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Fuzzy Semimaximal Submodules

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Abstract

Let R be a commutative ring with unity and an R-submodule N is called semimaximal if and only if M/N is a semisimple R-module. The main object of this work is to fuzzy this concept, study the basic properties and we investigate the sufficient conditions of F-submodules to be semimaximal. Also, the concepts of (simple, semisimple) F- submodules and quotient F- modules are introduced and given some properties.

Keywords: fuzzy simple (semisimple) modules, fuzzy quotient modules, fuzzy semimaximal submodule.

1.Introduction

Hatem in [1] introduced and studies semimaximal ideals of a ring and semimaximal submodules, where an ideal J of a ring R is called semimaximal if J is a finite intersection of maximal ideals [2]. We add many other results. Maysoun in [3] introduced the definition of fuzzy simple modules and fuzzy semisimple modules. Some properties of these concepts which are useful in next sections are given.

Moreover, a submodule N of an $\mathcal{R}$ −module $\mathcal{M}$ is said semimaximal if $\mathcal{M}/N$ is semismiple R- module. It is clear that every maximal ideal (submodule) is a semimaximal.

In [4] , we fuzzify the concept semimaximal ideal where a fuzzy ideal $\mathcal{H}$ is called a fuzzy semimaximal ideal if H is s a finite intersection of fuzzy maximal ideals also we study many properties this concept. In this paper, we fuzzify the concept semimaximal submodules in to fuzzy semimaximal submodules. Also, we give many basic properties of this notion.

Finally, (shortly fuzzy set, fuzzy submodule, fuzzy ideal and fuzzy module is F-set, F-submodule, F-ideal , and F-module).

1. Preliminaries

This section contains some definitions and properties of fuzzy set and fuzzy module.

Definition 1.1 [5]

Let S be a non-empty set and I be the closed interval [0,1] of the real line (real numbers). A F- set A in S (a F- subset of S) is a function from S in to I".

Definition 1.2[6]

Let $x_t : S \to [0,1]$ be are two F- set in S, where $x \in S, t \in [0,1]$, defined by:

$$x_t(y) = \begin{cases} t & if\ x = y \\ 0 & if\ x \neq y \end{cases}$$

for all $y \in S.$, Then x_t is called F- singleton .

Definition 1.3[6]

If A_1 *and* A_2 F sets in S , then:

1- $A_1 = A_2$ if and only if $A_1\ (x) = A_2\ (x),\ for\ all\ x \in S$.

2- $A_1 \leq A_2\ \ if\ and\ only\ if\ \ A_1\ (x) \leq A_2\ (x)\ ,\ for\ allx\ \in S$..

$If\ \ A_1 \leq A_2\ \ and\ there\ exists\ x$
$\in S\ such\ that\ \ A_1\ (x)$
$< A_2\ (x), then\ \ A_1\ \ is\ a\ proper\ F - subset\ \ of\ \ B\ and\ \ written\ A < B$.

By part (2), we can deduce that $x_t \leq A_1\ \ if\ and\ only\ if\ \ A_1\ (x) \geq t$".

Definition 1.4 [6],[7]

Let $\mathcal{M}$ be an R-module . A F- set $\mathcal{H}$ of $\mathcal{M}$ is F- module of an R-module M if :

1- $\mathcal{H}(x - y) \geq \min\{\mathcal{H}\ (x), \mathcal{H}(y), for\ all\ x, y \in \mathcal{M}\ \}$

2-$\mathcal{H}(rx) \geq \mathcal{H}(x) forall\ x \in \mathcal{M}\ \ and\ r \in R.$

3-$\mathcal{H}(0) = 1$"

Definition 1.5[6]

Let X_1 *and* X_2, be two F- modules of R-module $\mathcal{M}$. X_2 *is said* a F- submodule of X_1 , if $X_2 \leq X_1$.

Proposition 1.6 [6]

Let A be a F- set of an R-module $\mathcal{M}$. Then the level subset $\mathrm{A_t = \{x \in M, A(x) \geq t\}, t \in}$ $[0,1]$ is a submodule of $\mathcal{M}$ if and only if A is a F- submodule of $\mathcal{H}$ where $\mathcal{H}$ is a F- module of an R-module $\mathcal{M}$

Now, we shall give some properties of F- submodules , which are used in the next sections.

Definition 1.7[6]

Let A be a F-set of an R-module $\mathcal{M}$, then the submodule $\mathrm{A_t}\ of\ \mathcal{M}$ is called the level submodule of $\mathcal{M}$, where $t \in [0,1]$. "

Proposition 1.8[7],[8]

Let A be a F- module in $\mathcal{M}$, then we define $A_* = A_{A(0_M)} = \{\ x \in \mathcal{M}\ , A(x) = 1\} =$ $A(0_M) = 1$"

Proposition 1.9[11]

Let A be a fuzzy module of an R- module $\mathcal{M}$,then A_* is a submodule of $\mathcal{M}$." .

We add the following results :

Proposition 1.10

If $\mathcal{H}$ is a F- module of an R-module $\mathcal{M}$ and $\mathbb{N} \leq\ \mathcal{H}\ such\ that\ \mathcal{H}(0) = 1, then\ \mathbb{N}(0) = 1$.

Proof: it is clear by the definition of F- module .

Remark 1.11[11]

If A and B are F- submodules of F- module X such that $A \leq B$ *then* $A_* \leq B_*$

Proof:

Let $x \in A_*$, *then* $A(x) = A(0)$.

But $B(x) \geq A(x), \forall x \in M$, *hence* $B(x) \geq A(x) = A(0) = B(O)$. *Thus* $x \in B_{B(0)} = B_*$

Remark 1.12[11]

The convers of the above Remark is not true in general as the following example shows :

Let $X: Z \to [0,1]$, *define by* : $X(x) = 1$, $\forall x \in Z$,

Let $A(x) = \begin{cases} 1 & if\ x \in 4Z, \\ 0.9 & otherwise \end{cases}$

$$B(x) = \begin{cases} 1 & if\ x \in 2Z, \\ \frac{1}{2} & otherwise \end{cases}$$

A and B are F- submodules X and $A_* = 4Z$, $B_* = 2Z$

Hence , $A_* \leq B_*$. *But* $A \neq B$, *since* $A(3) = 0.9$, $B(3) = 1/2 = 0.5$.x

Remark

We assume that if $A_* = B_*$. Then , $A = B$ is called ***Condition (*)***

Maysoon in[11] introduced the following definition:

Definition 1.13[11]

Let X be a F- module of an R – module M , let A be F- submodule of X *Define* $X/A: M/A_*] \to [0,1]$ *by*| :

$$X/A(a + A_*) = \begin{cases} 1 & if\ a \in A_* \\ \sup\{X(a+b) & if\ b \in A_*, a \notin A_* \end{cases}$$

For all coset $a + A_* \in {}^{M}/_{A_*}$".

Proposition 1.14[11]

If X is a F − module of an Rmodule M and A is a submodule of X , then X/A is a F − module of M/A_*.

However , in [12] there exists a definition of quotient fuzzy module which is an equivalent to Definition 3.1 where $X(0) = 1$, as follows:

Proposition 1.15[12]

Let X be a F- module of an R- module $\mathcal{M}$ and A be a F- submodule of X. Define X/A: $M/A_* \to [0,1]$, such that

$X/A(a + A_*) = \sup\{X(a+b), a \in \mathcal{M}, b \in A_*\}$

Lemma 1.16[11]

If A be F- submodule of F- module X, then $X_*/A_* \leq (X/A)_*$.

Proposition 1.17[11]

Let X be a F- module of an R- module $\mathcal{M}$ such that $X(x) = 1, \forall x \in \mathcal{M}$. *Tthen* $(X/A)_* = X_*|A_*$, *for each* $A \leq X$.

Proposition 1.18[11]

If A , B are F- submodules of F- module X . Then $X/A \cap B \approx F -$ submodules in ($X/A \oplus X/B$).

Definition 1.19 [3]

A F- submodule of F- module X is called pure if for each F-ideal K of R , $KX \cap A = KA$.

Defintion1.20 [3]

A F-module X of an R-module M is called F-regular if every F-submodule of X is pure .

Definition 1.21[13]

Let A be a F- submodule of fuzzy module X is called an essential if $A \cap B \neq 0_1$, $for\ non-trivial\ F-submodule\ B\ of\ X$.

Definition 1.22 [3]

Let X be a F-module of an R- module M . X is called a multiplication F- module if and only if for each F- submodule A of X ,there exists a F- ideal K of R such that A= KX.

2.Fuzzy Simple (Semisimple) Modules

Recall that an R-module M is called simple if and only if has no proper non trivial submodules [2] and " M is called semisimple if and only if M is sum of simple submodules of M [2].

Maysoon in[3] introduced the definition of fuzzy simple modules and fuzzy semisimple modules . Some properties of these concepts which are useful in next sections are given. Moreover we add many other results .

Definition 2.1[3]

A F- module X is called simple if X has no nontrivial F- submodules .

In other words , X is simple if whenever $A \leq X$,either $A = X\ or\ A = 0_1$

Moreover, let $A \leq X$, A is a F- simple submodule of X if A is a F- simply module .

Remarks 2.2 [3]

If X is a F- module , then the following are held:

1) Every simple F- module is F_ regular F- module ,where is F_ regular is every F- submodule of X is pure .
2) If X is a simple F- module, then X_t is a simple module , ∀ t(0,1].

3)If X_t is a simple module , $\forall\ t \in (0,1]$,then is not necessarily that X is a simple $F-$ module ."

Proposition 2.3 [11]

Let X be a F- module of an R-module M and A be a F- submodule of X if A is simple ,then A_* is simple submodule in X_*."

Remark 2.4[11]

If A_* is a simple submodule in X_*, $then\ A\ is\ not\ F-simple\ submodule$ ".

Definition 2. 5 [2]

A $F-$ module X is called semisimple if X is sum of simple $F-$ submodule of X .

Next , we need the following lemma:

Lemma 2.6[11]

If X is a F- module of an R- module M and A is a F -direct summand in X ,then $A_*\ is\ a$ direct summand in X_*.

Proposition 2.7[11]

If X is a $F-$semisimple module , then X_* is a semisimple module .

Proposition 2.8[11]

If X_* is semisimple , then X semisimple when Condition $(*)$ hold."

Corollary 2.9[11]

Any F- submodule of semisimple fuzzy submodule is semisimple.

Proposition 2.10[11]

Ift X is a F- module of an R-module M ,where Condition (*) holds Then , the following statements are equivalent :

1- X is semisimple

2 - X has no proper essential F- submodule .

3- Every F- submodule of X is a direct summand of X ".

Proposition 2.11 [11]

If X is a F- module of an R –module M . Then , the following are equivalent:

1. Every fuzzy submodule of X is sum of fuzzy simple submodules.
2. X is a direct sum of fuzzy simple submodules of X .
3. Every fuzzy submodule of X is a direct summand of X

Proof: It is easy .

Proposition 2.12 [11]

The following are equivalent:

1. Every F- submodle of semisimple F- module is semisimple.
2. Every epimorphic image of semisimple F- module is semisimple .
3. Every sum of semisimple F- modules is semisimple .

Proof: It is easy .

3. Fuzzy Semimaximal Submodules

Definition 3 .1

If A is a F- submodule of F- module X, then A is called semimaximal if and only if X/A is a semisimple F- module.

Proposition 3. 2

If A is a semimaximal fuzzy submodule of fuzzy module X , then A_* is a semimaximal submodule of X_* .

Proof:

Since A is semimaxmal F − submodule , so X /A is semisimple .

Hence , $\frac{X}{A} = \underset{i\in\wedge}{\oplus} (C_i \mid A_i)$, where $C_i \mid A_i$ is simple F- submodules $\forall\, i \in \wedge$

Which implies $(XlA)_* = (\underset{i\in\wedge}{\oplus} (Ci|A))_* = \underset{i\in\wedge}{\oplus} (Ci\,|A))_*$

But $C_i \mid A_i$ *is* $F -$ *simple , implies* $(Ci|A_i\ )_*$ *is simple , by Propositio* 2 .3 $\forall\, i \in \wedge$ *.Hence,* $(XlA)_*$ *is semisimple . As* $X_*|A_* \leq (XlA)_*$ *by Lemma* 1.16 .

Then , $X_*|A_*$ *semisimple. Therefore* A_* *is* semimaxmal submodule of X_*

Proposition 3. 3

Let A be F- submodule of F- module X which satisfies $(X/A)_* = X_*/A_*$. If A_* is a semimaximal submodule of X_* , then A is a F-semimaximal submodule of X .

Proof:

Since A_* is a semimaxmal submodule of X_* *hence* , X_*/A_* *is* semisimple

Hence , X/A is semisimple F − module . Therefore A is a F −semimaximal_ submodule of X .

Remarks and Examples 3.4

1) Every F- maximal submodule of fuzzy module is a F- semimaxmal submodule.

Proof:

A is a F- maximal submodule of F- module X .Then X /A is simple , and so X /A is a semisimple .Hence , A is semimaximal.

The converse is not true in general see the following example :

Example:

Let $M = Z\ as\ Z-\ module\ .Define\ X(x) = 1, \forall\ x \in M$

$$Let\ A \leq X\ such\ that\ A(x) = \begin{cases} 1 & if\ x \in 6Z, \\ 0 & otherwise \end{cases}$$

Thus $X_*/\ A_* \ = M/\ A_* \ = \ Z/6Z \approx Z_6$ is semisemple and A is semimaximal.

But A_* is not a maximal in X_* .

Thus A is not maximal F- submodule by Prop 3.3.Hence $(X/A)_* = M / A_* = Z_6$ is semisemple , and $so\ by\ Prop$ 2.8 , X/A is semisemple ; that is A is a F $-$ semimaxmal submodule

2) A F- submodule of semimaximal F-module need not to be semimaximal .

For example :

Let A , $B \leq X$, such that $A(x) = \begin{cases} 1 & if\ x \in <\ \bar{2} > \\ 1/2 & otherwise \end{cases}$

$$B(x) = \begin{cases} 1 & if\ x\ \in <\ \bar{8} > \\ 1/3 & otherwise \end{cases}$$

, So $X_*\ /\ \ A_* \ = Z_{24}\ /<\ \bar{2} >\ \approx\ Z_{2_*}$ is semisimple .

So A_* is semimaximal .But $X_*/B_* = Z_{24}\ /<\ \bar{8} >\ \approx\ Z_8$ $,is\ not$ semisimple.

Hence, B_* is not semimaximal submodule .

Thus B is not semimaximal F- submodule by Proposition 3.3.

3) If A and B are F- submodules of F- module X such that $A \leq B \leq X\ and\ A$ is semimaximal in X .Then B is semimaximal in X .

Proof:

Since A is a fuzzy semimaximal submodule in X , X/A is semisimple .

Hence , $(X/A)/\ (X/B)$ is semisimple (image of semisimple Proposition 2.12)

By the second isomorphism theorem[14] , $X/A)/\ (X/B) \approx X/B$ is semisimple . Thus , B is a fuzzy semimaximal submodule .

4- Let A and B be two F- submodules of F- module X . If is A a semimaximal F -submodule of X , then $A + B$ is also semimaximal F- submodule of X.

Proof :

Clearly $A \leq A + B$ and , hence the result follows directly.

5) Let $\{A_i = 1,2,3, \dots n\}$ be a finite collection of semimaximal F- submodules of F- module X. Then ${}_{i=1}^{n}\oplus A_i$ $, i = \{1,2,3 \dots n\}$ is a semimaximal F- submodule.

Proof: It is clear by 4

6) Let A and B be two F- submodules of F- module X such that $A \leq B$. If A is semimaximal in B and B is semimaximal in X , then A is not necessary semimaximal of X ,as the following example shows :

Example:

Take $M = Z_{36}\ as\ Z-module$,let $X\ : M \rightarrow [0,1], define\ X(x) = 1, \forall x \in M,$

$$A(x) = \begin{cases} 1 & if\ x \in <\bar{9}> \\ 0 & otherwise \end{cases}$$

$$B(x) = \begin{cases} 1 & if\ x \in < \bar{3} > \\ 0 & otherwise \end{cases}$$

$X/B \quad : Z_{36}/ < \bar{3} > \cong Z_3 \rightarrow [0,1]$ $such\ that\ X/B\ (x) = 1, \forall x \in Z_3$

$X/A \quad : Z_{36}/ < \bar{9} > \cong Z_9 \rightarrow [0,1]$ $such\ that\ X/A\ (x) = 1, \forall x \in Z_9$

$Now\ , (X\ /B)_* = Z_3 \quad is\ simple\ \ , so\ that \quad X_*/B_*$
$= Z_3 \quad and\ hence\ \ B\ \ is \quad a\ F - semimaximal \quad submodule\ , by\ \ Prop$ 3.3

$But \quad \dfrac{X_*}{A_*} = Z_9\ is\ not\ \ semisimple\ \ submodule\ .$

$Therefore \quad A_* \ \ is\ not \quad a \quad semimaximal$
$submodule \ \ in\ \ X_*\ and\ \ hence\ A\ is\ not \ \ semimaximal\ \ in\ X.$

$Since \quad A\ \leq B\ and \quad B/A{:}\ Z_{36}\ / < \bar{9} \ > \rightarrow [0,1] \quad : \ defined\ by :$

$$\mathrm{B/A(x)} = \begin{cases} 1 & \mathrm{if\ x} \in\ < \bar{3}\ > / < \bar{9}\ > \cong\ \mathrm{Z}_3 \\ 0 & \text{otherwise} \end{cases}$$

Now , $(B\ /\mathrm{A})_* \quad \approx < \bar{3} \ > \ / \ < \bar{9} \ > \cong \ Z_3$ .But $B_*/A_* = < \bar{3} \ > / < \bar{9} \ > \cong \ Z_3$ $is\ simple$. Then A_* is semimaximl in B_* .

Proposition 3.5

Let A_i be a semimaximal F- submodule of F- module X_i , $i = 1,2,3, \dots \dots n$ Then ${}_{i=1}^{n}\oplus A_i$ is a semimaximal F- submodule of F- module ${}_{i=1}^{n}\oplus X_i$, $Provided$ $(\mathrm{X/A})_* = \ X_*/ \ \ A_* \quad .for\ each\ \ F-module\ X\ \ and\ \ A\ \leq X,\ \ \forall\ i =$ $1,2,3 \dots .n.$

Proof:

Since A_i is a semimaximal F- submodule of F- module X_i , $\forall i =$ $1,2,3, \dots .n. then \quad X_i/A_i$, is F- $semisimple\ module\ , for\ each\ i = 1,2,3 \dots .n.$
Hence ${}_{i=1}^{n}\oplus X_i\ /A_i$ isF-semisimple module and so by Remarks and Examples 3.4(5) $\left(\frac{{}_{i=1}^{n}\oplus X_i}{A_i}\right)_*$ is a semisimple .

B ut $\left(\dfrac{{}_{i=1}^{n}\oplus X_i}{A_i}\right)_* \ = \ {}_{i=1}^{n}\oplus\left(\left(\dfrac{X_i}{A_i}\right)_*\right)$ by[2]

$$= {}_{i=1}^{n}\oplus\ \left(\frac{(X_i)_*}{(A_i)_*}\right) \ \approx \ \frac{{}_{i=1}^{n}\oplus (X_i)_*}{{}_{i=1}^{n}\oplus(A_i)_*} = \frac{({}_{i=1}^{n}\oplus X_i)_*}{({}_{i=1}^{n}\oplus A_i)_*}$$

Hence $({}_{i=1}^{n}\oplus A_i)_*\ is\ a$ semimaximal submodule in $({}_{i=1}^{n}\oplus X_i)_*$.

By hypothesis $\left(\frac{{}_{i=1}^{n}\oplus X_i}{{}_{i=1}^{n}\oplus A_i}\right)_* = \frac{{}_{i=1}^{n}\oplus (X_i)_*}{{}_{i=1}^{n}\oplus (A_i)_*}$

Hence ${}_{i=1}^{n}\oplus A_i$ is a F- semimaximal submodule in ${}_{i=1}^{n}\oplus X_i$.

Remark 3.6

If *X* is a F- module of an *R*- module *M* ,then is not necessary that *X* has semimaximal F- submodule , for example .

Example :

Let $M = Z_{p^\infty}$ $(\ p\ is\ a\ prime\ number\ \ as\ Z - module$, let $X\ : M \rightarrow \ [0,1]\ , define\ by\ X(x) = 1\ ,\ \forall\ x \in M\ .$
Assume *X* has a semimaximal submodule say *A* . Let $A_* = \mathrm{N}$ is a a semimaximal submodule in $X_* = \ Z_{p^\infty} \ \ by\ Proposition$ 3.2 $;that\ is \quad X_*/N = \ Z_{p^\infty} \ /N$
is semisimple .

But $\dfrac{Z_{p^\infty}}{\mathrm{N}} \cong Z_{p^\infty}$ $, so\ that \ \ Z_{p^\infty} \ \ is$ semisimple, which is a contradiction .

Thus A is not semimaximal F- submodule of X.

The following proposition give sufficient condition but not necessary condition for F- submodule to be semimaximal.

Proposition 3.7

Let A be F- submodule of fuzzy module X such that A is intersection of a finite number of F- maximal submodules of X. Then, A is a F- semimaximal submodule .

Proof :

Let $A = A_1 \cap A_2 \cap \dots \dots \cap A_n$, A_i *is a F −* submodule of X , $\forall i = 1,2, \dots \dots . n$.

Hence,byProposition1.18, $X/A \cong F-$ *submdule of* $X|A_1 \oplus X|A_2 \dots \dots \dots .. \oplus X|A_n$

But is simple for each X/A_i , so $X/A_1 \oplus X / A_2 \dots \dots \dots . \oplus X| /A_n$ is semisimple and since a F- submodule of fuzzy semisimple is F- semisimple, Therefore X/A is a F- semisimple module

Thus A is F- semimaxmal submodule .

Remark 3.8

The converse of Proposition 3.7 is not true in general . We can give the following example :

Let $M = \oplus Z_p$ *as Z − module , p is a prime number*

Define $X : M \to [0,1]$ and $A: M \to [0,1]$ by $X(x) = 1$, $\forall x \in M$ *and* $A(x) = 1$, $\forall x \in Z_2$

But $X/A\,(x + A_*) = \begin{cases} 1 & if\ x \in A_* \\ \sup\{X(x+b) & if\ b \in A_*, x \notin A_* \end{cases}$

Thus $X/A\,(x + A_*) = 1, \forall\,(x + A_*) \in M|A_* = \oplus Z_p\,, p > 2$, is semisimple . Thus A is a F− semimaxmal submodule .

But A is not a finite intersection of F− maximal submodule .

Proposition 3.9

If X is a F- module of an R- module M and R is a semisimple ring ,then every F- submodule A of X is semimaximal .Provided $(XlA)_* = X_*|A_*$

Proof :

Let R be semisimple ring . Then M is a semisimple R-module .Since X is a F- module over M , then $X_* \leq M$ *and hence* X_* is semisimple which implies that X_* / A_* *is* semisimple. *Thus* A_* *is a* semimaximal submodule of X_*.

Then by Proposition 3.4 A is a F semimaximal submodule of X.

Proposition 3.10

The intersection of two semimaximal F- submodules is aslo semimaximal.

Proof :

If A , B are two semimaximal F- submodules of module X .Then X/A *and* X/B are semisimple . Therefore , $X/A \oplus X/B$ is semisimple .

As $X/ A \cap B \cong$ F− subodule of $X/A \oplus X/B$. But any F − submodule of $X/A \oplus X/B$ is semisimple , hence $X/A \cap B$ is semisimple

Therefore $A \cap B$ is a F -semimaximal submodule of X.

Proposition 3.11

If A , B are two F- submodules of F- module X such that A is semimaximal in X and B contains A . Then B is semimaximal in X .

Proof:

B/A is F- submodule of X/A since A ≤ B and $X/A \cong X/B$, by third isomorphism theorem.

But X /A is semisimple , imples $(X/A)/(B/A)$ is semisimple .

That is X/B is semisimple . Therefore B is semimaximal in X .

The following Corollary immediately consequence of Proposition 3.11

Corollary 3.12

If Å is a semimaximal F- submodule of F- module X and K be a F- ideal of a ring R . Then $[Å: K]$ is a F- submodule semimaximal.

Proof:

Since $[Å: K]$ is a fuzzy submodule of X containing Å , then result follows by Proposition 3.11

However the converse of Corollary 3.12. is not true in general ,for example

Example 3.13

Consider $M = Z_{12}$ $as\ Z - module\ and\ let\ X : M \to [0,1]$, $A : M \to [0,1], defined\ by$:

$where\ X(x) = 1 , \forall x \in M$,

$$A(x) = \begin{cases} 1 & if\ x \in \{0', , 4', 8'\}, \\ 1/2 & otherwise \end{cases}$$

A is a F- submodule of X. Let $K: Z \to [0,1]$ defined by:

$$K(x) = \begin{cases} 1 & if\ x \in 2Z \\ 1|3 & otherwise \end{cases}$$

K is a F- ideal of Z , then $X/A(x) = \begin{cases} 1 & if\ x \in Z_4 \\ 0 & otherwise \end{cases}$

Note that $\frac{X_*}{A_*} = Z_{12} /< \bar{4} >= Z_4$ is not semisimple . Hence A_* is not semimaximal **.**

Thus A is is not semimaximal Proposirion 3.4. However ,$[A_*: K_*] = [< \bar{4} >: 2Z] =< \bar{2} >$ is a semimaximal submodule of Z_4 .

But $[A : K]_* \leq [A_* : K_*]$ by [13].

Therefore $[A: K]$ is a semimaximal F- submodule of X .

Proposition 3.14

If A and B are two F- submodules of F- module X such that $B \leq A$ Then , A is semimaximal in X if and only if A/B is a semimaximal F- submodule of X/B .

Proof :

If A is semimaximal in X .Then X/A is semisimple ,which implies is semisimple , since $X/A \approx (X/B)/(A/B)$ .by [second isomorphism theorem [14] Hence A / B is a semimaximal F- submodule of X/B .The converse is smilarly.

Proposition 3.15

If K is semimaximal F- ideal of a ring R and X *is* be a F- module of R- module M, then KX is a semimaximal F- submodule , provided $(\frac{X}{KX})_* = X_* / K_* X_*$

Poorf:

Suppose K is semimaximal F- ideal a ring R , then K_* is a semimaximal ideal by [4] . Hence by[1] $K_* X_*$ is a semimaximal submodule and so $(KX)_*$ *is a* semimaximal submodule of X_*. Thus KX is a F-semimaximal submodule.

Remark 3.16

The following example shows that the converse of Proposition 3.15 is not true in general,

Consider $M = Z_6$ *as* $Z - module$, *define* $X: M \to [0,1]$ *by* $X(x) = 1 , \forall x \in M$

Define $K: Z \to [0,1]$, *by*: $K(x) = \begin{cases} 1 & if\ x \in 4Z , \\ 0 & otherwise \end{cases}$

X is a F- module and K is F- ideal of Z (it is easy to prove) .

Note that $X_* = Z_6$ and $K_* = 4Z$ which is not is semimaximal ideal .

Since , $(KX)_* = K_* X_*$ .Then $(KX)_* = (4Z)\, Z_6 = \{0', 2', 4'\} = < \bar{2} >$ and so .

$(X/KX)_* = Z_6 /< \bar{2} >$ (is simple)

Thus, $(KX)_*$ is a maximal submodule in X_*.

On the other hand , by Proposition 1.17 $(X/KX)_* = X_*/(KX)_*$

Thus KX is a semimaximal $F-$ submodule of X .

Corollary 3.17

X is be a multiplication F- module of an R module - M . If every F- ideal K in a ring R is semimaximal , then every F- submodule of X is semimaximal. Provided that $(X/A)_* = X_*/A_* , \forall A \leq X$

poorf: It is directly from Proposition 3.15 .

Proposition 3.18

Every epimorphic image of semimaximal F- submodule is a semimaximal F- submodule .

Proof:

Let $g : X/A \to Y / f(A)$ *such that* $g(x + A) = y + f(A) , \forall y \in Y$.

g is well-defined and g is an epimorphism .

Since X/A is semisimple . Hence , $g(X/A) = Y(X/A) = Y$ is semisimple .

Thus , $f(A)$ is a semimaximal fuzzy submodule.

Proposition 3.19

Let X be a finitely generated F- R- module M such that

is a semimaximal F- ideal , . $\forall A \leq X$. If = $(X/A)_* F - ann\, A$

$\frac{X_*}{A_*}$, $\forall A \leq X$, Then every F- submodule of X is semimaximal.

Proof:

Since X is a F- finitely generated R- module , then X_* is finitely generated R- module [15]. Since $(F - annA)_* \leq ann\, A_*$ *and* $F - annA$ *is* semimaximal F-ideal ,implies $(F - annA)_*$ is semimaximal *by*[4] , *so ann* A_* *is* semimaximal ideal .

Then by [1] , every submodule of X_* *is* semimaximal.

Hence , A_* is semimaximal . But this implies A is semimaximal , since $((X\ /A)_* = \frac{X_*}{A_*}$

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