Abstract

Let $R$ be a commutative ring with unity and an $R$-submodule $N$ is called semimaximal if and only if $M/N$ is a semisimple $R$-module. The main object of this work is to fuzzy this concept, study the basic properties and we investigate the sufficient conditions of $F$-submodules to be semimaximal. Also, the concepts of (simple, semisimple) $F$-submodules and quotient $F$-modules are introduced and given some properties.

Keywords: fuzzy simple (semisimple) modules, fuzzy quotient modules, fuzzy semimaximal submodule.

1. Introduction

Hatam in [1] introduced and studies semimaximal ideals of a ring and semimaximal submodules, where an ideal $J$ of a ring $R$ is called semimaximal if $J$ is a finite intersection of maximal ideals [2]. We add many other results. Maysoun in [3] introduced the definition of fuzzy simple modules and fuzzy semisimple modules. Some properties of these concepts which are useful in next sections are given. Moreover, a submodule $N$ of an $R$-module $M$ is said semimaximal if $M/N$ is semisimple $R$-module. It is clear that every maximal ideal (submodule) is a semimaximal.

In [4], we fuzzify the concept semimaximal ideal where a fuzzy ideal $H$ is called a fuzzy semimaximal ideal if $H$ is a finite intersection of fuzzy maximal ideals also we study many properties this concept. In this paper, we fuzzify the concept semimaximal submodules in to fuzzy semimaximal submodules. Also, we give many basic properties of this notion.

Finally, (shortly fuzzy set, fuzzy submodule, fuzzy ideal and fuzzy module is F-set, F-submodule, F-ideal , and F-module).
1. Preliminaries

This section contains some definitions and properties of fuzzy set and fuzzy module.

**Definition 1.1** [5]
Let $S$ be a non-empty set and $I$ be the closed interval $[0,1]$ of the real line (real numbers). A F-set $A$ in $S$ (a F-subset of $S$) is a function from $S$ in to $I$.

**Definition 1.2** [6]
Let $x_t : S \to [0,1]$ be are two F-set in $S$, where $x \in S$, $t \in [0,1]$, defined by:

$$x_t(y) = \begin{cases} t & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

for all $y \in S$. Then $x_t$ is called F-singleton.

**Definition 1.3** [6]
If $A_1$ and $A_2$ F-sets in $S$, then:

1. $A_1 = A_2$ if and only if $A_1(x) = A_2(x)$, for all $x \in S$.
2. $A_1 \leq A_2$ if and only if $A_1(x) \leq A_2(x)$, for all $x \in S$.

If $A_1 \leq A_2$ and there exists $x \in S$ such that $A_1(x) < A_2(x)$, then $A_1$ is a proper F-subset of $B$ and written $A < B$.

By part (2), we can deduce that $x_t \leq A_1$ if and only if $A_1(x) \geq t$.

**Definition 1.4** [6], [7]
Let $\mathcal{M}$ be an $R$-module. A F-set $\mathcal{H}$ of $\mathcal{M}$ is F-module of an $R$-module $M$ if:

1. $\mathcal{H}(x-y) \geq \min \{\mathcal{H}(x),\mathcal{H}(y), \text{ for all } x, y \in \mathcal{M}\}$
2. $\mathcal{H}(rx) \geq \mathcal{H}(x) \text{ for all } x \in \mathcal{M} \text{ and } r \in R$.
3. $\mathcal{H}(0) = 1$.

**Definition 1.5** [6]
Let $X_1$ and $X_2$, be two F-modules of R-module $\mathcal{M}$. $X_2$ is said a F-submodule of $X_1$, if $X_2 \leq X_1$.

**Proposition 1.6** [6]
Let $A$ be a F-set of an $R$-module $\mathcal{M}$. Then the level subset $A_t = \{x \in M, A(x) \geq t \}, t \in [0,1]$ is a submodule of $\mathcal{M}$ if and only if $A$ is a F-submodule of $\mathcal{H}$ where $\mathcal{H}$ is a F-module of an $R$-module $\mathcal{M}$.

Now, we shall give some properties of F-submodules, which are used in the next sections.

**Definition 1.7** [6]
Let $A$ be a F-set of an $R$-module $\mathcal{M}$, then the submodule $A_t$ of $\mathcal{M}$ is called the level submodule of $\mathcal{M}$, where $t \in [0,1]$.

**Proposition 1.8** [7], [8]
Let $A$ be a fuzzy module in $\mathcal{M}$, then we define $A_t = A_{A_0(M)} = \{x \in M, A(x) = 1\} = A(0_M) = 1$.

**Proposition 1.9** [11]
Let $A$ be a fuzzy module of an $R$-module $\mathcal{M}$, then $A_t$ is a submodule of $\mathcal{M}$.

We add the following results:

**Proposition 1.10**
If $\mathcal{H}$ is a F-module of an $R$-module $\mathcal{M}$ and $\mathcal{N} \leq \mathcal{H}$ such that $\mathcal{H}(0) = 1$, then $\mathcal{N}(0) = 1$.

Proof: it is clear by the definition of F-module.
Remark 1.11[11]
If $A$ and $B$ are $F$-submodules of $F$-module $X$ such that $A \leq B$ then $A_* \subseteq B_*$.
Proof:
Let $x \in A_*$, then $A(x) = A(0)$.
But $B(x) \geq A(x), \forall x \in M$, hence $B(x) \geq A(x) = A(0) = B(0)$. Thus $x \in B_{(0)} = B_*$.

Remark 1.12[11]
The converse of the above Remark is not true in general as the following example shows:
Let $X: Z \to [0,1]$, define by: $X(x) = 1, \forall x \in Z$,
Let $A(x) = \begin{cases} 1 & \text{if } x \in 4Z, \\ 0.9 & \text{otherwise} \end{cases}$
Let $B(x) = \begin{cases} 1 & \text{if } x \in 2Z, \\ \frac{1}{2} & \text{otherwise} \end{cases}$
A and $B$ are $F$-submodules $X$ and $A_* = 4Z, B_* = 2Z$.
Hence, $A_* \leq B_*$. But $A \neq B$, since $A(3) = 0.9, B(3) = 1/2 = 0.5$. $x$

Remark
We assume that if $A_* = B_*$. Then, $A = B$ is called **Condition (*)**

Mayssoon in[11] introduced the following definition:

**Definition 1.13[11]**
Let $X$ be a $F$-module of an $R$-module $M$, let $A$ be $F$-submodule of $X$
Define $X/A: [M/A_*, \to [0,1]$ by:

$$X/A(a + A_*) = \begin{cases} \sup \{X(a + b) \} & \text{if } b \in A_*, a \not\in A_* \\ 1 & \text{if } a \in A_* \end{cases}$$

For all coset $a + A_* \in M/A_*$.

**Proposition 1.14[11]**
If $X$ is a $F$-module of an $R$-module $M$ and $A$ is a submodule of $X$
, then $X/A$ is a $F$-module of $M/A_*$.
However, in [12] there exists a definition of quotient fuzzy module which is an equivalent to Definition 3.1 where $X(0) = 1$, as follows:

**Proposition 1.15[12]**
Let $X$ be a $F$-module of an $R$-module $M$ and $A$ be a $F$-submodule of $X$.
Define $X/A: [M/A_*, \to [0,1]$, such that
$$X/A(a + A_*) = \sup \{X(a + b), a \in M, b \in A_*\}$$

**Lemma 1.16[11]**
If $A$ be $F$-submodule of $F$-module $X$, then $X/A_* \leq (X/A)_*$.

**Proposition 1.17[11]**
Let $X$ be a $F$-module of an $R$-module $M$ such that $X(x) = 1, \forall x \in M$.
Then $(X/A)_* = X|A_*$, for each $A \leq X$.

**Proposition 1.18[11]**
If $A, B$ are $F$-submodules of $F$-module $X$. Then $X/A \cap B \approx F$-submodules in $(X/A \oplus X/B)$. 

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Definition 1.19 [3]
A F- submodule of F-module X is called pure if for each F-ideal K of R, KX ∩ A = KA.

Definition 1.20 [3]
A F-module X of an R-module M is called F-regular if every F-submodule of X is pure.

Definition 1.21 [13]
Let A be a F-submodule of fuzzy module X is called an essential if $A ∩ B = 0$, for non-trivial F-submodule B of X.

Definition 1.22 [3]
Let X be a F-module of an R-module M. X is called a multiplication F-module if and only if for each F-submodule A of X, there exists a F-ideal K of R such that A = KX.

2. Fuzzy Simple (Semisimple) Modules

Recall that an R-module M is called simple if and only if has no proper non trivial submodules [2] and "M is called semisimple if and only if M is sum of simple submodules of M [2].

Maysoon in [3] introduced the definition of fuzzy simple modules and fuzzy semisimple modules. Some properties of these concepts which are useful in next sections are given. Moreover we add many other results.

Definition 2.1 [3]
A F-module X is called simple if X has no nontrivial F-submodules.
In other words, X is simple if whenever $A ≤ X$, either $A = X$ or $A = 0$.
Moreover, let $A ≤ X$, A is a simple submodule of X if $A$ is a simple module.

Remarks 2.2 [3]
If $X$ is a F-module, then the following are held:
1) Every simple F-module is F-regular F-module, where is F-regular is every F-submodule of X is pure.
2) If X is a simple F-module, then $X_t$ is a simple module, $∀ t (0,1]$.
3) If $X_t$ is a simple module, $∀ t ∈ (0,1]$, then is not necessarily that $X$ is a simple F-module."

Proposition 2.3 [11]
Let $X$ be a F-module of an R-module M and A be a F-submodule of X if A is simple, then A is simple submodule in $X_t$."

Remark 2.4 [11]
If $A_*$ is a simple submodule in $X_*$, then $A$ is not F-semisimple submodule.".

Definition 2.5 [2]
A F-module X is called semisimple if X is sum of simple F-submodule of X.

Next, we need the following lemma:
Lemma 2.6 [11]
If X is a F-module of an R-module M and A is a F-direct summand in X, then $A_*$ is a direct summand in $X_*$. 

Proposition 2.7 [11]
If X is a F-semisimple module, then X is a semisimple module. 

Proposition 2.8 [11]
If $X_*$ is semisimple, then $X$ semisimple when Condition (*) hold."
Corollary 2.9[11]
Any F- submodule of semisimple fuzzy submodule is semisimple.

Proposition 2.10[11]
If $X$ is a F- module of an $R$-module $M$, where Condition (*) holds Then, the following statements are equivalent:
1- $X$ is semisimple
2- $X$ has no proper essential F- submodule.
3- Every F- submodule of $X$ is a direct summand of $X$.

Proposition 2.11 [11]
If $X$ is a F- module of an $R$–module $M$. Then, the following are equivalent:
1. Every fuzzy submodule of $X$ is sum of fuzzy simple submodules.
2. $X$ is a direct sum of fuzzy simple submodules of $X$.
3. Every fuzzy submodule of $X$ is a direct summand of $X$.

Proof: It is easy.

The following are equivalent:
1. Every F- submodule of semisimple F- module is semisimple.
2. Every epimorphic image of semisimple F- module is semisimple.
3. Every sum of semisimple F- modules is semisimple.

Proof: It is easy.

3. Fuzzy Semimaximal Submodules
Definition 3.1
If $A$ is a F- submodule of F- module $X$, then $A$ is called semimaximal if and only if $X/A$ is a semisimple F- module.

Proposition 3.1
If $A$ is a semimaximal fuzzy submodule of fuzzy module $X$, then $A_\ast$ is a semimaximal submodule of $X_\ast$.

Proof:
Since $A$ is semimaximal F - submodule, so $X/A$ is semisimple. Hence, $X/A = \bigoplus_{i\in\hat{}} (C_i | A_i)$, where $C_i | A_i$ is simple F- submodules $\forall i \in \hat{}$.
Which implies $(X/A)_\ast = \bigoplus_{i\in\hat{}} (C_i | A_i)_\ast$.

But $C_i | A_i$ is F- simple, implies $(C_i | A_i)_\ast$ is simple, by Proposition 2.3 $\forall i \in \hat{}$.
Hence, $(X/A)_\ast$ is semisimple. As $X_\ast|A_\ast \leq (X/A)_\ast$, by Lemma 1.16.
Then, $X_\ast|A_\ast$ is semisimple. Therefore $A_\ast$ is semimaximal submodule of $X_\ast$.

Proposition 3.2
Let $A$ be F- submodule of F- module $X$ which satisfies $(X/A)_\ast = X_\ast/A_\ast$. If $A_\ast$ is a semimaximal submodule of $X_\ast$, then $A$ is a F- semimaximal submodule of $X$.

Proof:
Since $A_\ast$ is a semimaximal submodule of $X$, hence, $X_\ast/A_\ast$ is semisimple.
Hence, $X/A$ is semisimple F – module. Therefore $A$ is a F – semimaximal submodule of $X$.

Remarks and Examples 3.4
1) Every F- maximal submodule of fuzzy module is a F- semimaximal submodule.
Proof:

A is a F-
maximal submodule of F-
module X . Then X/A is simple , and so X/A is a semisimple . Hence , A is semimaximal.

The converse is not true in general see the following example :

Example:

Let M = Z as Z - module . Define X(x) = 1, ∀ x ∈ M

Let A ≤ X such that A(x) = { 1 if x ∈ 6Z,
0 otherwise

Thus X*/A* = M/A* = Z/6Z ≈ Z_6 is semisimple and A is semimaximal.

But A* is not a maximal in X*

Thus A is not maximal F- submodule by Prop 3.3. Hence (X/A)* = M/A* = Z_6 is semisimple , and so by Prop 2.8 , X/A is semisimple ; that is A is a F-
semimaximal submodule

2) A F- submodule of semimaximal F-module need not to be semimaximal .

For example :

Let A , B ≤ X, such that A(x) = { 1 if x ∈ < 2 >
1/2 otherwise

B(x) = { 1 if x ∈ < 8 >
1/3 otherwise

So X*/A* = Z_{24} / < 2 > ≈ Z_2* is semisimple.

So A* is semimaximal . But X*/B* = Z_{24} / < 8 > ≈ Z_8 is not semisimple.

Hence, B* is not semimaximal submodule .

Thus B is not semimaximal F- submodule by Proposition 3.3.

3) If A and B are F- submodules of F- module X such that A ≤ B ≤ X and A is semimaximal in X . Then B is semimaximal in X .

Proof:

Since A is a fuzzy semimaximal submodule in X , X/A is semisimple .

Hence , (X/A)/(X/B) is semisimple (image of semisimple Proposition 2.12)

By the second isomorphism theorem[14] , X/A)/(X/B) ≈ X/B is semisimple . Thus , B is a fuzzy semimaximal submodule .

4- Let A and B be two F- submodules of F- module X . If is A a semimaximal F- submodule of X , then A + B is also semimaximal F- submodule of X .

Proof :

Clearly A ≤ A + B and , hence the result follows directly.

5) Let \{A_i = 1,2,3,...,n\} be a finite collection of semimaximal F- submodules of F-
module X . Then \bigoplus_{i=1}^n A_i , i = \{1,2,3...n\} is a semimaximal F- submodule.

Proof: It is clear by 4

6) Let A and B be two F- submodules of F- module X such that A ≤ B . If A is semimaximal in B and B is semimaximal in X , then A is not necessary semimaximal of X , as the following example shows :

Example:

Take M = Z_36 as Z - module , let X : M → [0,1], define X(x) = 1, ∀ x ∈ M ,

A(x) = \begin{cases} 
1 & \text{if } x \in < 9 > \\
0 & \text{otherwise}
\end{cases}
\[ B(x) = \begin{cases} 1 & \text{if } x \in (-\frac{3}{2}, \frac{3}{2}) \\ 0 & \text{otherwise} \end{cases} \]

\[ X/B : Z_{36}/(-\frac{3}{2}, \frac{3}{2}) \cong Z_3 \rightarrow [0,1] \text{ such that } X/B(x) = 1, \forall x \in Z_3 \]

\[ X/A : Z_{36}/(-\frac{9}{2}, \frac{9}{2}) \cong Z_9 \rightarrow [0,1] \text{ such that } X/A(x) = 1, \forall x \in Z_9 \]

Now, \((X /B)_* = Z_3\) is simple, so that \(X_/B\) = \(Z_3\) and hence \(B\) is a \(F\) - semimaximal submodule, by Prop 3.3

But \(X_/A\) is not a semisimple submodule.

Therefore, \(A\) is not a semimaximal submodule in \(X\), and hence \(A\) is not semimaximal in \(X\).

Since \(A \leq B\) and \(B/A: Z_{36}/(-\frac{9}{2}, \frac{9}{2}) \rightarrow [0,1]\) : defined by:

\[ B/A(x) = \begin{cases} 1 & \text{if } x \in (-\frac{3}{2}, \frac{3}{2}) \cap (-\frac{9}{2}, \frac{9}{2}) = Z_3 \\ 0 & \text{otherwise} \end{cases} \]

Now, \((B /A)_* \approx (-\frac{3}{2}, \frac{3}{2}) \cap (-\frac{9}{2}, \frac{9}{2}) \cong Z_3\). But \(B_/A_*= (-\frac{3}{2}, \frac{3}{2}) \cap (-\frac{9}{2}, \frac{9}{2}) \cong Z_3\) is simple. Then \(A_*\) is semimaximal in \(B_*\).

**Proposition 3.5**

Let \(A_i\) be a semimaximal \(F\)-submodule of \(F\)-module \(X_i\), \(i = 1,2,3,...,n\)

Then \(\bigoplus_{i=1}^n A_i\) is a semimaximal \(F\)-submodule of \(F\)-module \(\bigoplus_{i=1}^n X_i\),

Provided \((X/A)_* = X_/A_\) for each \(F\)-module \(X\) and \(A \leq X\), \(\forall i = 1,2,3,...,n\).

Proof:

Since \(A_i\) is a semimaximal \(F\)-submodule of \(F\)-module \(X_i\), \(\forall i = 1,2,3,...,n\), then \(X_/A_i\) is \(F\)-semisimple module, for each \(i = 1,2,3,...,n\).

Hence \(\bigoplus_{i=1}^n X_i /A_i\) is \(F\)-semisimple module and so by Remarks and Examples 3.4(5) \(\left(\bigoplus_{i=1}^n X_i /A_i\right)_*\) is a semisimple.

But \(\left(\bigoplus_{i=1}^n X_i /A_i\right)_* = \bigoplus_{i=1}^n \left(\frac{X_i}{A_i}\right)_*\) by[2]

\[ (\bigoplus_{i=1}^n \left(\frac{X_i}{A_i}\right)_*)_* = \bigoplus_{i=1}^n \left(\frac{X_i}{A_i}\right)_* \cong \bigoplus_{i=1}^n \left(\frac{X_i}{A_i}\right)_* = \frac{\left(\bigoplus_{i=1}^n X_i\right)_*}{\left(\bigoplus_{i=1}^n A_i\right)_*} \]

Hence \(\left(\bigoplus_{i=1}^n A_i\right)_*\) is a semimaximal submodule in \(\left(\bigoplus_{i=1}^n X_i\right)_*\).

By hypothesis \(\frac{\left(\bigoplus_{i=1}^n X_i\right)_*}{\left(\bigoplus_{i=1}^n A_i\right)_*} = \frac{\left(\bigoplus_{i=1}^n X_i\right)_*}{\left(\bigoplus_{i=1}^n A_i\right)_*} \)

Hence \(\left(\bigoplus_{i=1}^n X_i\right)_*\) is a \(F\) - semimaximal submodule in \(\bigoplus_{i=1}^n X_i\).

**Remark 3.6**

If \(X\) is a \(F\)-module of an \(R\)-module \(M\), then it is not necessary that \(X\) has semimaximal \(F\)-submodule, for example.

**Example**:

Let \(M = Z_{p\infty}\) (\(p\) is a prime number as \(Z\)-module), let \(X : M \rightarrow [0,1]\), define by \(X(x) = 1\), \(\forall x \in M\).

Assume \(X\) has a semimaximal submodule say \(A\). Let \(A_* = N\) is a semimaximal submodule in \(X_* = Z_{p\infty}\) by Proposition 3.2; that is \(X_/N = Z_{p\infty}/N\) is semisimple.

But \(\frac{Z_{p\infty}}{N}\cong Z_{p\infty}\), so that \(Z_{p\infty}\) is semisimple, which is a contradiction.
Thus $A$ is not semimaximal $F$-submodule of $X$.

The following proposition gives sufficient condition but not necessary condition for $F$-submodule to be semimaximal.

**Proposition 3.7**

Let $A$ be $F$-submodule of fuzzy module $X$ such that $A$ is intersection of a finite number of $F$-maximal submodules of $X$. Then, $A$ is a $F$-semimaximal submodule.

**Proof:**

Let $A = A_1 \cap A_2 \cap \ldots \cap A_n$, $A_i$ is a $F$-submodule of $X$, $\forall i = 1, 2, \ldots, n$.

Hence, by Proposition 1.18, $X/A \cong F$-submodule of $X|A_1 \oplus X|A_2 \oplus \ldots \oplus X|A_n$.

But is simple for each $X/A_i$, so $X/A_1 \oplus X/A_2 \oplus \ldots \oplus X/A_n$ is semisimple and since a $F$-submodule of fuzzy semisimple is $F$-semisimple, therefore $X/A$ is a $F$-semisimple module.

Thus $A$ is $F$-semimaximal submodule.

**Remark 3.8**

The converse of Proposition 3.7 is not true in general. We can give the following example:

Let $M = \bigoplus Z_p$ as $Z$-module, $p$ is a prime number.

Define $X : M \to [0,1]$ and $A : M \to [0,1]$ by $X(x) = 1$, $\forall x \in M$ and $A(x) = 1$, $\forall x \in Z_2$.

But $X/A (x + A_n) = \begin{cases} 1 & \text{if } x \in A_n \\ \sup \{X(x + b) \} & \text{if } b \in A_n, x \notin A_n \end{cases}$

Thus $X/A (x + A_n) = 1, \forall (x + A_n) \in M/A_n = \bigoplus Z_p, p > 2$ is semisimple. Thus $A$ is a $F$-semimaximal submodule.

But $A$ is not a finite intersection of $F$-maximal submodule.

**Proposition 3.9**

If $X$ is a $F$-module of an $R$-module $M$ and $R$ is a semisimple ring, then every $F$-submodule $A$ of $X$ is semimaximal. Provided $(X|A)_* = X|A_*$

**Proof:**

Let $R$ be semisimple ring. Then $M$ is a semisimple $R$-module. Since $X$ is a $F$-module over $M$, then $X_* \leq M$ and hence $X_*$ is semisimple which implies that $X_*/A_*$ is semisimple. Thus $A_*$ is a semimaximal submodule of $X_*$. Then by Proposition 3.4 $A$ is a $F$-semimaximal submodule of $X$.

**Proposition 3.10**

The intersection of two semimaximal $F$-submodules is also semimaximal.

**Proof:**

If $A$, $B$ are two semimaximal $F$-submodules of module $X$. Then $X/A$ and $X/B$ are semisimple. Therefore, $X/A \oplus X/B$ is semisimple.

As $X/A \cap B \cong F$-submodule of $X/A \oplus X/B$. But any $F$-submodule of $X/A \oplus X/B$ is semisimple, hence $X/A \cap B$ is semisimple.

Therefore $A \cap B$ is a $F$-semimaximal submodule of $X$. 

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Proposition 3.11
If $A$, $B$ are two F- submodules of F- module $X$ such that $A$ is semimaximal in $X$ and $B$ contains $A$. Then $B$ is semimaximal in $X$.

Proof:
$B/A$ is F- submodule of $X/A$ since $A \subseteq B$ and $X/A \cong X/B$, by third isomorphism theorem.

But $X/A$ is semisimple, implies $(X/A)/(B/A)$ is semisimple.

That is $X/B$ is semisimple. Therefore $B$ is semimaximal in $X$.

The following Corollary immediately consequence of Proposition 3.11

Corollary 3.12
If $\mathcal{A}$ is a semimaximal F- submodule of F- module $X$ and $K$ be a F- ideal of a ring $R$. Then $[\mathcal{A}: K]$ is a F- submodule semimaximal.

Proof:
Since $[\mathcal{A}: K]$ is a fuzzy submodule of $X$ containing $\mathcal{A}$, then result follows by Proposition 3.11

However the converse of Corollary 3.12 is not true in general, for example

Example 3.13
Consider $M = Z_{12}$ as Z-module and let $X : M \rightarrow [0,1]$, $A : M \rightarrow [0,1]$ defined by:
where $X(x) = 1$, $\forall x \in M$,

$A(x) = \begin{cases} 
1 & \text{if } x \in \{0', 4', 8'\}, \\
\frac{1}{2} & \text{otherwise}
\end{cases}$

$A$ is a F- submodule of $X$. Let $K : Z \rightarrow [0,1]$ defined by:

$K(x) = \begin{cases} 
1 & \text{if } x \in 2Z \\
\frac{1}{3} & \text{otherwise}
\end{cases}$

$K$ is a F- ideal of $Z$, then $X/A(x) = \begin{cases} 
1 & \text{if } x \in Z_4 \\
0 & \text{otherwise}
\end{cases}$

Note that $\frac{X}{A} = Z_{12}/<4> = Z_4$ is not semisimple. Hence $A_*$ is not semimaximal.

Thus $A$ is not semimaximal Proposition 3.4. However $[A_* : K_*] = [<4> : 2Z] = <\bar{2}>$ is a semimaximal submodule of $Z_4$.

But $[A : K] \leq [A_* : K_*]$ by [13].

Therefore $[A : K]$ is a semimaximal F- submodule of $X$.

Proposition 3.14
If $A$ and $B$ are two F- submodules of F- module $X$ such that $B \leq A$ Then, $A$ is semimaximal in $X$ if and only if $A/B$ is a semimaximal F- submodule of $X/B$.

Proof:
If $A$ is semimaximal in $X$. Then $X/A$ is semisimple, which implies is semisimple, since $X/A \cong (X/B)/(A/B)$ by second isomorphism theorem [14]. Hence $A/B$ is a semimaximal F- submodule of $X/B$. The converse is similarly.
Proposition 3.15

If \( K \) is semimaximal \( F \)-ideal of a ring \( R \) and \( X \) is a \( F \)-module of \( R \)-module \( M \), then \( KX \) is a semimaximal \( F \)-submodule, provided \( \left( \frac{X}{KX} \right)_* = X_* / K_* X_* \).

Proof:
Suppose \( K \) is semimaximal \( F \)-ideal a ring \( R \), then \( K_* \) is a semimaximal ideal by [4]. Hence by[1] \( K, X_* \) is a semimaximal submodule and so \((KX)_* \) is a semimaximal submodule of \( X_* \). Thus \( KX \) is a \( F \)-semimaximal submodule.

Remark 3.16

The following example shows that the converse of Proposition 3.15 is not true in general.

Consider \( M = Z_6 \) as \( Z-module \), define \( X: M \to [0,1] \) by \( X(x) = 1, \forall x \in M \).

Define \( K: Z \to [0,1] \), by: \( K(x) = \begin{cases} 1 & \text{if } x \in 4Z, \\ 0 & \text{otherwise} \end{cases} \)

\( X \) is a \( F \)-module and \( K \) is \( F \)-ideal of \( Z \) (it is easy to prove).

Note that \( X_* = Z_6 \) and \( K_* = 4Z \) which is not a semimaximal ideal.

Since \( (KX)_* = K_* X_* \). Then \((KX)_* = (4Z) Z_6 = \{0', 2', 4'\} = <\ 2\ > \) and so \( (X/KX)_* = Z_6 / <\ 2\ > \) (is simple)

Thus, \((KX)_* \) is a maximal submodule in \( X_* \).

On the other hand, by Proposition 1.17 \((X/KX)_* = X_/ (KX)_* \).

Thus \( KX \) is a semimaximal \( F \)-submodule of \( X \).

Corollary 3.17

\( X \) is a multiplication \( F \)-module of an \( R \)-module \( M \). If every \( F \)-ideal \( K \) in a ring \( R \) is semimaximal, then every \( F \)-submodule of \( X \) is semimaximal. Provided that \( (X / A)_* = X_* / A_* \), \( \forall A \leq X \).

Proof: It is directly from Proposition 3.15.

Proposition 3.18

Every epimorphic image of semimaximal \( F \)-submodule is a semimaximal \( F \)-submodule.

Proof:
Let \( g: X/A \to Y / f(A) \) such that \( g(x + A) = y + f(A), \forall y \in Y \).

\( g \) is well-defined and \( g \) is an epimorphism.

Since \( X/A \) is semisimple. Hence, \( g(X/A) = Y(X/A) = Y \) is semisimple.

Thus, \( f(A) \) is a semimaximal fuzzy submodule.

Proposition 3.19

Let \( X \) be a finitely generated \( F \)-\( R \)-module \( M \) such that
\( \frac{X}{A_*} \), \( \forall A \leq X \), Then every \( F \)-submodule of \( X \) is semimaximal.

Proof:
Since \( X \) is a \( F \)-finitely generated \( R \)-module, then \( X_* \) is finitely generated \( R \)-module [15]. Since \((F - \text{ann}A)_* \leq \text{ann} A, and F - \text{ann}A \text{ is semimaximal } \text{F-ideal} \), implies \((F - \text{ann}A)_* \text{ is semimaximal by[4]}, so \text{ann} A \text{ is semimaximal ideal} \).

Then by [1], every submodule of \( X \) is semimaximal.
Hence, $A$ is semimaximal. But this implies $A$ is semimaximal, since $\left((X/A), \frac{X}{A}\right)$.

References

15. Inaam, M.A. Hadi; Maysoun, A. Hamil, Cancellation and Weakly Cancellation Fuzzy Modules, Journal of Basrah Researches (Sciences), 2011, 37, 4.D.