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The Necessary Condition for Optimal Boundary Control Problems for Triple Elliptic Partial Differential Equations

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Abstract

In this work, we prove that the triple linear partial differential equations (PDEs) of the elliptic type (TLEPDEs) with a given classical continuous boundary control vector (CCBCVr) has a unique "state" solution vector (SSV) by utilizing the Galerkin's method (GME). Also, we prove the existence of a classical continuous boundary optimal control vector (CCBOCVr) ruled by the TLEPDEs. We study the existence solution for the triple adjoint equations (TAJEs) related with the triple state equations (TSEs). The Fréchet derivative (FDe) for the objective function is derived. At the end we prove the necessary "conditions" theorem (NCTh) for optimality for the problem.

Keywords: boundary optimal control, triple linear partial differential equations of elliptic type, Fréchet derivative, necessary conditions.

1. Introduction

In many scopes, the optimal control problem (OCPr) has a significant base of life problems, different examples for applications of such problems are studied in medicine [1], in aircraft [2], in electric power [3], in economic growth [4], and many other fields.

This role push many investigators to study the OCPr for nonlinear ordinary differential equations (NONODEs) as [5], or for different types of linear PDEs (LPDEs) hyperbolic, parabolic and elliptic as in [6,7] and [8] respectively. However, many others interested to study the OCPr for couple nonlinear PDEs (CNONLPDEs) of these three types [9,10] and [10], whilst [11,12] and [13] studied these three types of the CNONLPDEs but involved a Neumann boundary control (NBC). On the other hand, [14,15], and [16] in 2019 studied OCPr for triple PDEs (TPDEs) of the three types, while [17] studied OCPr involving NBC

"OCPrNBC" for TPDEs of parabolic type (TPDEsP). All these investigations push us to seek the OCPrNBC governed by the TLEPDEs.

In this paper and at first, we prove that the TLEPDEs with a given CCBCVr has a unique SSV utilizing the GME. Second we prove the existence theorem of CCBOCVr ruled by the TLEPDEs. We study the existence for the solution of the TAJE related with the TSEs. The FDe of the objective function is derived. At the end, the NCTh of optimality of is demonstrated.

2. Problem Description

Let Ω be a bounded and open connected subset in R^2 with "Lipshitz boundary" $\partial\Omega$, the OCPr is considered by the "state vector equation" which consists of the TLEPDEs with the NBC.

$A_1y_1 + y_1 - y_2 - y_3 = f_1(x), \text{in } \Omega$ (1)

$A_2y_2 + y_1 + y_2 + y_3 = f_2(x), \text{in } \Omega$ (2)

$A_3y_3 + y_1 - y_2 + y_3 = f_3(x), \text{in } \Omega$ (3)

$\sum_{i,j=1}^{2} a_{1_{ij}} \frac{\partial y_1}{\partial n_1} = u_1 , \text{on } \partial\Omega$ (4)

$\sum_{i,j=1}^{2} a_{2_{ij}} \frac{\partial y_2}{\partial n_2} = u_2 , \text{on } \partial\Omega$ (5)

$\sum_{i,j=1}^{2} a_{3_{ij}} \frac{\partial y_3}{\partial n_3} = u_3 , \text{on } \partial\Omega$ (6)

where

$A_r y_r = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{r_{ij}}(x)\frac{\partial y_r}{\partial x_j}\right), r = 1,2,3,\ a_{r_{ij}} = a_{r_{ij}}(x_{ij}) \in L^\infty(\Omega)$, and $(u_1, u_2, u_3) = \left(u_1(x), u_2(x), u_3(x)\right) \in \left(L_2(\partial\Omega)\right)^3$ is the NBC vector (NBCV), $(y_1, y_2, y_3) = \left(y_1(x), y_2(x), y_3(x)\right) \in \left(H^1(\Omega)\right)^3$ is the SSV corresponding to NBCV, $(f_1, f_2, f_3) = \left(f_1(x), f_2(x), f_3(x)\right) \in \left(L_2(\Omega)\right)^3$ is given functions, for all $X \in \Omega$, and $n_l, \forall\, l = 1,2,3$, is a unit vector normal on Σ.

The controls are defined in the set $\vec{W} \subset \left(L_2(\partial\Omega)\right)^3$, with

$\vec{W} = \left\{(u_1, u_2 , u_3) \in \left(L_2(\partial\Omega)\right)^3 \,\middle|\, (u_1, u_2 , u_3) \in \vec{U} \subset R^3 \text{ a.e in } \partial\Omega\right\}$ Where $\vec{U}$ is a convex set.

The objective functional is defined

$$\operatorname{Min}_{\vec{u}\in\vec{W}} G_o(\vec{u}) = \frac{1}{2}\|y_1 - y_{1d}\|^2_{L_2(\Omega)} + \frac{1}{2}\|y_2 - y_{2d}\|^2_{L_2(\Omega)} + \frac{1}{2}\|y_3 - y_{3d}\|^2_{L_2(\Omega)} + \frac{\alpha}{2}\|u_1\|^2_{L_2(\partial\Omega)} + \frac{\alpha}{2}\|u_2\|^2_{L_2(\partial\Omega)} + \frac{\alpha}{2}\|u_3\|^2_{L_2(\partial\Omega)} \quad (7)$$

Let $\vec{V} = (V)^3 = \left(H^1(\Omega)\right)^3$. The symbols $(v , v)_{L_2(\Omega)}$, and $\| V \|_{L^2(\Omega)}$ $\left(\| V \|_{L^2(\partial\Omega)}\right)$ are the inner product (IP) and the norm in $L_2(\Omega)$ $(L_2(\partial\Omega))$, by $(v , v)_{H^1(\Omega)}$, $\|V\|_{H^1(\Omega)}$ the IN and the norm in $H^1(\Omega)$, By $(\vec{v} , \vec{v})_{L_2(\Omega)} = \sum_{i=1}^{2}(v_i , v_i)$ and $\| \vec{v} \|_{(L_2(\Omega))^3} = \sum_{i=1}^{3}\| v_i\|_{L_2(\Omega)}$ the IP and the norm in $\left(L_2(\Omega)\right)^3$, by $(\vec{v} , \vec{v})_{L_2(\Omega)} = \sum_{i=1}^{3}(v_i , v_i)$ and $\| \vec{v} \|_{\left(H^1(\Omega)\right)^3} = \sum_{i=1}^{3}\| v_i\|_{H^1(\Omega)}$ the IP and the norm in $\vec{V}$ and $\vec{V}^*$ is the dual of $\vec{V}$.

3. Weak Formulation:

The weak form (WFO) for (1-3) is obtained by multiplying their both sides by $v_1 \in V$, $v_2 \in V$ and $v_3 \in V$ respectively, then integrating them and then using the generalized Green's theorem is applied for the terms that contain the derivatives of order two, to get:

$a_1(y_1, v_1) - (y_2 + y_3 , v_1)_{L_2(\Omega)} = (f_1 , v_1)_{L_2(\Omega)} + (u_1 , v_1)_{L_2(\partial\Omega)} , \forall\, v_1 \in V$ (8)

$a_2(y_2, v_2) + (y_1 + y_3, v_2)_{L_2(\Omega)} = (f_2, v_2)_{L_2(\Omega)} + (u_2, v_2)_{L_2(\partial\Omega)}, \forall v_2 \in V$ (9)

$a_3(y_3, v_3) + (y_1 - y_2, v_3)_{L_2(\Omega)} = (f_3, v_3)_{L_2(\Omega)} + (u_3, v_3)_{L_2(\partial\Omega)}, \forall v_3 \in V$ (10)

Adding (8) , (9) and (10) , to get :

$a(\vec{y}, \vec{v}) = F(\vec{v}), \forall \vec{v} \in V$ (11)

where

$$a(\vec{y}, \vec{v}) = a_1(y_1, v_1) - (y_2 + y_3, v_1)_{L_2(\Omega)} + a_2(y_2, v_2)$$
$$(y_1 + y_3, v_2)_{L_2(\Omega)} + a_3(y_3, v_3) + (y_1 - y_2, v_3)_{L_2(\Omega)} \quad (12a)$$

$a_r(y_r, v_r) = \int_\Omega \left(\sum_{i,j=1}^{2} a_{r_{ij}} \frac{\partial y_r}{\partial x_i} \frac{\partial v_r}{\partial x_j} + y_r v_r\right) dx$, with

$a_r(y_r, v_r) \geq C_{1r} \|y_r\|^2_{H^1(\Omega)}$, where $C_{1r} \geq 0$, $r = 1, 2, 3$

$|a_r(y_r, v_r)| \leq C_{2r} \|y_r\|^2_{H^1(\Omega)} \|V_r\|^2_{H^1(\Omega)}$, where $C_{2r} \geq 0$, $r = 1, 2, 3$

$a(\vec{y}, \vec{y}) \geq \alpha_1 \|\vec{y}\|^2_{(H^1(\Omega))^3}$

$|a(\vec{y}, \vec{v})| \leq \alpha_2 \|\vec{y}\|_{(H^1(\Omega))^3} \|\vec{v}\|_{(H^1(\Omega))^3}$

Where $\|\vec{y}\|^2_{(H^1(\Omega))^3} = \|\vec{y}\|^2_{(L_2(\Omega))^3} + \|\nabla \vec{y}\|^2_{(L_2(\Omega))^3}$, and

$$F(\vec{v}) = (f_1, v_1)_{L_2(\Omega)} + (u_1, v_1)_{L_2(\partial\Omega)} + (f_2, v_2)_{L_2(\Omega)} + (u_2, v_2)_{L_2(\partial\Omega)} + (f_3, v_3)_{L_2(\Omega)} +$$
$$(u_3, v_3)_{L_2(\partial\Omega)} \quad (12b)$$

Assumptions (A):

a) $a(\vec{y}, \vec{v})$ is coercive , i.e , $a(\vec{y}, \vec{y}) \geq C\|\vec{y}\|^2_{(H^1(\Omega))^3}$.

b) $|a(\vec{y}, \vec{y})| \leq C_1 \|\vec{y}\|^2_{(H^1(\Omega))^3} \|\vec{v}\|_{(H^1(\Omega))^3}$ where $c_1 > 0$.

c) $|F(\vec{v})| \leq C_2 \|\vec{v}\|_{(L_2(\Omega))^3}$, $\forall \vec{v} \in V$, $c_2 > 0$.

To find the solution of (11), the GME is applied, and an approximation (APP) subspace $\vec{V}_n \subset \vec{V}$ ($\vec{V}_n$ is the set of continuous function in Ω) is chose, thus (11) will be in the following APP form:

$a(\vec{y}_n, \vec{v}) = F(\vec{v}), \forall \vec{y}_n, \vec{v} \in \vec{V}_n$ (13)

Theorem 3.1: If $\vec{u} \in (L_2(\partial\Omega))^3$, is a given NBCV, then problem (13) has a unique APP solution(APPS) $\vec{y}_n \in \vec{V}_n$

Proof: let $\{\vec{\varphi}_1, \vec{\varphi}_2, \dots\dots\dots, \vec{\varphi}_n\}$ span $\vec{V}_n$, then the APPS of (13) is written by:

$\vec{y}_n = \sum_{j=1}^{n} d_j \vec{\varphi}_j (x_1, x_2)$ (14)

where $\vec{\emptyset}_j = \left(\left(4\,\ell \bmod (4 - \ell)\right)_{\varphi_k}, \left(4 \bmod (\ell + 1)\right)_{\varphi_k}, \left((4 + \ell^2) \bmod (\ell)\right)_{\varphi_k}\right), \ell = 1, 2, 3$

$j = k + n(\ell - 1)$ and $d_j = d_{\ell_k}$ is unknown constant , $\forall j = 1, 2, \dots\dots, n$, with $n = 3N$

By substituting (14) in (13) , with $\vec{v} = \vec{\varphi}_i$, we get:

$\sum_{j=1}^{n} d_j\, a(\vec{\varphi}_j, \vec{\varphi}_i) = F(\vec{\varphi}_i)$, $\forall i = 1, 2, \dots, n$ (15)

It is clear that (15) is equivalent to the algebraic system.

$A_{n\times n} D_{n\times 1} = b_{n\times 1}$ (16)

where

$A_{n\times n} = (a_{ij})_{n\times n}$, $a_{ij} = a(\vec{\varphi}_j, \vec{\varphi}_i)$, $b_{n\times 1} = (b_1, b_2, \dots. b_n)^T$, $b_i = F(\vec{\varphi}_i)$

and $D_{n\times 1} = (d_1, d_2, \dots, d_n)^T$, i, j = 1 , 2 , ..., n.
Since $A_{n\times n}\, D_{n\times 1} = 0 \Longrightarrow a\left(\sum_{j=1}^{n} d_j\, \vec{\varphi}_j, \vec{\varphi}_i\right) = 0$, then from using(A – a)

$$c\left\|\sum_{j=1}^{n} d\vec{\varphi}_j\right\|^2_{(H^1(\Omega))^2} \le \sum_{i=1}^{n} d_i a\left(\sum_{j=1}^{n} d_j\vec{\varphi}_j, \vec{\varphi}_i\right) = 0$$

The uniqueness (16) is obtained from its corresponding homogeneous system.

Proposition 3.1 [6]: For any $\vec{v}$ in $\vec{V}$, $\vec{V}_n$ has a sequence $\{\vec{\varphi}_n\}$ with $\vec{\varphi}_n \in \vec{V}_n$, $\forall$ n for which $\vec{\varphi}_n \rightarrow \vec{V}$ strongly in $\vec{V}$.

Now, by Theorem 3.1, the following sequence of the WFO has a sequence for the solutions $\{\vec{y}_n\}_{n=1}^{\infty}$

$$a(\vec{y}_n, \vec{\varphi}_n) = F(\vec{\varphi}_n),\ \forall \vec{y}_n, \vec{\varphi}_n \in \vec{V}_n, \forall n \qquad (17)$$

Theorem 3.2: The sequence $\{\vec{y}_n\}_{n=1}^{\infty}$ converges to $\vec{y}$ strongly in $\left(H^1(\Omega)\right)^3$.

Proof: we have $\vec{y}_n$ is a solution of (17), then by (A – a & c) :
$\|\vec{y}_n\|_{\left(H^1(\Omega)\right)^3} \le \bar{C}_1$, where $\bar{C}_1 > 0$, $\forall n$

From the Alaoglu's theorem (Agth)[8], $\{\vec{y}_n\}$ has a subsequence. It is not loss of generality to say again $\{\vec{y}_n\}$ for which $\vec{y}_n \rightarrow \vec{y}$, weakly in $\vec{V}$.

Now, let $\vec{v} \in \vec{V}$ be fixed, then $L_{\vec{V}}(\vec{w}) = a(\vec{w}, \vec{v})$ is a bounded linear functional i.e, $L_{\vec{V}} \in \vec{V}$.

To prove the sequence of the solutions $\{\vec{y}_n\}_{n=1}^{\infty}$ of the WFO (17) converges to the solution of the WFO(11).

Step 1: since $\vec{y}_n \rightarrow \vec{y}$ weakly in $\vec{V}$ and by Proposition3.1, $\vec{\varphi}_n \rightarrow \vec{V}$ strongly in $\vec{V}$, then

$$|a(\vec{y}_n, \vec{\varphi}_n) - a(\vec{y}, \vec{v})| \le |a(\vec{y}_n, \vec{\varphi}_n - \vec{v}) + a(\vec{y}_n - \vec{y}, \vec{v})| \le$$
$$C_1\|\vec{y}_n\|_{\left(H^1(\Omega)\right)^3}\|\vec{\varphi}_n - \vec{v}\|_{\left(H^1(\Omega)\right)^3} + C_2\|\vec{y}_n - \vec{y}\|_{\left(H^1(\Omega)\right)^3}\|\vec{v}\|_{\left(H^1(\Omega)\right)^3} \rightarrow 0 \qquad (18)$$

Hence

$$a(\vec{y}_n, \vec{\varphi}_n) \rightarrow a(\vec{y}, \vec{v}) \qquad (19)$$

Step 2: since $\vec{\varphi}_n \rightarrow \vec{v}$ strongly in $\vec{V} \Longrightarrow \vec{\varphi}_n \longrightarrow \vec{v}$ weakly in $\vec{V}$, then $F(\vec{\varphi}_n) \longrightarrow F(\vec{v})$

The above two steps give the following

$$a(\vec{y}, \vec{v}) = F(\vec{v}),\ \forall \vec{v} \in \vec{V}$$

which means $\vec{y}$ is a solution in (11).

Now, to prove $\vec{y}_n \rightarrow \vec{y}$ strongly in $\vec{V}$, by using (A − a), it follows that :

$$C\|\vec{y} - \vec{y}_n\|_{\left(H^1(\Omega)\right)^3} \le a(\vec{y} - \vec{y}_n, \vec{y}) - a(\vec{y}, \vec{y}_n) + a(\vec{y}_n, \vec{y}_n)$$
$$= a(\vec{y} - \vec{y}_n, \vec{y}) = L_{\vec{y}}(\vec{y} - \vec{y}_n) \rightarrow 0$$

Thus $\{\vec{y}_n\}$ converges to $\vec{y}$ strongly in $\left(H^1(\Omega)\right)^3$.

4. Existence of a CCBOCVr:

Lemma 4.1: The operator $\vec{u} - \vec{y}_{\vec{u}}$ is Lipschitz continuous from $\left(L_2(\partial\Omega)\right)^3$ into $\left(L_2(\Omega)\right)^3$ and is satisfied

$$\|\overrightarrow{\Delta y}\|_{\left(L_2(\Omega)\right)^3} \le C_3\|\overrightarrow{\Delta u}\|_{\left(L_2(\partial\Omega)\right)^3}, \text{with } C_3 > 0 .$$

Proof: Let $\acute{u}_1 , \acute{u}_2 , \acute{u}_3$ be controls of the WFO (11) $\acute{y}_1$, $\acute{y}_2$ and $\acute{y}_3$ be their corresponding SSV, subtracting the obtaining WFO from (11), by letting $\Delta y_1 = \acute{y}_1 - y_1$ and $\Delta u = \acute{u}_1 - u$ with $\vec{v} = \overrightarrow{\Delta y}$, to get

$$a(\overrightarrow{\Delta y}, \overrightarrow{\Delta y}) = (\Delta u_1 , \Delta y_1)_{(L_2(\partial\Omega))} + (\Delta u_2 , \Delta y_2)_{(L_2(\partial\Omega))} + (\Delta u_3 , \Delta y_3)_{(L_2(\partial\Omega))} \quad (20)$$

which gives after using(A-a), the Cauchy inequality(CSIn) and then the trace operator to obtain

$$C\|\overrightarrow{\Delta y}\|^2_{(H^1(\Omega))^3} \le \left|a(\overrightarrow{\Delta y}, \overrightarrow{\Delta y})\right| \le C_1\|\overrightarrow{\Delta u}\|_{(L_2(\partial\Omega))^3}\|\overrightarrow{\Delta y}\|_{(H^1(\Omega))^3}$$

Then,

$$\|\overrightarrow{\Delta y}\|_{(H^1(\Omega))^3} \le C_2\|\overrightarrow{\Delta u}\|_{(L_2(\partial\Omega))^3} \text{ where } C_2 = \frac{C_1}{C}$$

Since $\|\overrightarrow{\Delta y}\|_{(L_2(\Omega))^3} \le C\|\overrightarrow{\Delta y}\|_{(H^1(\Omega))^3}$, then the above inequality becomes

$$\|\overrightarrow{\Delta y}\|_{(L_2(\Omega))^3} \le C_3\|\overrightarrow{\Delta u}\|_{(L_2(\partial\Omega))^3} \text{ , where } C_3 = C \cdot C_2 \quad (21)$$

Lemma 4.2 [3]: The norm $\| \ \|_{L_2(\Omega)}$ (or the norm $\| \ \|_{L_2(\partial\Omega)}$) is weakly lower semi continuous (WELSC)

Lemma 4.3: The objective function (7) is WELSC.

Proof: The norm $\| \ \|_{L_2(\partial\Omega)}$ is WELSC (by lemma 4.2), but when $\vec{u}_n \to \vec{u}$ weakly in $(L_2(\Omega))^3$, then by using lemma 4.1 gives $\vec{y}_n \to \vec{y} = \vec{y}_{\vec{u}}$ weakly in $(L_2(\Omega))^3$, then by using lemma 4.2 , $\|\vec{y} - \vec{y}_n\|^2_{L_2(\Omega)}$ is WELSC, i.e , $G_0(\vec{u})$ is WELSC.

Lemma 4.4 [3]: The norm $\| \ \|_{L_2(\Omega)}$ ($\| \ \|_{L_2(\partial\Omega)}$) is strictly convex (SC) .

Remark 4.1: by applying lemma 4.4, $G_0(\vec{u})$ is (SC).

Theorem 4.1: If U_i , $\forall i = 1,2,3$ is bounded, then there is a CCBOCVr for the problem (8).

Proof: Since U_i , $\forall i = 1,2,3$ is bounded , then W_i ($\forall i = 1,2,3$) is a bounded and then $\vec{W}$ is bounded Since $G_0(\vec{u}) \ge 0$, then there is a minimum sequence $\{\vec{u}_n\} = \{(u_{1n}, u_{2n}, u_{3n})\} \in \vec{W}$, for each n , such that:

$$\lim_{n\to\infty} G_0(\vec{u}_n) = \inf_{\vec{w}\in\vec{W}} G_0(\vec{w})$$

From the coercive property of $G_0(\vec{u})$, and its infimum, there exists a constant $C > 0$ such that $\|\vec{u}_n\|_{(L_2(\partial\Omega))^3} \le C$, $\forall$ n (22)

Then by Agth, the sequence $\{\vec{u}_n\}$ has a subsequence. It is not loss of generality to say again $\{\vec{u}_n\}$ for which $\vec{u}_n \to \vec{u}$ weakly in $(L_2(\partial\Omega))^3$.

From theorem3.1, for each control $\vec{u}_n = (u_{1n}, u_{2n}, u_{3n})$ the TEPDEsE has a unique APPS $\vec{y}_n = \vec{y}_{u_n}$.

To prove (for each n) $\{\vec{y}_n\}$, is bounded in $\vec{V}$, using (A – a & c) , CSIn and the trace operator , to get

$$C\|\vec{y}_n\|^2_{(H^1(\Omega))^3} \le a(\vec{y}_n, \vec{y}_n) = F(\vec{y}_n)$$

$$\le \ell_1\|y_{1n}\|_{L_2(\Omega)} + c_1\|y_{1n}\|_{H^1(\Omega)} + \ell_2\|y_{2n}\|_{L_2(\Omega)} + c_2\|y_{2n}\|_{H^1(\Omega)} + \ell_3\|y_{3n}\|_{L_2(\Omega)} + c_3\|y_{3n}\|_{H^1(\Omega)} \le s\|\vec{y}_n\|_{H^1(\Omega)}$$

then $\|\vec{y}_n\|_{(H^1(\Omega))^3} \le a$, for each n with $a = \frac{s}{C} > 0$.

where $r_1 = \max(\ell_1, c_1)$, $r_2 = \max(\ell_2, c_2)$, $r_3 = (\ell_3, c_3)$ and $s = \max(r_1, r_2, r_3)$.

Then by Agth $\{\vec{y}_n\}$ has a subsequence. It is not loss of generality to say again $\{\vec{y}_n\}$ for which $\vec{y}_n \to \vec{y}$ weakly in $\vec{V}$, since $\forall n\, \vec{y}_n$ satisfies the WFO (11) for each, or

$$a(\vec{y}_n,\vec{v}) = F_n(\vec{v}) = (f_1, v_1)_{L_2(\Omega)} + (u_{1n}, v_1)_{L_2(\partial\Omega)} + (f_2, v_2)_{L_2(\Omega)} + (u_{2n}, v_2)_{L_2(\partial\Omega)} + (f_3, v_3)_{L_2(\Omega)} + (u_{3n}, v_3)_{L_2(\partial\Omega)} \quad (23)$$

To show (23) converges to

$$a(\vec{y},\vec{v}) = F(\vec{v}) \quad (24)$$

First, since $y_{in} \to y_i$ weakly in $L_2(\Omega)\ \forall\, i$ (for $y_{in} \to y_i$ weakly in V_i), then by CSIn, one has:

$$\Big| a_1(y_{1n}, v_1) - (y_{2n} + y_{3n}, v_1)_{L_2(\Omega)} + a_2(y_{2n}, v_2) + (y_{1n} + y_{3n}, v_2)_{L_2(\Omega)} + a_3(y_{3n}, v_3) + (y_{1n} - y_{2n}, v_3)_{L_2(\Omega)} - a_1(y_1, v_1) + (y_2 + y_3, v_1)_{L_2(\Omega)} - a_2(y_2, v_2) - (y_1 + y_3, v_2)_{L_2(\Omega)} - a_3(y_3, v_3) - (y_1 - y_2, v_3)_{L_2(\Omega)}$$
$$\le (C_1\| y_{1n} - y_1\|_{H^1(\Omega)} + \| y_{2n} - y_2\|_{L_2(\Omega)} + \| y_{3n} - y_3\|_{L_2(\Omega)})\| v_1\|_{L_2(\Omega)}$$
$$+(C_2\| y_{2n} - y_2\|_{H^1(\Omega)} + \| y_{1n} - y_1\|_{L_2(\Omega)} + \| y_{3n} - y_3\|_{L_2(\Omega)})\| v_2\|_{L_2(\Omega)}$$
$$+(C_3\| y_{3n} - y_3\|_{H^1(\Omega)} + \| y_{1n} - y_1\|_{L_2(\Omega)} + \| y_{2n} - y_2\|_{L_2(\Omega)})\| v_3\|_{L_2(\Omega)} \to 0$$

Second, we have $\vec{u}_n \to \vec{u}$ weakly in $\left(L^2(\partial\Omega)\right)^3$, then the terms in the right hand side of (23) converges to the those in the right hand side of (24) .

Thus (23) converges to (24) .

But, we have $\vec{u}_n \to \vec{u}$ weakly in $\left(L_2(\partial\Omega)\right)^3$ and $G_0(\vec{u})$ is WELSC, then

$$G_0(\vec{u}) \le \lim_{n\to\infty} \inf_{\vec{u}_n \in \vec{W}} G_0(\vec{u}_n) = \lim_{n\to\infty} G_0(\vec{u}_n) = \inf_{\vec{w}\in\vec{W}} G_0(\vec{w})$$

Then

$G_0(\vec{u}) = \inf_{\vec{w}\in\vec{W}} G_0(\vec{w})$, i.e, $\vec{u}$ a CCBOCVr.

Applying Remark 4.1, gives us $\vec{u}$ which is unique.

5. **The** NCTh **for Optimality:**

Theorem 5.1: The TAJEs$(z_1, z_2, z_3) = \left(z_{1u_1}, z_{2u_2}, z_{3u_3}\right)$ of the WFO of the TSEs (1-6) are given by:

$$A_1 z_1 + z_1 + z_2 + z_3 = (y_1 - y_{1d}), \text{ in } \Omega \quad (25)$$
$$A_2 z_1 - z_1 + z_2 - z_3 = (y_2 - y_{2d}), \text{ in } \Omega \quad (26)$$
$$A_3 z_1 - z_1 + z_2 + z_3 = (y_3 - y_{3d}), \text{ in } \Omega \quad (27)$$
$$\frac{\partial z_1}{\partial n_1} = 0, \text{ in } \partial\Omega \quad (28)$$
$$\frac{\partial z_2}{\partial n_2} = 0, \text{ in } \partial\Omega \quad (29)$$
$$\frac{\partial z_3}{\partial n_3} = 0, \text{ in } \partial\Omega \quad (30)$$

Then the FDe of G_0 is given by :

$$\left(G_0'(\vec{u}), \overrightarrow{\Delta u} \right)_{L_2(\partial\Omega)} = \left(\vec{z} + \propto \vec{u}, \overrightarrow{\Delta u}\right)_{L_2(\partial\Omega)}$$

Proof: Rewriting the TAJEs (25-30) by its WFO, adding them, then substituting $\vec{v} = \overrightarrow{\Delta y}$, once get the following WFO which has a unique solution $\vec{z} = \vec{z}_{\vec{u}}$:

$a_1(z_1, \Delta y_1) + (z_2 + z_3, \Delta y_1)_{L_2(\Omega)} + a_2(z_2, \Delta y_2) - (z_1 + z_3, \Delta y_1)_{L_2(\Omega)} + a_3(z_3, \Delta y_3) - (z_1 - z_2, \Delta y_3)_{L_2(\Omega)} = (y_1 - y_{1d}, \Delta y_1)_{L_2(\Omega)} + (y_2 - y_{2d}, \Delta y_2)_{L_2(\Omega)} +$

$$(y_3 - y_{3d}\,, \Delta y_3\,)_{L_2(\Omega)} \tag{31}$$

Utilizing (12a&b) in (11), then substituting $\vec{v} = \vec{z}$ once and once again $\vec{v} = \vec{z}$ and setting $\vec{y} + \overrightarrow{\Delta y}$ instead of $\vec{y}$, then subtracting the second obtained equation from the first one, to get

$$a_1(\Delta y_1, z_1\,) - (\Delta y_2, z_1\,)_{L_2(\Omega)} - (\Delta y_3, z_1\,)_{L_2(\Omega)} + a_2(\Delta y_2, z_2\,) + (\Delta y_1, z_2\,)_{L_2(\Omega)} + (\Delta y_3, z_2\,)_{L_2(\Omega)} + a_3(\Delta y_3, z_3\,) + (\Delta y_1, z_3\,) - (\Delta y_2, z_3\,) = (\Delta u_1\,, z_1\,)_{L_2(\partial\Omega)} + (\Delta u_2\,, z_2\,)_{L_2(\partial\Omega)} + (\Delta u_3\,, z_3\,)_{L_2(\partial\Omega)} \tag{32}$$

Subtracting (32) from (31), to get

$$\left(\overrightarrow{\Delta u}\,, \vec{z}\,\right)_{L_2(\partial\Omega)} = \left(\vec{y} - \vec{y}_d\,, \overrightarrow{\Delta y}\,\right)_{L_2(\Omega)} \tag{33}$$

Now, for the cost function, we have

$$G_0(\vec{u} + \overrightarrow{\Delta u}) - G_0(\vec{u}) = (y_1 - y_{1d}\,, \Delta y_1\,)_{L_2(\Omega)} + (y_2 - y_{2d}\,, \Delta y_2\,)_{L_2(\Omega)} + (y_3 - y_{3d}\,, \Delta y_3\,)_{L_2(\Omega)} + \frac{1}{2}\left\|\overrightarrow{\Delta y}\right\|^2_{(L_2(\Omega))^3} + \frac{\propto}{2}\left\|\overrightarrow{\Delta u}\right\|^2_{(L_2(\partial\Omega))^3} \tag{34}$$

From (33) & (34), we get

$$G_0\left(\vec{u} + \overrightarrow{\Delta u}\right) - G_0(\vec{u}) = \left(\vec{z} + \propto \vec{u}\,, \overrightarrow{\Delta u}\right)_{L_2(\partial\Omega)} + \frac{1}{2}\left\|\overrightarrow{\Delta y}\right\|^2_{(L_2(\Omega))^3} + \frac{\propto}{2}\left\|\overrightarrow{\Delta u}\right\|^2_{(L_2(\partial\Omega))^3} \tag{35}$$

From lemma 4.1, it yield that

$$\frac{1}{2}\left\|\overrightarrow{\Delta y}\right\|^2_{(L_2(\Omega))^3} + \frac{\propto}{2}\left\|\overrightarrow{\Delta u}\right\|^2_{(L_2(\partial\Omega))^3} = \epsilon(\overrightarrow{\Delta u})\left\|\overrightarrow{\Delta u}\right\|^2_{(L_2(\partial\Omega))^3} \tag{36}$$

Where $\epsilon\left(\overrightarrow{\Delta u}\right) \longrightarrow 0$, as $\left\|\overrightarrow{\Delta u}\right\|^2_{(L_2(\partial\Omega))^3} \longrightarrow 0$ with $\epsilon\left(\overrightarrow{\Delta u}\right) = \epsilon_1\left(\overrightarrow{\Delta u}\right) + \epsilon_2\left(\overrightarrow{\Delta u}\right)$

Then from the FDe of G_0 , and (35-36), once get:

$\left(G'_0(\vec{u}), \overrightarrow{\Delta u}\right) = \left(\vec{z} + \propto \vec{u}\,, \overrightarrow{\Delta u}\right)_{L_2(\partial\Omega)}$.

Theorem 5.2: The (CCBOC) of (1-6) is:

$G'_0(\vec{u}) = \vec{z} + \propto \vec{u} = 0$ with $\vec{y} = \vec{y}_{\vec{u}}$ and $\vec{z} = \vec{z}_{\vec{u}}$.

Proof: If $\vec{u}$ is an optimal control of the problem, then

$G_0(\vec{u}) = \min_{\vec{w} \in \vec{W}} G_0(\,\vec{w}\,)\,, \forall\, \vec{w} \in \left(L_2(\partial\Omega)\right)^3$

i.e., $G'_0(\vec{u}) = 0 \Longrightarrow \vec{z} = -\propto \vec{u}\,,\ \overrightarrow{\Delta u} = \vec{w} - \vec{u}$

The necessary condition for optimality is:

$(\vec{z} + \propto \vec{u}\,, \vec{u}) \leq (\vec{z} + \propto \vec{u}\,, \vec{w})\,, \forall\, \vec{w} \in \left(L_2(\partial\Omega)\right)^3$.

6. Conclusions

The existence and uniqueness theorem for the SSV of the TLEPDEs is proved successfully using the GME when the CCBCVr is given. The proof of the existence CCBOCVr ruled by the considered TLEPDEs is demonstrated. The studding of the existence solution of the TAJEs related with the TLEPDEs is demonstrated. The FDe is derived. Finally the NCTh of optimality for the considered problem is demonstrated.

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