor (K) in other words, the stresses on subsequent components are attenuated by a factor ‘k’, called attenuation factor that is generally assumed to be a constant for all the components or to be a parameter having different fixed values for different components, or it may be simply a random variable. This system was first proposed and studied by Pandit and Sriwastav (1975).Rekha and Chechu Raju (1999) presented a closed form solution of stress attenuated reliability function for n-cascade system with exponential stress and standby strengths following Rayleigh and exponential distributions. Sundar (2012)has done a case study of cascade reliability with Rayleigh distribution. Devi (2016) studied Cascade System Reliability with Stress and Strength Follow Lindley Distribution. Dong and Cui studied System Reliability under Cascading Failure Models. Doloi and Gogoi (2017) presented a cascade reliability model for exponential and Lindley distributions. Reddy(2016) studied Cascade and System Reliability for Exponential Distributions. In (2019) Kanaparthi et el studied Cascade and System Reliability for the New Rayleigh-Pareto Distribution

The aim of this study is to compare among three estimation methods (MLE ,UMVUE and PTSSs) in estimate the cascade reliability model(2+1) based on inverted exponential distribution (IE) with unknown scale parameter. For this purpose, we use Thamson shrinkage technique through the equation below:

$$\tilde{\alpha} = \emptyset(\hat{\alpha})(\hat{\alpha} - \alpha_0) + \alpha_0 \tag{1}$$

Where α_0 is a prior knowledge(estimation) about the parameter α and $0 \leq \emptyset(\hat{\alpha}) \leq 1$ is a shrinkage weight factor to assign the degree of belief in α_0; also $\hat{\alpha}$ is the classical estimator of α (MLE or **U**MVUE). Several authors have studied the estimator defined in (1) for a special distribution for different parameters and suitable regions (R) as well as for estimate the parameters of linear regression model. See [13-16].

2. Statistical Model

In life distribution, if a random variable X following an exponential distribution then the variable $Y = \frac{1}{X}$ has an inverted exponential distribution (IE) . The Invers Exponential Distribution IE(α) has the following probability density function p.d.f and cumulative distribution function c.d.f. for $x > 0$:

$$f(x;\alpha) = \frac{\alpha}{x^2} exp\left(\frac{-\alpha}{x}\right) \quad ; \alpha > 0 \tag{2}$$

$$F(x;\alpha) = exp\left(\frac{-\alpha}{x}\right) \quad ; \alpha > 0 \tag{3}$$

Killer and Kamath (1982) defined and studied IE for the first time. Lin et al (1989) descripted IE distribution as a lifetime model. It can be seen that IE distribution was widely used in the analytic study specially in engineering, biology and medicine fields Oguntunde et el(2017). Bayes estimators of the parameter and reliability function of the IE distribution were obtained by Singh et al(2013). Oguntunde et al. (2014) proposed exponentiated GIE (EGIE) distribution. Singh et al (2015) estimated the stress strength reliability parameter of IE distribution. Fatima and Ahmad (2018) used a Bayesian Approximation Techniques of Inverse Exponential Distribution with Applications in Engineering.

For cascade model (2+1), assume that X_1, X_2 and X_3 are the strengths with two fundamental components and one standby component is i.i.d with p.d.f follows IED(α) with scale parameter α_i ; $i = 1,2,3$, when activated faces the stresses random variables Y_1, Y_2 and Y_3 are imposed on the strengths components and followed IED(β) with scale parameter β_j; $j = 1,2,3$. In cascade system, after every failure the stress gets modified by attenuation factor (k) such that: $Y_2 = kY_1, Y_3 = kY_2 = k^2Y_1, \dots, Y_i = k^{i-1}Y_i; k > 1$

And we suppose a factor (m) to modified the strength such that: $X_2 = mX_1, X_3 = mX_2 = m^2X_1, \dots, X_i$

$$= m^{i-1}X_1 \text{ ; m} > 0$$

The real reliability function for the (2+1) cascade model is given by:

$$R = p[X_1 \geq Y_1, X_2 \geq Y_3] + p[X_1 < Y_1, X_2 \geq Y_2, X_3 \geq Y_3] + p[X_1 \geq Y_1, X_2 < Y_2, X_3 \geq Y_3] \tag{4}$$

$$R = R_1 + R_2 + R_3 \tag{5}$$

$$R_1 = p[X_1 \geq Y_1, X_2 \geq Y_2] \tag{6}$$

$$= \int_0^\infty [\bar{F}_{x_1}(y_1)]g(y_1)dy_1 \int_0^\infty [\bar{F}_{x_2}(y_2)]g(y_2)dy_2$$

$$= \int_0^\infty \left[1 - exp\left(\frac{-\alpha_1}{y_1}\right)\right]\frac{\beta_1}{{y_1}^2}exp\left(\frac{-\beta_1}{y_1}\right) dy_1 \int_0^\infty \left[1 - exp\left(\frac{-\alpha_2}{y_2}\right)\right]\frac{\beta_2}{{y_2}^2}exp\left(\frac{-\beta_2}{y_2}\right) dy_2$$

So we get

$$R_1 = \left[\frac{\alpha_1}{\alpha_1+\beta_1}\right]\left[\frac{\alpha_2}{\alpha_2+\beta_2}\right] \tag{7}$$

$$R_2 = p[X_1 < Y_1, X_2 \geq Y_2, X_3 \geq Y_3] \tag{8}$$

$$= p[X_1 < Y_1, mX_1 \geq kY_1]\, p[X_2 \geq Y_2]p[X_2 \geq Y_2]$$

where

$$p[X_1 < Y_1, mX_1 \geq kY_1] = \int_0^\infty [F_{x_1}(y_1)][\bar{F}_{x_1}(\tfrac{k}{m}\, y_1)]g(y_1)dy_1 \tag{9}$$

$$= \int_0^\infty exp\left(\frac{-\alpha_1}{y_1}\right)\left[1 - exp\left(\frac{-\alpha_1}{\frac{k}{m}y_1}\right)\right]\frac{\beta_1}{{y_1}^2}exp\left(\frac{-\beta_1}{y_1}\right)$$

Then $p[X_1 < Y_1, mX_1 \geq kY_1] = \frac{(\frac{k}{m}\alpha_1)\beta_1}{\left(\frac{k}{m}\alpha_1+\beta_1\right)\left(\alpha_1+\frac{k}{m}\alpha_1+\beta_1\right)}$ (10)

And

$$p[X_2 \geq Y_2] = \int_0^\infty [\bar{F}_{x_2}(y_2)]g(y_2)dy_2 = \frac{\alpha_2}{\alpha_2+\beta_2} \tag{11}$$

So

$$R_2 = \left[\frac{(\frac{k}{m}\alpha_1)\beta_1}{\left(\frac{k}{m}\alpha_1+\beta_1\right)\left(\alpha_1+\frac{k}{m}\alpha_1+\beta_1\right)}\right]\left(\frac{\alpha_2}{\alpha_2+\beta_2}\right) \tag{12}$$

By the same way finding R_3

$$R_3 = p[X_1 \geq Y_1, X_2 < Y_2, X_3 \geq Y_3] \tag{13}$$

$$= \left[\frac{\frac{k}{m}\alpha_2\beta_2}{\left(\frac{k}{m}\alpha_2+\beta_2\right)\left(\alpha_2+\frac{k}{m}\alpha_2+\beta_2\right)}\right]\left(\frac{\alpha_1}{\alpha_1+\beta_1}\right) \tag{14}$$

Substitution (6), (11) and (13) in (4)

$$R = \left[\frac{\alpha_1}{\alpha_1+\beta_1}\right]\left[\frac{\alpha_2}{\alpha_2+\beta_2}\right] + \left[\frac{\frac{k}{m}\alpha_1\beta_1}{\left(\frac{k}{m}\alpha_1+\beta_1\right)\left(\alpha_1+\frac{k}{m}\alpha_1+\beta_1\right)}\right]\left(\frac{\alpha_2}{\alpha_2+\beta_2}\right) + \left[\frac{\frac{k}{m}\alpha_2\beta_2}{\left(\frac{k}{m}\alpha_2+\beta_2\right)\left(\alpha_2+\frac{k}{m}\alpha_2+\beta_2\right)}\right]\left(\frac{\alpha_1}{\alpha_1+\beta_1}\right) \tag{15}$$

3. Maximum Likelihood Estimation (MLE)

The Maximum likelihood method is an important and commonly, since it contained properties for good estimate. Suppose$X_i; i = 1,2,3$ strength random sample follows IED(α) with the sample size n.The likelihood function is given by;

$$l = L(x_1, x_2, \ldots, x_n; \alpha)$$

$$= \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{\alpha}{x_i{}^2} exp\left(\frac{-\alpha}{x_i}\right)$$

$$= \alpha^n \prod_{i=1}^{n} \frac{1}{x_i{}^2}\ exp\left(\frac{-\alpha}{\sum_{i=1}^{n} x_i}\right) \tag{16}$$

Taking logarithm of (16) and then differentiating the result partially with respect to α:

$$\frac{\partial lnL}{\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \frac{1}{x_i} \tag{17}$$

Equalizing (17) to zero to get the estimated scale parameter of IED

$$\hat{\alpha}_{MLE} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} \tag{18}$$

Hence, for strength random samples, $X_{1_{i_1}} \sim IE(\alpha_1); i_1 = 1,2, \ldots, n_1$, $X_{2_{i_2}} \sim IE(\alpha_2); i_2 = 1,2, \ldots, n_2$ and $X_{3_{i_3}} \sim IE(\alpha_3); i_3 = 1,2, \ldots, n_3$ with samples size n_1 , $n_2$and n_3 respectively MLE for unknown parameters $\alpha_1, \alpha_2$and α_3 is :

$$\hat{\alpha}_{\Lambda(MLE)} = \frac{n_\Lambda}{T_\Lambda} \quad ; T_\Lambda = \sum_{i_\Lambda=1}^{n_\Lambda} \frac{1}{x_{\Lambda_{i_\Lambda}}} \quad ; \Lambda = 1,2,3 \tag{19}$$

By the same way for the stress random variables$Y_{1_{j_1}} \sim IE(\beta_1); j_1 = 1,2, \ldots, m_1$, $Y_{2_{j_2}} \sim IE(\beta_2); j_2 = 1,2, \ldots, m_2$and $Y_{3_{j_3}} \sim IE(\beta_3); j_3 = 1,2, \ldots, m_3$ with samples size m_1 , $m_2$and m_3 respectively, the MLE estimator for unknown parameters $\beta_1, \beta_2$and β_3 will be as follows:

$$\hat{\beta}_{\Lambda(MLE)} = \frac{m_\Lambda}{H_\Lambda}\ ;\ H_\Lambda = \sum_{j_\Lambda=1}^{m_\Lambda} \frac{1}{y_{\Lambda_{j_\Lambda}}},\ \ \Lambda = 1,2,3 \tag{20}$$

Substitution (19) and (20) in (15) the MLE for Cascade reliability, invariability will be as:

$$\hat{R}_{MLE} = \left[\frac{\hat{\alpha}_{1(MLE)}}{\hat{\alpha}_{1(MLE)}+\hat{\beta}_{1(MLE)}}\right]\left[\frac{\hat{\alpha}_{2(MLE)}}{\hat{\alpha}_{2(MLE)}+\hat{\beta}_{2(MLE)}}\right] + \left[\frac{\frac{k}{m}\hat{\alpha}_{1(MLE)}\hat{\beta}_{1(MLE)}}{\left(\frac{k}{m}\hat{\alpha}_{1(MLE)}+\hat{\beta}_{1(MLE)}\right)\left(\hat{\alpha}_{1(MLE)}+\frac{k}{m}\hat{\alpha}_{1(MLE)}+\hat{\beta}_{1(MLE)}\right)}\right]\left(\frac{\hat{\alpha}_{2(MLE)}}{\hat{\alpha}_{2(MLE)}+\hat{\beta}_{2(MLE)}}\right) + \left[\frac{\frac{k}{m}\hat{\alpha}_{2(MLE)}\hat{\beta}_{2(MLE)}}{\left(\frac{k}{m}\hat{\alpha}_{2(MLE)}+\hat{\beta}_{2(MLE)}\right)\left(\hat{\alpha}_{2(MLE)}+\frac{k}{m}\hat{\alpha}_{2(MLE)}+\hat{\beta}_{2(MLE)}\right)}\right]\left(\frac{\hat{\alpha}_{1(MLE)}}{\hat{\alpha}_{1(MLE)}+\hat{\beta}_{1(MLE)}}\right) \tag{21}$$

4. Uniformly Minimum Variance Unbiased Estimators

The UMVUE method depends on minimizing the mean square error among unbiased estimators. The unbiased estimator $\hat{\alpha}$ of α is called (UMVUE) if and only if Var $(\hat{\alpha}) \leq$ Var$(\hat{\alpha}_{ub})$ for any $x \in X$ and any other unbiased estimator of α, see Devor and Berk (2012). We could find the UMVU of the scale parameters $\alpha_{\Lambda,} and\ \beta_\Lambda$; Λ =1,2,3 of the stress $X_{\Lambda_{i_\Lambda}}; i_\Lambda = 1,2, \ldots, n_\Lambda$ and strength $Y_{\Lambda_{j_\Lambda}} j_\Lambda = 1,2, \ldots, m_\Lambda$;of cascade reliability model with IED that belongs to exponential family densities function as it shown below :

$Q(x; \sigma) = a(\sigma)b(x)\exp\left(\sum_{j=1}^{m} \rho_j(\sigma) r_j(x)\right)$, where $a(\sigma), b(x) > o$, and $\sigma\sigma_1, \sigma_2, \ldots, \sigma_r$ with $\gamma_j < \sigma_j < \delta_j$ and each of $\sigma, \gamma_j and \delta_j$ are constant .

Let $a(\sigma) = \alpha_1$; $b(x) = 1/x^2$; $\rho_j(\sigma) = -\alpha_1$; $r_j(x) = \frac{1}{x}$

Thus, T_Λ is a complete sufficient statistic for (α_Λ) for Λ =1, 2,3.

To find the distribution of $T_1 = \sum_{i=1}^{n_1} \frac{1}{x_{1_{i_1}}}$, suppose $Z_1 = \frac{1}{X}$ consequently, $X = \frac{1}{Z_1}$

$$\wp(z_1) = f(X = \frac{1}{Z_1}) \frac{dx}{dz_1} \tag{22}$$

Substitute (2) in (22)

$$\wp(z_1) = \alpha_1 \exp(-\alpha_1 z_1) \tag{23}$$

Thus $Z_1 \sim Exp(\alpha_1)$,and $T_1 \sim \Gamma(n_1, \alpha_1)$

$$U(t_1) = \frac{\alpha_1^{n_1}}{\Gamma(n_1)} t^{n_1 - 1}. \exp(-\alpha_1 t_1) \quad ; \quad t_1 > 0\,, \alpha_1 > 0\,,\ n_1 > 0 \tag{24}$$

then $E\left(\frac{1}{T_1}\right) = \frac{\alpha_1}{n_1 - 1}$

So the unbiased estimator of (α_1) is ($\frac{n_1-1}{T_1}$), therefore according to Lehmann-Scheffe theorem the (UMVUE) of (α_1) is

$$\hat{\alpha}_{1(UMVU)} = \frac{n_1 - 1}{T_1}$$

Hence

$$\hat{\alpha}_{\Lambda(UMVU)} = \frac{n_\Lambda - 1}{T_\Lambda} \ ; T_\Lambda = \sum_{i_\Lambda=1}^{n_\Lambda} \frac{1}{x_{\Lambda_{i_\Lambda}}} \quad , k1,2,3 \tag{25}$$

By the same way, we can obtain (UMVUE) of (β_K) as below:

$$\hat{\beta}_{\Lambda(UMVU)} = \frac{m_\Lambda - 1}{H_\Lambda} \quad ; H_\Lambda = \sum_{j_\Lambda=1}^{m_\Lambda} \frac{1}{y_{\Lambda_{j_\Lambda}}} \quad , \Lambda = 1,2,3 \tag{26}$$

Substituting (25), (26) in (15) to obtain UMVU estimator for cascade reliability model of IED as the following :

$$\begin{aligned} \hat{R}_{(UMVU)} & \left[\frac{\hat{\alpha}_{1(UMVU)}}{\hat{\alpha}_{1(UMVU)} + \hat{\beta}_{1(UMVU)}}\right]\left[\frac{\hat{\alpha}_{2(UMVU)}}{\hat{\alpha}_{2(UMVU)} + \hat{\beta}_{2(UMVU)}}\right] \\ & + \left[\frac{\frac{k}{m}\hat{\alpha}_{1(UMVU)}\hat{\beta}_{1(UMVU)}}{\left(\frac{m}{k}\hat{\alpha}_{1(UMVU)} + \hat{\beta}_{1(UMVU)}\right)\left(\hat{\alpha}_{1(UMVU)} + \frac{m}{k}\hat{\alpha}_{1(UMVU)} + \hat{\beta}_{1(UMVU)}\right)}\right]\left(\frac{\hat{\alpha}_{2(UMVU)}}{\hat{\alpha}_{2(UMVU)} + \hat{\beta}_{2(UMVU)}}\right) \\ & + \left[\frac{\frac{k}{m}\hat{\alpha}_{2(UMVU)}\hat{\beta}_{2(UMVU)}}{\left(\frac{k}{m}\hat{\alpha}_{2(UMVU)} + \hat{\beta}_{2(UMVU)}\right)\left(\hat{\alpha}_{2(UMVU)} + \frac{k}{m}\hat{\alpha}_{2(UMVU)} + \hat{\beta}_{2(UMVU)}\right)}\right]\left(\frac{\hat{\alpha}_{1(UMVU)}}{\hat{\alpha}_{1(UMVU)} + \hat{\beta}_{1(UMVU)}}\right) \end{aligned} \tag{27}$$

5. Preliminary Test Single Shrinkage Estimator (PTSSSE).

In this section, we use (1) to estimate the cascade reliability parameters $\alpha_\Lambda, \beta_\Lambda; \Lambda = 1,2,3$ of $X_\Lambda \sim IED$ and $Y_\Lambda \sim IED$ respectively. First in order to find PTSSSE of α_Λ choose $\emptyset(\hat{\alpha}_\Lambda)$ as follows:

$$\emptyset(\hat{\alpha}_\Lambda) = \begin{cases} \vartheta_1(\hat{\alpha}_\Lambda) & if\, \hat{\alpha}_\Lambda \in R_{pt_1} \\ \vartheta_2(\hat{\alpha}_\Lambda) & if\, \hat{\alpha}_\Lambda \notin R_{pt_1} \end{cases} \tag{28}$$

Where $R_{pt_1}$is a preliminary test region of acceptances of size (λ_1) for testing hypotheses $H_0: \alpha_\Lambda = \alpha_{\Lambda_0}$ vs. $H_A: \alpha_\Lambda \neq \alpha_{\Lambda_0}$ using the test statistic $\tau_1 = \frac{(n_\Lambda - 1)\hat{\alpha}_\Lambda}{\alpha_{\Lambda_0}}$; $\hat{\alpha}_\Lambda = \hat{\alpha}_{\Lambda(UMVU)}$ thus, by rewriting equation(1) as below:

$$\tilde{\alpha}_1 = \begin{cases} \vartheta_1(\hat{\alpha}_\Lambda)\left(\hat{\alpha}_\Lambda - \alpha_{\Lambda_0}\right) + \alpha_{\Lambda_0} & , if\, \hat{\alpha}_\Lambda \in R_{pt_1} \\ \vartheta_2(\hat{\alpha}_\Lambda)\left(\hat{\alpha}_\Lambda - \alpha_{\Lambda_0}\right) + \alpha_{\Lambda_0} & , if\, \hat{\alpha}_\Lambda \notin R_{pt_1} \end{cases} \tag{29}$$

Where $\vartheta_r(\hat{\alpha}_\Lambda); r = 1,2$ represents the shrinkage weight factor which may be a function of $\hat{\alpha}_\Lambda$ or may be constants with the condition $0 \leq \vartheta_r(\hat{\alpha}_\Lambda) \leq 1$. Using the form (29), we proposed two preliminary test single stage shrinkage estimators $\tilde{\alpha}_{sss}$ when a prior knowledge α_{Λ_0} is available for α_Λ

$$\tilde{\alpha}_{sss_{\Lambda_1}} = \begin{cases} \alpha_{\Lambda_0} & , if\, \hat{\alpha}_\Lambda \in R_{pt_1} \\ \hat{\alpha}_\Lambda & , if\, \hat{\alpha}_\Lambda \notin R_{pt_1} \end{cases} \tag{30}$$

Then $\vartheta_1(\hat{\alpha}_\Lambda) = 0$ and $\vartheta_2(\hat{\alpha}_\Lambda) = 1$ in (28)

And

$$\tilde{\alpha}_{sss_{\Lambda_2}} = \begin{cases} \alpha_{\Lambda_0} & , if\, \hat{\alpha}_\Lambda \in R_{pt_1} \\ c_1(\hat{\alpha}_\Lambda - \alpha_{\Lambda_0}) & , if\, \hat{\alpha}_\Lambda \notin R_{pt_1} \end{cases} \tag{31}$$

Thus $\vartheta_1(\hat{\alpha}_\Lambda) = 0$ and $\vartheta_2(\hat{\alpha}_\Lambda) = c_1$ in (28) while:

$R_{pt_1} = (a_1, a_2)$; $a_1 = \frac{\alpha_{\Lambda_0}}{2n_\Lambda} X^2_{1_{1-\lambda/2,2n_\Lambda}}$

And $a_2 = \frac{\alpha_{\Lambda_0}}{2n_\Lambda} X^2_{1_{\lambda/2,2n_\Lambda}}$

where $X^2_{1_{1-\lambda/2,2n_\Lambda}}$, $X^2_{1_{\lambda/2,2n_\Lambda}}$ are respectively the lower and upper $100(\lambda/2)$ precedential point of chi-square distribution with $(2n_\Lambda)$ degree of freedom.

Similarly for parameters β_Λ ; $\Lambda = 1,2,3$ of stress random variables $Y_\Lambda \sim IED$ choose $\emptyset_2(\hat{\beta}_\Lambda)$ as below:

$$\emptyset_2(\hat{\beta}_\Lambda) = \begin{cases} \Theta_1(\hat{\beta}_\Lambda) & if\, \hat{\beta}_\Lambda \in R_{pt_2} \\ \Theta_2(\hat{\beta}_\Lambda) & if\, \hat{\beta}_\Lambda \notin R_{pt_2} \end{cases} \tag{32}$$

Where R_{pt_2} is a preliminary test region of acceptances of size (λ_2) for testing hypotheses Ѣ$_0$: $\beta_\Lambda = \beta_{\Lambda_0}$ vs. Ѣ$_A$: $\beta_\Lambda \neq \beta_{\Lambda_0}$ using the test statistic $\tau_2 = \frac{(m_\Lambda - 1)\hat{\beta}_\Lambda}{\beta_{\Lambda_0}}$; $\hat{\beta}_\Lambda = \hat{\beta}_{\Lambda(UMVU)}$ thus, we get:

$$\tilde{\beta}_\Lambda = \begin{cases} \Theta_1(\hat{\beta}_\Lambda)(\hat{\beta}_\Lambda - \beta_{\Lambda_0}) + \beta_{\Lambda_0} & , if\, \hat{\beta}_\Lambda \in R_{pt_2} \\ \Theta_2(\hat{\beta}_\Lambda)(\hat{\beta}_\Lambda - \beta_{\Lambda_0}) + \beta_{\Lambda_0} & , if\, \hat{\beta}_\Lambda \notin R_{pt_2} \end{cases} \tag{33}$$

Such that $0 \leq \Theta_r(\hat{\beta}_\Lambda) \leq 1, r = 1,2$ and $\Theta_r(\hat{\beta}_\Lambda)$ represents as shrinkage weight factor which may be constants or functions of $\hat{\beta}_\Lambda$. We proposed two preliminary test single stage shrinkage estimators $\tilde{\beta}_{sss}$ to estimate the reliability parameters when a prior knowledge β_{Λ_0} is available for β_Λ

$$\tilde{\beta}_{sss_{\Lambda_1}} = \begin{cases} \beta_{\Lambda_0} & , if\, \hat{\beta}_\Lambda \in R_{pt_2} \\ \hat{\beta}_\Lambda & , if\, \hat{\beta}_\Lambda \notin R_{pt_2} \end{cases} \tag{34}$$

Then $\Theta_1(\hat{\beta}_\Lambda) = 0$ and $\Theta_2(\hat{\beta}_\Lambda) = 1$ in (28)

And

$$\tilde{\beta}_{sss_{\Lambda_2}} = \begin{cases} \beta_{\Lambda_0} & , if\, \hat{\beta}_\Lambda \in R_{pt_2} \\ c_2(\hat{\beta}_\Lambda - \beta_{k_0}) & , if\, \hat{\beta}_\Lambda \notin R_{pt_2} \end{cases} \tag{35}$$

Thus $\Theta_1(\hat{\beta}_\Lambda) = 0$ and$\Theta_2(\hat{\beta}_\Lambda) = c_2$ in (28)

Such that $R_{pt_2} = (s_1, s_2)$ while $s_1 = \frac{\beta_{\Lambda_0}}{2m_\Lambda} Y^2_{\Lambda_{1-\lambda_2/2,2m_\Lambda}}$ and $s_2 = \frac{\beta_{\Lambda_0}}{2m_\Lambda} Y^2_{\Lambda_{\lambda_2/2,2m_k}}$

$Y^2_{1_{1-\lambda_2/2,2m_\Lambda}}$, $Y^2_{1_{\lambda_2/2,2m_\Lambda}}$ are respectively the lower and upper $100(\lambda_2/2)$ precedential point of chi-square distribution with $(2m_k)$ degree of freedom.

Substituting (30),(31),(34) and (35) in (15) to obtain preliminary test single Shrinkage estimator for cascade reliability model(2+1) of IED as the following

$$\widehat{R}_{sss} = \left[\frac{\widetilde{\alpha}_{1sss}}{\widetilde{\alpha}_{1sss}+\widetilde{\beta}_{1sss}}\right]\left[\frac{\widetilde{\alpha}_{2sss}}{\widetilde{\alpha}_{2sss}+\widetilde{\beta}_{2sss}}\right] + \left[\frac{\frac{k}{m}\widetilde{\alpha}_{1sss}\widetilde{\beta}_{1sss}}{\left(\frac{k}{m}\widetilde{\alpha}_{1sss}+\widetilde{\beta}_{1sss}\right)\left(\widetilde{\alpha}_{1sss}+\frac{k}{m}\widetilde{\alpha}_{1sss}\widetilde{\beta}_{1sss}\right)}\right]\left(\frac{\widetilde{\alpha}_{2sss}}{\widetilde{\alpha}_{1sss}+\widetilde{\beta}_{2sss}}\right) + \left[\frac{\frac{k}{m}\widetilde{\alpha}_{2sss}\widetilde{\beta}_{2sss}}{\left(\frac{k}{m}\widetilde{\alpha}_{2sss}+\widetilde{\beta}_{2sss}\right)\left(\widetilde{\alpha}_{2sss}+\frac{k}{m}\widetilde{\alpha}_{2sss}+\widetilde{\beta}_{2(MLE)}\right)}\right]\left(\frac{\widetilde{\alpha}_{1sss}}{\widetilde{\alpha}_{1sss}+\widetilde{\beta}_{1sss}}\right) \quad (36)$$

6. Simulation Study

In this section, we present some results based on Monte Carlo simulations to compare the performance of different methods: MLE, UMVUE and PTSSSE of Cascade Reliability Model(2+1) using different sample (20, 40, and 60). For this purpose the following steps of Monte Carlo simulation are used based on Mean Squared Errors criteria with 1000 replicates:

Step1: Generate random samples which follow the continuous uniform distribution defined on the interval (0,1), as $u_{i1}, u_{i2}, \dots, u_{ij}$ and $w_1, w_2, \dots, w_m$,for all $i= 1,2,\dots,k \ and \ j = 1,2,\dots,n_i$. respectively.,

Step2: Applying an inverse transformation approach to generate random variables follows IED as follows:

$F(x) = exp\left(\frac{-\alpha}{x}\right)$: $Ui = exp\left(\frac{-\alpha}{x}\right)$: $x_i = [\alpha_1/-\ln(Ui)]^{\frac{1}{2}}$

And, by the same method, we get:

$y_i = [\alpha_2/-\ln(Vj)]^{\frac{1}{2}}$

Step3: Recall the R from equation (15).

Step4: find the cascade reliability R of the MLE using equation (21).

Step5: find the uniformly minimum variance unbiased method of R using equation (27).

Step6: Compute (PTSSSE) of cascade reliability using equations (36).

Step7: Calculate the MSE based on (L=1000) trials as follows:

MSE $= \frac{1}{L}\sum_{i=1}^{L}(\hat{R}_i - R)^2$

7. Results of Simulation

In this section, the simulation results are used to determine the best outcome of the conceder estimation methods (MLE, UMVUE, PTSSSE) of cascade reliability model (2+1) estimator based on one parameter IED. In the cascade reliability model (2+1) of estimate the system reliability $R = p[X_1 \geq Y_1, X_2 \geq Y_3] + p[X_1 < Y_1, X_2 \geq Y_2, X_3 \geq Y_3] + p[X_1 \geq Y_1, X_2 < Y_2, X_3 \geq Y]_3$)).The following tables of mean square error show, at most the orders rank of the estimators as follows: $\widehat{R}_{MLE}$, $\widehat{R}_{(UMVU)}$and $\widehat{R}_{sss}$ respectively, that means $\widehat{R}_{sss}$ is the best than the others estimators. The following tables (1-9) will present the simulation results.

Table 1: $\alpha_1 = 1$, $\alpha_2 = 1, \beta_1 = 1.5$, $\beta_2 = 1.5$, $R = 0.800000000$

k	m	(n1, n2,m1,m2)	UM	PT	MLE
1.5	0.2	(20, 20, 20, 20)	0.800040031839184	0.800000399098201	0.800055821748313
		(60,60, 20, 40)	0.800040032049879	0.800000862941454	0.800045359839790
1.8	0.5	(40, 20, 60, 60)	0.800040032014180	0.799999583412720	0.800053589528536
		(20,40,60, 20)	0.800040031806876	0.799999000261255	0.800019751832772
1.3	0.7	(40, 60, 20, 40)	0.800040032055934	0.799999163925098	0.800033275056292
		(60, 20, 40, 60)	0.800040032013649	0.800000092631102	0.800056080018052
1.1	0.9	(40, 40, 40, 40)	0.800040032025629	0.800000881956533	0.800050772085346
		(60,60,60, 60)	0.800040032025629	0.799999795970744	0.800040301978738

Table2: $MSE\ \alpha_1 = 1\ , \alpha_2 = 1,\ \beta_1 = 1.5\ \ , \beta_2 = 1.5$

k	m	(n1, n2,m1,m2)	UM	PT	MLE
1.5	0.2	(20, 20, 20, 20)	0.000000001602548	0.000000000868428	0.000000071936176
		(60,60, 20, 40)	0.000001602565018	0.000000456950252	0.000042789507494
1.8	0.5	(40, 20, 60, 60)	0.000001602562159	0.000000397091475	0.000042894580602
		(20,40,60, 20)	0.000000001602546	0.000000000960463	0.000000345959007
1.3	0.7	(40, 60, 20, 40)	0.000001602565503	0.000000433528185	0.000032505999070
		(60, 20, 40, 60)	0.000001602562117	0.000000382471540	0.000044937037854
1.1	0.9	(40, 40, 40, 40)	0.000001602563075	0.000000389473349	0.000032823139814
		(60,60,60, 60)	0.000001602563075	0.000000306936893	0.000014943488773

Table 3: $\alpha_1 = 1.5\ , \alpha_2 = 1.5\ ,\ \beta_1 = 2\ ,\ \beta_2 = 2\ ,\ R = 0.7857142857$

k	m	(n1, n2,m1,m2)	UM	PT	MLE
1.5	0.2	(20, 20, 20, 20)	0.785734705446280	0.785714149871148	0.785720768248288
		(60,60, 20, 40)	0.785734705585085	0.785713939601543	0.785737729080948
1.8	0.5	(40, 20, 60, 60)	0.785734705520629	0.785715533029282	0.785750736020131
		(20,40,60, 20)	0.785734705528084	0.785714520693756	0.785718895194405
1.3	0.7	(40, 60, 20, 40)	0.785734705594252	0.785712470091343	0.785725567713847
		(60, 20, 40, 60)	0.785734705530414	0.785714534388167	0.785748611031530
1.1	0.9	(40, 40, 40, 40)	0.785734705546020	0.785713842247919	0.785728999913387
		(60,60,60, 60)	0.785734705546020	0.785714339660318	0.785729005582190

Table 4: $\alpha_1 = 1.5\ \ , \alpha_1 = 1.5\ \ , \beta_1 = 2\ \ ,\ \ \beta_2 = 2$

k	m	(n1,n2,m1,m2)	UM	PT	MLE
1.5	0.2	(20, 20, 20, 20)	0.000000000416965	0.000000001318706	0.000000291526563
		(60,60, 20, 40)	0.000000416971124	0.000000578485132	0.000026615438193
1.8	0.5	(40, 20, 60, 60)	0.000000416968491	0.000000456480257	0.000020849454688
		(20,40,60, 20)	0.000000000416969	0.000000001195640	0.000000123002043
1.3	0.7	(40, 60, 20, 40)	0.000000000416971	0.000000000680612	0.000000045033693
		(60, 20, 40, 60)	0.000000416968890	0.000000466212429	0.000037108268999
1.1	0.9	(40, 40, 40, 40)	0.000000416969528	0.000000576798620	0.000054516476000
		(60,60,60, 60)	0.000000416969528	0.000000383695325	0.000028051216264

Table 5 $\alpha_1 = 3\ ,\ \ \alpha_2 = 3\ , \beta_1 = 1\ \ , \beta_2 = 1, R = 0.6250000000$

k	m	(n1,n2,m1,m2)	UM	PT	MLE
1.5	0.2	(20, 20, 20, 20)	0.624937468791855	0.624999705238945	0.624932272202343
		(60,60, 20, 40)	0.624937468537429	0.624999441035243	0.624953403330766
1.8	0.5	(40, 20, 60, 60)	0.624937468714946	0.625000056038205	0.624940516955036
		(20,40,60, 20)	0.624937469058640	0.624999904721153	0.624913690218326
1.3	0.7	(40, 60, 20, 40)	0.624937468534079	0.624998720248892	0.624943162050123
		(60, 20, 40, 60)	0.624937468736565	0.624998458153044	0.624939686353734
1.1	0.9	(40, 40, 40, 40)	0.624937468734366	0.625002127090182	0.624942577777575
		(60,60,60, 60)	0.624937468734366	0.625000176762664	0.624942314238157

Table 6: $\alpha_1 = 3$, $\alpha_2 = 3$, $\beta_1 = 1$, $\beta_2 = 1$

k	m	n1,n2,m1,m2)	UM	PT	MLE
1.5	0.2	(20, 20, 20, 20)	0.000000003910152	0.000000003234843	0.000000239325138
		(60,60, 20, 40)	0.000000003910184	0.000000001870957	0.000000085454692
1.8	0.5	(40, 20, 60, 60)	0.000000003910162	0.000000001122373	0.000000045087534
		(20,40,60, 20)	0.000000003910119	0.000000002522772	0.000000181399978
1.3	0.7	(40, 60, 20, 40)	0.000000003910184	0.000000001857193	0.000000189712498
		(60, 20, 40, 60)	0.000000003910159	0.000000001040114	0.000000030728521
1.1	0.9	(40, 40, 40, 40)	0.000000003910159	0.000000001408268	0.000000056503599
		(60,60,60,60	0.000000003910159	0.000000000964106	0.000000038498170

8. Conclusion

the cascade reliability model (2+1) $R = p[X_1 \geq Y_1, X_2 \geq Y_3] + p[X_1 < Y_1, X_2 \geq Y_2, X_3 \geq Y_3] + p[X_1 \geq Y_1, X_2 < Y_2, X_3 \geq Y_3]$ based on inverted exponential distribution were used in this paper to verify the performance of different estimators which are ; Maximum likelihood estimation, Unbiased estimation method and Preliminary test single stage shrinkage (PTSSS) estimator using different samples $(20, 40,$ and $60)$. The Monte Carlo Simulation was exhibited to analyses and comparison between these methods. the results show that the performance of (PTSSS) estimator was appropriate behavior and it is efficient estimator than the others in the sense of MSE at most. While $\hat{R}_{(UMVU)}$ had the second rank and followed by $\hat{R}_{sss}$.

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