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Generalize Partial Metric Spaces

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Abstract

The purpose of this research is to introduce a concept of general partial metric spaces as a generalization of partial metric space, give some results and properties and find relations between the general partial metric space, partial metric spaces and D-metric spaces.

Keywords: partial metric space, D-metric space, general partial metric space.

1 Introduction and preliminaries

Metric spaces are very important in mathematics introduced and studied by the French mathematicians. Many researchers tried to generalized the metric space to different types for example, b-metric space defined by Stefan Czerwik [1], G-metric space defined by Mustafa and Sims [2], 2-metric space defined by Gahler [3], D*-metric space[4], partial metric space defined by Mathews[5] and D-metric space defined by Dhage [11]

First , give some definitions and properties of the partial metric spaces that can be found in [5-10]

A non-empty set Y is said to be partial metric space if there exists a function $p: Y^2 \to [0, \infty)$ satisfies the following condition:

p1 $\alpha = \beta \leftrightarrow p(\alpha, \alpha) = p(\alpha, \beta) = p(\beta, \beta)$

p2 $p(\alpha, \alpha) \leq p(\alpha, \beta)$

p3 $p(\alpha, \beta) = p(\beta, \alpha)$

p4 $p(\alpha, \beta) \leq p(\alpha, \mu) + p(\mu, \beta) - p(\mu, \mu)$

$\forall \alpha, \beta$ and $\mu \in Y$,where p is a partial metric on Y .

A basic example of a partial metric space is (R^+, p), where $p(\alpha, \beta) = \max\{\alpha, \beta\}$ $\forall \alpha, \beta \in R^+$

Also, the partial metric space p on Y generates a T_0 topology τ_p on Y, which has as a base of family open balls $\{B_p(\alpha,\epsilon): \alpha \in Y,\ \epsilon > 0\}$, where $B_p(\alpha,\varepsilon) = \{\beta \in Y: p(\alpha,\beta) < p(\alpha,\alpha) + \epsilon\}\ \forall\ \alpha \in \mathrm{Y}$ and $\epsilon > 0$. (see [9])

The sequence $\{\alpha_n\}$ in a partial metric space (Y,p) converge sequence if
$lim_{n\to\infty}\, p(\alpha\,,\alpha_n) = p(\alpha,\alpha)$
The sequence $\{\alpha_n\}$ in a partial metric space (Y,p) is said to be Cauchy sequences if
$\lim_{n,m\to\infty} p(\alpha_n,\alpha_m)$ exists (finite).
The partial metric space (Y,p) is said to be complete if every cauchy sequence $\{\alpha_n\}$ convergent to a point α in Y.
A mapping $F:(Y,p) \to (Y`,p`)$ is said to be continuous at $\alpha_0 \in$Y, if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mathrm{F}(\mathrm{B}_p(\alpha_0,\delta)) \subseteq \mathrm{B}_p(\mathrm{F}\alpha_0,\epsilon). \text{ (See [10])}$$

If p is a partial metric space, then the function $p^s: Y^2 \to R^+$ defined by

$$p^s(\alpha\,,\beta) = 2p(\alpha,\beta) - p(\alpha,\alpha) - p(\beta,\beta)$$

is a metric on Y (See [6])

Not that, the sequence $\{\alpha_n\}$ is a Cauchy sequence in a partial metric space (Y,p) if and only if $\{\alpha_n\}$ is a Cauchy sequence in the metric space (Y,p^s). (See [9], [10])

A partial metric space (Y,p) is said complete if and only if the metric space (Y,p^s) is complete. Furthermore, $\lim_{n\to\infty} p^s(\alpha_n,\alpha) = 0$ if and only if

$$p(\alpha,\alpha) = lim_{n\to\infty} p(\alpha_n,\alpha) = lim_{n,m\to\infty} p(\alpha_n,\alpha_m). (see [9], [10])$$

Second, we recall definition of D-metric space that can be found in [11], [12], [13]

A non-empty set Y is said to be D-metric space if there exists a function $D: Y^3 \to [0,\infty)$ satisfies the following conditions:

D1. $D(\alpha,\beta,\gamma) = 0 \Leftrightarrow \alpha = \beta = \gamma$
D2. $D(\alpha,\beta,\gamma) = D(\beta,\alpha,\gamma) = D(\gamma,\alpha,\beta) = D\ (\beta,\gamma,\alpha) = \ \dots\ (S\beta mmetr\beta)$
D3. $D(\alpha,\beta,\gamma) \le D(\mu\,,\beta,\gamma) + D(\alpha,\mu,\gamma) + D(\alpha,\beta,\mu)$

$\forall \alpha,\beta,\gamma$ *and* $\mu \in Y$, where D is D-metric on Y.

Y if for given $\epsilon > 0$, A sequence $\{\alpha_n\}$ in D-metric space (Y,D) is said converge to $\alpha \in$ there exists a positive integer $\mathfrak{m}_0$ such that $D(\alpha_n,\alpha_m,\alpha) < \epsilon\ \forall\ m \ge \mathfrak{m}_0, n \ge \mathfrak{m}_0$. [13]

A sequence $\{\alpha_n\}$ in D-metric space (Y,D) is said Cauchy if for given $\epsilon > 0$; there exists an positive integer $\mathfrak{m}_0$ such that $\mathrm{D}(\alpha_n,\alpha_m,\alpha_l) < \epsilon\ \forall \mathrm{m} \ge \mathfrak{m}_0, \mathrm{n} \ge \mathfrak{m}_0, \mathrm{l} \ge \mathfrak{m}_0$.[13]

A sequence $\{\alpha_n\}$ in D-metric space (Y,D) is said to be complete if every Cauchy sequence in Y converges to a point α in Y.

A sequence $\{\alpha_n\}$ in D-metric space (Y,D) converges strongly to an element α in Y if $(i) D(\alpha_n,\alpha_m,\alpha) \to 0\ as\ n,m \to \infty$. $(ii)\ \{D(\beta,\beta,\alpha_n)\}$ converges to $D(\beta,\beta,\alpha)\ \forall\ \beta \in Y$. [13]

A sequence $\{\alpha_n\}$ in D-metric space (Y,D) is said to be very strongly converges to an element α in Y if

$(i) D(\alpha_n, \alpha_m, \alpha) \to 0 \;\; as\, n, m \to \infty. (ii)\{D(\beta, z, \alpha_n)\}\; converges\; to\; D(\beta, z, \alpha) \;\; \forall \beta, z \in Y$. [13]

2. Generalize Partial Metric Spaces

Definition 1

A non-empty set Y is said to be a general partial metric space if there exists a function $D_p: Y^3 \to [0, \infty)$ satisfy the following condition:

Dp1. $Dp(\alpha, \alpha, \alpha) = D_p(\alpha, \beta, \gamma) = D_p(\beta, \beta, \beta) = D_p(\gamma, \gamma, \gamma) \Leftrightarrow \alpha = \beta = \gamma$
Dp2. $D_p(\alpha, \alpha, \alpha) \le D_p(\alpha, \beta, \gamma)$
Dp3. $D_p(\alpha, \beta, \gamma) = D_p(\alpha, \gamma, \beta) = D_p(\beta, \alpha, \gamma) = D_p(\beta, \gamma, \alpha) = \ldots (S\beta mmetr\beta)$
Dp4. $D_p(\alpha, \beta, \gamma) \le D_p(\mu, \beta, \gamma) + D_p(\alpha, \mu, \gamma) + D_p(\alpha, \beta, \mu) - D_p(\mu, \mu, \mu)$

$\forall\; \alpha, \beta, \gamma\; and\; \mu \in Y$, where D_p is general partial metric space on Y.

So, given an example of a general partial metric space we obtain

Example 2

Let $Y = [0, \infty)$ and define a function D_p on Y^3 by $D_p(\alpha, \beta, \gamma) = ma\alpha\{\alpha, \beta, \gamma\}$

Then D_p is a general partial metric space on Y.

Solution:

1) since $ma\alpha\{\alpha, \beta, \gamma\} = ma\alpha\{\alpha, \alpha, \alpha\} = ma\alpha\{\beta, \beta, \beta\} = ma\alpha\{\gamma, \gamma, \gamma\}$ if and only if $\alpha = \beta = \gamma$ then $D_p(\alpha, \beta, \gamma) = D_p(\alpha, \alpha, \alpha) = D_p(\beta, \beta, \beta) = D_p(\gamma, \gamma, \gamma)$ if and only if $\alpha = \beta = \gamma$

2) $D_p(\alpha, \alpha, \alpha) = ma\alpha\{\alpha, \alpha, \alpha\} = \alpha \le ma\alpha\{\alpha, \beta, \gamma\} = D_p(\alpha, \beta, \gamma)$.

3) Trivial

4) Since $ma\alpha\{\alpha, \beta, \gamma\} \le ma\alpha\{\mu, \beta, \gamma\} + ma\alpha\{\alpha, \mu, \gamma\} + ma\alpha\{\alpha, \beta, \mu\} - ma\alpha\{\mu, \mu, \mu\}$

Then $D_p(\alpha, \beta, \gamma) + D_p(\mu, \mu, \mu) \le D_p(\mu, \beta, \gamma) + D_p(\alpha, \mu, \gamma) + D_p(\alpha, \beta, \mu)$ □

Definition 3

Let (Y, D_p) be a general partial metric space, then

(1) A sequence $\{\alpha_n\}$ in (Y, D_p) converges to a point $\alpha \in Y$ if

$\lim_{n, m \to \infty} D_p(\alpha_n, \alpha_m, \alpha) = D_p(\alpha, \alpha, \alpha)$

(2) A sequence $\{\alpha_n\}$ in (Y, D_p) is Cauchy sequence if

$\lim_{n, m, l \to \infty} D_p(\alpha_n, \alpha_m, \alpha_l)$ exists (finite)

(3) A general partial metric space (Y, D_p) is said to be complete if every Cauchy sequence is converge to a point α in Y.

(4) A sequence $\{\alpha_n\}$ in (Y, D_p) is a strongly converge to α if

i. $D_p(\alpha_n, \alpha_m, \alpha) \rightarrow D_p(\alpha, \alpha, \alpha)$ $as\ n, m \rightarrow \infty$
ii. $D_p(\beta, \beta, \alpha_n) \rightarrow D_p(\beta, \beta, \alpha)$ $as\ n \rightarrow \infty$ $\forall \beta \in Y$

(5) A sequence $\{\alpha_n\}$ in (Y, D_p) is a very strongly converge to α if

i. $D_p(\alpha_n, \alpha_m, \alpha) \rightarrow D_p(\alpha, \alpha, \alpha)$ $as\ n, m \rightarrow \infty$
ii. $D_p(\beta, \gamma, \alpha_n) \rightarrow D_p(\beta, \gamma, \alpha)$ $as\ n \rightarrow \infty$ $\forall \beta, \gamma \in Y$

(6) For $\alpha \in Y$ and $\epsilon > 0$, the open ball of the general partial metric space with center α and radius ϵ is $B_{D_p}(\alpha, \epsilon) = \{\beta \in Y : D_p(\alpha, \beta, \beta) < D_p(\alpha, \alpha, \alpha) + \epsilon\}$.

(7) A mapping $F: (Y, D_p) \rightarrow (Y', D_p')$ is said to be continuous at α if for each open ball

$B_{D_p}(F(\alpha), \epsilon')$ in (Y', D_p') there exists a ball $B_p(\alpha, \epsilon)$ in (Y, D_p) such that

$F(B_{D_p}(\alpha, \epsilon)) \subseteq B_{D_p}(F(\alpha), \epsilon')$.

Not that, if $\{\alpha_n\}$ is a converge sequence in(Y, D_p) then the converge point is not unique.

Example4

Let $Y = [0, \infty)$ with $D_p(\alpha, \beta, \gamma) = ma\alpha\{\alpha, \beta, \gamma\}$ then (Y, D_p) is a general partial metric observe that if the sequence $\{1 + \frac{1}{n^2}\}, \alpha \geq 1$ then $D_p(\alpha_n, \alpha_m, \alpha) = lim_{n,m \rightarrow \infty} ma\alpha\{1 + \frac{1}{n^2}, 1 + \frac{1}{m^2}, \alpha\} = \alpha = D_p(\alpha, \alpha, \alpha)$.
Hence, every $\alpha \in [1, \infty)$ is a convergent point for the sequence $\{1 + \frac{1}{n^2}\}$.
Thus, the converge point is not unique.

Theorem 5

Every converge sequence in (Y, D_p) is a Cauchy sequence.

Proof:

Let (Y, D_p) be a general partial metric space and $\{\alpha_n\}$ is a converge sequence to α and $\epsilon > 0$.

Since $\{\alpha_n\}$ is converge to α then there exists $k \in N$ such that

$|D_p(\alpha_n, \alpha_m, \alpha) - D_p(\alpha, \alpha, \alpha)| < \epsilon \ \forall n, m > k$

So that

$$D_p(\alpha_n, \alpha_m, \alpha_l) \leq D_p(\alpha, \alpha_m, \alpha_l) + D_p(\alpha_n, \alpha, \alpha_l) + D_p(\alpha_n, \alpha_m, \alpha) - D_p(\alpha, \alpha, \alpha)$$
$$\leq 2D_p(\alpha, \alpha, \alpha) + \epsilon \qquad \forall n, m, l > K$$

Hence, $\lim_{n,m,l \rightarrow \infty} D_p(\alpha_n, \alpha_m, \alpha_l)$ exists, thus $\{\alpha_n\}$ is a Cauchy sequence. □

Remark 6

It is clear from the definition that every strong converge sequence is a converge but the opposite is not true, as we see in example (4) that a sequence $\{\alpha_n\} = \{1 + \frac{1}{n^2}\}$ is a converge to 2 but not strongly a converge to 2, to see this

Take $1 < \beta < 2, lim_{n\to\infty} D_p(\beta, \beta, \alpha_n) = \beta \neq 2 = D_p(\beta, \beta, 2)$

Thus, $\alpha_n = \{1 + \frac{1}{n^2}\}$ is not strongly converging to 2.

Theorem 7

If $\{\alpha_n\}$ is a strongly converge, then the converge point is unique.

Proof

Let $\{\alpha_n\}$ be a strongly converge to w, z

$\{D_p(\beta, \beta, \alpha_n)\}$is real sequence converge to $D_p(\beta, \beta, w)$, $D_p(\beta, \beta, z)$ since the converge point is unique

$\therefore D_p(\beta, \beta, w) = D_p(\beta, \beta, z) \quad \forall \beta$

Take $\beta = z$ then $D_p(z, z, w) = D_p(z, z, z)$ ... 1

Take $\beta = w$ then $D_p(w, w, w) = D_p(w, w, z)$ ... 2

by defying $D_p(z, z, z) \leq D_p(\beta, \beta, z) \quad \forall \beta$

If $\beta = w$ then $D_p(z, z, z) \leq D_p(w, w, z) = D_p(w, w, w)$

$\therefore D_p(z, z, z) \leq D_p(w, w, w)$ ... 3

Also, $D_p(w, w, w) \leq D_p(\beta, \beta, w) \quad \forall \beta$

If $\beta = z$ then $D_p(w, w, w) \leq D_p(z, z, w) = D_p(z, z, z)$

$\therefore D_p(w, w, w) \leq D_p(z, z, z)$ ... 4

$b\beta$ 3, 4; $we\ have\ D_p(w, w, w) = D_p(z, z, z)$

$\therefore D_p(w, w, z) = D_p(w, w, w) = D_p(z, z, z), thus\ w = z.$ □

Example 8

Let $Y = [0, \infty)$ and $D_p(\alpha, \beta, \gamma) = ma\alpha\{\alpha, \beta, \gamma\}$ then (Y, D_p) is general partial metric space as we see in example 2, if $\alpha_n = 2^{\frac{1}{n}}$, then the sequence $\{\alpha_n\}$ is a very strongly converge to1. Indeed $D_p(\alpha_n, \alpha_m, \alpha_l) = ma\alpha\left\{2^{\frac{1}{n}}, 2^{\frac{1}{m}}, 1\right\} \to 1 = D_p(1,1,1)\ as\ n, m \to \infty$

$Also\ D_p(\beta, \gamma, \alpha_n) = ma\alpha\{\beta, \gamma, 2^{\frac{1}{n}}\} \longrightarrow ma\alpha\{\beta, \gamma, 1\}\ as\ n \to \infty \quad \forall \beta, \gamma \in Y$

Remark 9

It is clear from definition that every very strong converge sequence is a strongly converge, so that If $\{\alpha_n\}$ is very strongly converge then the converge point is unique.

Theorem 10

Let (Y, D_p) be general partial metric space, if $D_p(\alpha, \beta, \gamma) = 0$ then $\alpha = \beta = \gamma$

Proof

We have $D_p(\alpha,\alpha,\alpha) \leq D_p(\alpha,\beta,\gamma) = 0$, $D_p(\beta,\beta,\beta) \leq D_p(\alpha,\beta,\gamma) = 0$ *and*

$$D_p(\gamma,\gamma,\gamma) \leq D_p(\alpha,\beta,\gamma) = 0$$

$$\Rightarrow D_p(\alpha,\alpha,\alpha) = D_p(\beta,\beta,\beta) = D_p(\gamma,\gamma,\gamma) = D_p(\alpha,\beta,\gamma) = 0$$

Therefore $\alpha = \beta = \gamma$. □

Remark 11

If $\alpha = \beta = \gamma$, then $D_p(\alpha,\beta,\gamma)$ may not be zero.

3. Relations between D-metric, partial metric and general partial metric spaces

Theorem 12

Let (Y, D_p) be a general partial metric space, then the functions $D^g: Y^3 \to [0,\infty)$ given by $D^g(\alpha,\beta,\gamma) = D_p(\alpha,\alpha,\beta) + D_p(\alpha,\alpha,\gamma) + D_p(\beta,\beta,\alpha) + D_p(\beta,\beta,\gamma) + D_p(\gamma,\gamma,\alpha) + D_p(\gamma,\gamma,\beta) - 2D_p(\alpha,\alpha,\alpha) - 2D_p(\beta,\beta,\beta) - 2D_p(\gamma,\gamma,\gamma)$. (1.1)

Is a D-metric space on Y

Proof

1) Since $D_p(\alpha,\alpha,\beta) - D_p(\alpha,\alpha,\alpha) \geq 0, D_p(\alpha,\alpha,\gamma) - D_p(\alpha,\alpha,\alpha) \geq 0$,

$D_p(\beta,\beta,\alpha) - D_p(\beta,\beta,\beta) \geq 0$,

$D_p(\beta,\beta,\gamma) - D_p(\beta,\beta,\beta) \geq 0$,

$D_p(\gamma,\gamma,\alpha) - D_p(\gamma,\gamma,\gamma) \geq 0$ *and* $D_p(\gamma,\gamma,\beta) - D_p(\gamma,\gamma,\gamma) \geq 0$ *so* $D^g(\alpha,\beta,\gamma) \geq 0$.

2) *If* $D^g(\alpha,\beta,\gamma) = 0$

$$\textit{then } D_p(\alpha,\alpha,\beta) + D_p(\alpha,\alpha,\gamma) + D_p(\beta,\beta,\alpha) + D_p(\beta,\beta,\gamma) + D_p(\gamma,\gamma,\alpha) + D_p(\gamma,\gamma\ \beta) - 2D_p(\alpha,\alpha,\alpha) - 2D_p(\beta,\beta,\beta) - 2D_p(\gamma,\gamma\ \gamma) = 0$$

Take $D_p(\alpha,\alpha,\beta) + D_p(\alpha,\alpha,\gamma) - 2D_p(\alpha,\alpha,\alpha) = 0$

$\Rightarrow 2D_p(\alpha,\alpha,\alpha) = D_p(\alpha,\alpha,\beta) + D_p(\alpha,\alpha,\gamma) \ldots 1$

$D_p(\alpha,\alpha,\gamma) + D_p(\beta,\beta,\alpha) - 2D_p(\alpha,\alpha,\alpha) = 0$

$\Rightarrow 2D_p(\alpha,\alpha,\alpha) = D_p(\alpha,\alpha,\gamma) + D_p(\beta,\beta,\alpha) \ldots 2$

From 1&2, *we get* $D_p(\alpha,\alpha,\beta) + D_p(\alpha,\alpha,\gamma) = D_p(\alpha,\alpha,\gamma) + D_p(\beta,\beta,\alpha)$

then $D_p(\alpha,\alpha,\beta) = D_p(\beta,\beta,\alpha) \ldots 3$

Since $D_p(\alpha,\alpha,\alpha) = D_p(\alpha,\alpha,\beta) + D_p(\alpha,\beta,\beta) = D_p(\alpha,\alpha,\beta) + D_p(\alpha,\alpha,\beta)$

$$= 2D_p(\alpha,\alpha,\beta)$$

$D_p(\alpha,\alpha,\alpha) = D_p(\alpha,\alpha,\beta) \ldots 4$

Now, take $D_p(\beta,\beta,\gamma) + D_p(\beta,\beta,\alpha) - 2D_p(\beta,\beta,\beta) = 0$

$\Rightarrow 2D_p(\beta,\beta,\beta) = D_p(\beta,\beta,\gamma) + D_p(\beta,\beta,\alpha) \dots 5$

$D_p(\alpha,\alpha,\beta) + D_p(\beta,\beta,\gamma) - 2D_p(\beta,\beta,\beta) = 0$

$\Rightarrow 2D_p(\beta,\beta,\beta) = D_p(\alpha,\alpha,\beta) + D_p(\beta,\beta,\gamma) \dots 6$

$From\ 5\&6, we\ get\ D_p(\beta,\beta,\alpha) + D_p(\beta,\ \beta,\gamma) = D_p(\alpha,\alpha,\beta) + D_p(\beta,\ \beta,\gamma)$

$then\ D_p(\beta,\beta,\alpha) = D_p(\alpha,\alpha,\beta) \dots 7$

$Since\ 2D_p(\beta,\ \beta,\ \beta) = D_p(\beta,\ \beta,\ \alpha) + D_p(\alpha,\ \alpha,\ \beta) = D_p(\alpha,\ \alpha,\ \beta) + D_p(\alpha,\ \alpha,\ \beta)$

$$= 2D_p(\alpha,\alpha,\beta)$$

$\Rightarrow D_p(\beta,\beta,\beta) = D_p(\alpha,\alpha,\beta) \dots 8$

$From\ 4\&8, we\ get D_p(\alpha,\alpha,\alpha) = D_p(\alpha,\ \alpha,\ \beta) = D_p(\beta,\beta,\beta)$

$so\ by\ definition\ \ \alpha = \beta \dots 9$

$and\ take\ D_p(\gamma,\gamma,\alpha) + D_p(\gamma,\gamma,\ \beta) - 2D_p(\gamma,\gamma,\gamma) = 0$

$\Rightarrow 2D_p(\gamma,\gamma,\gamma) = D_p(\gamma,\gamma,\ \alpha) + D_p(\gamma,\gamma,\ \beta)\ if\ \ \beta = \alpha$

$Then\ 2D_p(\gamma,\gamma,\gamma) = D_p(\gamma,\gamma,\ \alpha) + D_p(\gamma,\gamma,\alpha) = 2D_p(\gamma,\gamma,\alpha)$

$\Rightarrow D_p(\gamma,\gamma,\gamma) = D_p(\gamma,\gamma,\ \alpha) \dots 10$

$and\ D_p(\alpha,\alpha,\beta) + D_p(\alpha,\gamma,\gamma) - 2D_p(\alpha,\alpha,\alpha) = 0$

$\Rightarrow 2D_p(\alpha,\alpha,\alpha) = D_p(\alpha,\alpha,\beta) + D_p(\alpha,\gamma,\gamma)\ if\ \ \beta = \alpha$

$Then, 2D_p(\alpha,\alpha,\alpha) = D_p(\alpha,\alpha,\alpha) + D_p(\alpha,\gamma,\gamma)$

$\Rightarrow D_p(\alpha,\alpha,\alpha) = D_p(\alpha,\gamma,\gamma) \dots 11$

$From\ 10\&11, we\ get D_p(\gamma,\gamma,\gamma) = D_p(\gamma,\gamma,\alpha) = D_p(\alpha,\alpha,\alpha)$

$so\ by\ definition\ \ \alpha = \gamma \dots 12$

$Then\ by\ 9\&12\ we\ get\ \alpha = \beta = \gamma.$

3) $Trivial$

4) $by\ definition\ since\ D_p(\mu,\mu,\beta) + D_p(\mu,\mu,\gamma) - 2D_p(\mu,\mu,\mu) \geq 0$

$D_p(\beta,\beta,\mu) + D_p(\beta,\beta,\mu) - 2D_p(\beta,\beta,\beta) \geq 0$

$D_p(\gamma,\gamma,\mu) + D_p(\gamma,\gamma,\mu) - 2D_p(\gamma,\gamma,\gamma) \geq 0$

$D_p(\alpha,\alpha,\mu) + D_p(\alpha,\ \alpha,\mu) - 2D_p(\alpha,\ \alpha,\ \alpha) \geq 0$,

$D_p(\mu,\mu,\alpha) + D_p(\mu,\mu,\gamma) - 2D_p(\mu,\mu,\mu) \geq 0$

$D_p(\mu,\mu,\alpha) + D_p(\mu,\mu,\ \beta) - 2D_p(\mu,\mu,\mu) \geq 0$

When combined, it is more than and equal to zero and when adding these values

$D_p(\alpha,\alpha,\beta), D_p(\alpha,\alpha,\gamma), D_p(\beta,\beta,\alpha), D_p(\beta,\beta,\gamma), D_p(\gamma,\gamma,\alpha), D_p(\gamma,\gamma,\beta), -2D_p(\alpha,\alpha,\alpha),$

$-2D_p(\beta,\beta,\beta), -2D_p(\gamma,\gamma,\gamma)$ *to two parties, we get*

$$D_p(\alpha,\alpha,\beta) + D_p(\alpha,\alpha,\gamma) + D_p(\beta,\beta,\alpha) + D_p(\beta,\beta,\gamma) + D_p(\gamma,\gamma,\alpha) + D_p(\gamma,\gamma,\beta) - 2D_p(\alpha,\alpha,\alpha) - 2D_p(\beta,\beta,\beta) - 2D_p(\gamma,\gamma,\gamma)$$

$$\leq D_p(\alpha,\alpha,\beta) + D_p(\alpha,\alpha,\gamma) + D_p(\beta,\beta,\alpha) + D_p(\beta,\beta,\gamma) + D_p(\gamma,\gamma,\alpha) + D_p\gamma,\gamma,\beta) - 2D_p(\alpha,\alpha,\alpha) - 2D_p(\beta,\beta,\beta) - 2D_p(\gamma,\gamma,\gamma) + D_p(\mu,\mu,\beta) + D_p(\mu,\mu,\gamma) + D_p(\beta,\beta,\mu) + D_p(\beta,\beta,\mu) + D_p(\gamma,\gamma,\mu) + D_p(\gamma,\gamma,\mu) + D_p(\alpha,\alpha,\mu) + D_p(\alpha,\alpha,\mu) + D_p(\mu,\mu,\alpha) + D_p(\mu,\mu,\gamma) + D_p(\mu,\mu,\alpha) + D_p(\mu,\mu,\beta)$$

$$-2D_p(\mu,\mu,\mu) - 2D_p(\beta,\beta,\beta) - 2D_p(\gamma,\gamma,\gamma) - 2D_p(\alpha,\alpha,\alpha) - 2D_p(\mu,\mu,\mu) - 2D_p(\mu,\mu,\mu)$$
$$= [D_p(\mu,\mu,\beta) + D_p(\mu,\mu,\gamma) + D_p(\beta,\beta,\mu) + D_p(\beta,\beta,\gamma) + D_p(\gamma,\gamma,\mu) + D_p(\gamma,\gamma,\beta) - 2D_p(\mu,\mu,\mu) - 2D_p(\beta,\beta,\beta) - 2D_p(\gamma,\gamma,\gamma)] + [D_p(\alpha,\alpha,\mu) + D_p(\alpha,\alpha,\gamma) + D_p(\mu,\mu,\alpha) + D_p(\mu,\mu,\gamma) + D_p(\gamma,\gamma,\alpha) + D_p(\gamma,\gamma,\mu) - 2D_p(\alpha,\alpha,\alpha) - 2D_p(\mu,\mu,\mu) - 2D_p(\gamma,\gamma,\gamma)] + [D_p(\alpha,\alpha,\beta) + D_p(\alpha,\alpha,\mu) + D_p(\beta,\beta,\alpha) + D_p(\beta,\beta,\mu) + D_p(\mu,\mu,\alpha) + D_p(\mu,\mu,\beta) - 2D_p(\alpha,\alpha,\alpha) - 2D_p(\beta,\beta,\beta) - 2D_p(\mu,\mu,\mu)]$$
$$\Rightarrow D^g(\alpha,\beta,\gamma) \leq D^g(\mu,\beta,\gamma) + D^g(\alpha,\mu,\gamma) + D^g(\alpha,\beta,\mu)$$ □

Corollary 13

Let (Y, D_p) be a general partial metric space, the function $D^g: Y^3 \longrightarrow [0,\infty)$ given by

$$D^g(\alpha,\beta,\gamma) = D_p(\alpha,\beta,\gamma) + D_p(\alpha,\alpha,\beta) + D_p(\alpha,\alpha,\gamma) + D_p(\beta,\beta,\alpha) + D_p(\beta,\beta,\gamma) + D_p(\gamma,\gamma,\alpha) + D_p(\gamma,\gamma,\beta) - 2D_p(\alpha,\alpha,\alpha) - 2D_p(\beta,\beta,\beta) - 3D_p(\gamma,\gamma,\gamma)$$

is $D-$ *metric space.* (1.2)

Proof: the prove is similar to theorem 12

Remark 14

It is clear from definition that every D-metric space (Y, D) is a general partial metric space(Y, D_p)**,** but the converse is not true. as we saw in example 2, by definition of D-metric space if $\alpha = \beta = \gamma$ then $D(\alpha,\beta,\gamma) = 0$
Suppose $\alpha = 5 = \beta = \gamma$ then $D_p(5,5,5) = ma\alpha\{5,5,5\} = 5 \neq 0$

Lemma 15

Let (Y, D_p) be a general partial metric space if $\{D_p(\alpha_m,\alpha_m,\alpha_m)\} \rightarrow \alpha$ *as* $m \rightarrow \infty$ and $\{D^g(\alpha_n,\alpha_m,\alpha_l)\}$ is a Cauchy sequence as $n, m, l \rightarrow \infty$, then $D_p(\alpha_n,\alpha_m,\alpha_l) \rightarrow \alpha$ as $n, m, l \longrightarrow \infty$ where D^g define in corollary 13.

Proof

Since $D_p(\alpha_n,\alpha_m,\alpha_l) \rightarrow \alpha$ *as* $m \longrightarrow \infty$, then from every$\epsilon > 0$ there exists $n_0 \in N$ such that

$\left|D_p(\alpha_m,\alpha_m,\alpha_m)-\alpha\right|<\frac{\epsilon}{2}\quad \forall\, m\ >\ n_0$, and $D^g(\alpha_n,\alpha_m,\alpha_l)\ <\ \frac{\varepsilon}{2}\quad \forall\, n\,,m,l\ >\ n_0$

$$\frac{\epsilon}{2}>D^g(\alpha_n,\alpha_m,\alpha_l)$$

$$= D_p(\alpha_n,\alpha_m,\alpha_l)+D_p(\alpha_n,\alpha_n,\alpha_m)+D_p(\alpha_n,\alpha_n,\alpha_l)+D_p(\alpha_m,\alpha_m,\alpha_n)+D_p(\alpha_m,\alpha_m,\alpha_l)+D_p(\alpha_l,\alpha_l,\alpha_n)+D_p(\alpha_l,\alpha_l,\alpha_m)-2D_p(\alpha_n,\alpha_n,\alpha_n)-2D_p(\alpha_m,\alpha_m,\alpha_m)-3D_p(\alpha_l,\alpha_l,\alpha_l).$$

$$\Rightarrow\ D_p(\alpha_n,\alpha_m,\alpha_l)-D_p(\alpha_m,\alpha_m,\alpha_m)\ <\ \frac{\epsilon}{2}$$

So that $\left|D_p(\alpha_n,\alpha_m,\alpha_l)-\alpha\right|=\left|D_p(\alpha_n,\alpha_m,\alpha_l)-D_p(\alpha_m,\alpha_m,\alpha_m)+D_p(\alpha_m,\alpha_m,\alpha_m)-\ \alpha\right|\le \left|D_p(\alpha_n,\alpha_m,\alpha_l)-D_p(\alpha_m,\alpha_m,\alpha_m)\right|+\left|D_p(\alpha_m,\alpha_m,\alpha_m)-\ \alpha\right|<\frac{\epsilon}{2}+\frac{\epsilon}{2}<\ \epsilon$

Hence $D_p(\alpha_n,\alpha_m,\alpha_l)\ \to\ \alpha\ \ as\, n,m,l\longrightarrow\infty$ □

Theorem 16

Let (Y,D_p) be a general partial metric space, then

i) A sequence $\{\alpha_n\}$ is a Cauchy sequence in a general partial metric space (Y,D_p) if and only if $\{\alpha_n\}$ is a Cauchy sequence in (Y,D^g).

ii) A general partial metric space (Y,D_p) is complete if (Y,D^g) is complete.

Where D^g define in corollary 13

Proof *i*

First, we must prove that each Cauchy sequence in (Y,D_p) is Cauchy in (Y,D^g).

Then, there exists $\alpha\in R$ such that, $\forall\epsilon>0$ there is $n_0\in N$ with

$\left|D_p(\alpha_n,\alpha_m,\alpha_l)-\ \alpha\right|<\frac{\epsilon}{14}$ $\forall$ n, m, l $\ge n_0$. Hence,

$|D^g(\alpha_n,\alpha_m,\alpha_l)|=\left|D_p(\alpha_n,\alpha_m,\alpha_l)+D_p(\alpha_n,\alpha_n,\alpha_m)+D_p(\alpha_n,\alpha_n,\alpha_l)+D_p(\alpha_m,\alpha_m,\alpha_n)+D_p(\alpha_m,\alpha_m,\alpha_l)+D_p(\alpha_l,\alpha_l,\alpha_n)+D_p(\alpha_l,\alpha_l,\alpha_m)-2D_p(\alpha_n,\alpha_n,\alpha_n)-2D_p(\alpha_m,\alpha_m,\alpha_m)-3D_p(\alpha_l,\alpha_l,\alpha_l)\right|\le\left|D_p(\alpha_n,\alpha_m,\alpha_l)-\ \alpha\right|+\left|D_p(\alpha_n,\alpha_n,\alpha_m)-\ \alpha\right|+\left|D_p(\alpha_n,\alpha_n,\alpha_l)-\ \alpha\right|+\left|D_p(\alpha_m,\alpha_m,\alpha_n)-\ \alpha\right|+\left|D_p(\alpha_m,\alpha_m,\alpha_l)-\ \alpha\right|+\left|D_p(\alpha_l,\alpha_l,\alpha_n)-\ \alpha\right|+\left|D_p(\alpha_l,\alpha_l,\alpha_m)-\alpha\right|-2\left|D_p(\alpha_n,\alpha_n,\alpha_n)-\ \alpha\ \right|-2\left|D_p(\alpha_m,\alpha_m,\alpha_m)-\ \alpha\right|-3\left|D_p(\alpha_l,\alpha_l,\alpha_l)-\ \alpha\right|\frac{\varepsilon}{14}<\varepsilon$ $\forall$ n, m, l $\ge n_0$.

Hence, $\{\alpha_n\}$ is a Cauchy sequence in (Y,D^g).

Conversely, now we must prove $\{\alpha_n\}$ is Cauchy sequence in (Y,D_p)

Since $\{\alpha_n\}$ is a Cauchy sequence in (Y,D^g) so $\forall\epsilon>0,\exists\, n_0\in\ N$ such that

$$D^g(\alpha_n,\alpha_m,\alpha_l)<\frac{\epsilon}{2}\ \ \forall n\,,m,l>\ n_0$$

$$\frac{\epsilon}{2}\ >D^g(\alpha_n,\alpha_m,\alpha_l)$$

$$= D_p(\alpha_n,\alpha_m,\alpha_l)+D_p(\alpha_n,\alpha_n,\alpha_m)+D_p(\alpha_n,\alpha_n,\alpha_l)+D_p(\alpha_m,\alpha_m,\alpha_n)+D_p(\alpha_m,\alpha_m,\alpha_l)+D_p(\alpha_l,\alpha_l,\alpha_n)+D_p(\alpha_l,\alpha_l,\alpha_m)-2D_p(\alpha_n,\alpha_n,\alpha_n)-2D_p(\alpha_m,\alpha_m,\alpha_m)-3D_p(\alpha_l,\alpha_l,\alpha_l)$$

$\Rightarrow\ D_p(\alpha_n,\alpha_m,\alpha_l) - D_p(\alpha_n,\alpha_n,\alpha_n)\ \le\ D^g(\alpha_n,\alpha_m,\alpha_l)\ < \frac{\epsilon}{2}$

By compensation $D_p(\alpha_n,\alpha_n,\alpha_n)$ to two parties, we have

$D_p(\alpha_n,\alpha_m,\alpha_l)\ \le\ D^g(\alpha_n,\alpha_m,\alpha_l)\ +\ D_p(\alpha_n,\alpha_n,\alpha_n)\ < \frac{\epsilon}{2} + D_p(\alpha_n,\alpha_n,\alpha_n)$

$And\ since\ D_p(\alpha_m,\alpha_m,\alpha_m)\ \le\ D_p(\alpha_n,\alpha_m,\alpha_l)\ so$

$$D_p(\alpha_m,\alpha_m,\alpha_m)\ \le\ D_p(\alpha_n,\alpha_m,\alpha_l)\ \le\ D^g(\alpha_n,\alpha_m,\alpha_l)\ +\ D_p(\alpha_n,\alpha_n,\alpha_n) < \frac{\epsilon}{2} + D_p(\alpha_n,\alpha_n,\alpha_n)$$

$\Rightarrow\ D_p(\alpha_m,\alpha_m,\alpha_m)\ \le \frac{\epsilon}{2} + D_p(\alpha_n,\alpha_n,\alpha_n)\quad \forall\, n\,,m\ >\ n_0$

$Let\ \alpha_n =\ D_p(\alpha_n,\alpha_n,\alpha_n)\ \in\ R\ such\ that\ |\ \alpha\ _m -\ \alpha_n| < \frac{\epsilon}{2}$

$\therefore$ { α_m } is a Cauchy sequence, $\therefore \{\ \alpha_n\}\ \rightarrow\ \ \alpha\ , \therefore D_p(\alpha_m,\alpha_m,\alpha_m) \rightarrow\ \ \alpha\ \in\ R$

Then by lemma 15, $D_p(\alpha_m,\alpha_m,\alpha_m)$ is Cauchy sequence in $(\ Y, D_p)$. □

ii :

If $\{\alpha_n\}$ is a Cauchy sequence in (Y, D_p), then it is a Cauchy sequence in (Y, D^g) and since D-metric (Y, D^g) is complete then there exists $\alpha \in\ \ Y$ such that

$\lim_{n\,,m\rightarrow\infty} D^g(\alpha_n,\alpha_m,\alpha)\ =\ 0$, hence

$$lim_{n\,,m\to\infty}[D_p(\alpha_n,\alpha_m,\alpha)\ +\ D_p(\alpha_n,\alpha_n,\alpha_m)\ +\ D_p(\alpha_n,\alpha_n,\alpha)\ +\ D_p(\alpha_m,\alpha_m,\alpha_n)\ +\ D_p(\alpha_m,\alpha_m,\alpha)\ +\ D_p(\alpha\,,\alpha\,,\alpha_n)\ +\ D_p(\alpha,\alpha\,,\alpha_m) - 2D_p(\alpha_n,\alpha_n,\alpha_n) - 2D_p(\alpha_m,\alpha_m,\alpha_m) - 3D_p(\alpha\,,\alpha\,,\alpha)]\ =\ 0$$

There for $lim_{n\ m\rightarrow\infty}[D_p(\alpha_n,\alpha_m,\alpha) - D_p(\alpha,\alpha,\alpha)]\ =\ 0$

$\Rightarrow lim_{n\ \ m\to\infty}\, D_p(\alpha_n,\alpha_m,\alpha)\ = D_p(\alpha\,,\alpha,\alpha)$ hence (Y, D_p) is converge

Thus (Y, D_p) is complete.

Corollary 17

we can get from the proof of theorem 16, (ii), $lim_{n,m\rightarrow\infty}[D_p(\alpha_n,\alpha_m,\alpha) - D_p(\alpha,\alpha,\alpha)] = lim_{n\,,m\rightarrow\infty)}[D_p(\alpha_n,\alpha_m,\alpha) - D_p(\alpha_m,\alpha_m,\alpha_m)]$ $=$ $lim_{n\,,m\rightarrow\infty}[D_p(\alpha_n,\alpha_m\alpha) - D_p(\alpha_n,\alpha_n,\alpha_n)]\ =\ 0,\ such\ that$ $lim_{n\,,m\rightarrow\infty}\, D_p(\alpha_n,\alpha_m,\alpha)$ $=$ $lim_{m\rightarrow\infty)}\ D_p(\alpha_m,\alpha_m,\alpha_m) = lim_{n\to\infty}\, D_p(\alpha_n,\alpha_n,\alpha_n) =\ D_p(\alpha\,,\alpha\,,\alpha).$

Propositions 18

If $\{\alpha_n\}$ is Cauchy sequence in (Y, D^g), then $lim_{n\,,m\rightarrow\infty}\, D_p(\alpha_n,\alpha_n,\alpha_m)$ $= lim_{n\to\infty}\, D_p(\alpha_n,\alpha_n,\alpha_n)$.

Proof

Since $\{\alpha_n\}$ is Cauchy sequence in (Y, D^g) then $lim_{n\,,m,l\to\infty}\, D^g(\alpha_n,\alpha_m,\alpha_l)\ =\ 0$

And $D^g(\alpha_n, \alpha_m, \alpha_l) =$
$D_p(\alpha_n, \alpha_n, \alpha_m) + D_p(\alpha_n, \alpha_n, \alpha_l) + D_p(\alpha_m, \alpha_m, \alpha_n) + D_p(\alpha_m, \alpha_m, \alpha_l) + D_p(\alpha_l, \alpha_l, \alpha_n) + D_p(\alpha_l, \alpha_l, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_n) - 2D_p(\alpha_m, \alpha_m, \alpha_m) - 2D_p(\alpha_l, \alpha_l, \alpha_l)$.

And $D_p(\alpha_n, \alpha_n, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_n) \leq D^g(\alpha_n, \alpha_m, \alpha_l)$ then

$lim_{n,m \to \infty} D_p(\alpha_n, \alpha_n, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_n) \to 0$

Similarly $lim_{n,m \to \infty} D_p(\alpha_n, \alpha_n, \alpha_m) - D_p(\alpha_m, \alpha_m, \alpha_m) \to 0$

Since $D_p(\alpha_m, \alpha_m, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_n) = D_p(\alpha_m, \alpha_m, \alpha_m) + D_p(\alpha_n, \alpha_n, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_n)$ $then$ $lim_{n,m\to\infty}[D_p(\alpha_m, \alpha_m, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_n)] = lim_{n,m\to\infty}[D_p(\alpha_m, \alpha_m, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_m)] + lim_{n,m\to\infty}[D_p(\alpha_n, \alpha_n, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_n)] \longrightarrow 0$

So that $lim_{n,m\to\infty}[D_p(\alpha_m, \alpha_m, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_n)] \longrightarrow 0$

Let $\alpha_n = D_p(\alpha_n, \alpha_n, \alpha_n)$

$\therefore |\alpha_m - \alpha_n| \to 0$ as $n, m \longrightarrow \infty$

Hence $\{\alpha_n\}$ is a Cauchy sequence in R, therefor $\{D_p(\alpha_n, \alpha_n, \alpha_n)\}$ converge to α.

Also , $lim_{n\to\infty} D_p(\alpha_n, \alpha_n, \alpha_m) =$
$lim_{n,m\to\infty}[D_p(\alpha_n, \alpha_n, \alpha_m) + D_p(\alpha_n, \alpha_n, \alpha_n) - D_p(\alpha_n, \alpha_n, \alpha_n)]$
Then $[D_p(\alpha_n, \alpha_n, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_n)] \longrightarrow 0$ so $lim_{n,m\to\infty} D_p(\alpha_n, \alpha_n, \alpha_m) = \alpha$

Thus $lim_{n,m\to\infty} D_p(\alpha_n, \alpha_n, \alpha_m) = lim_{n\to\infty} D_p(\alpha_n, \alpha_n, \alpha_n)$. □

Theorem 19

If (Y, p) be partial metric space then

$$D_p(\alpha, \beta, \gamma) = p(\alpha, \beta) + p(\alpha, \gamma) + p(\beta, \gamma) - p(\alpha, \alpha) - P(\beta, \beta) - p(\gamma, \gamma) \quad (1.3)$$

is general partial metric space.

1)$Since$ $p(\alpha, \beta) - p(\alpha, \alpha) \geq 0$, $p(\alpha, \gamma) - p(\gamma, \gamma) \geq 0$, $p(\beta, \gamma) - p(\beta, \beta) \geq 0$

then $D_p(\alpha, \beta, \gamma) \geq 0$

2)let $D_p(\alpha, \beta, \gamma) = D_p(\alpha, \alpha, \alpha) = D_p(\beta, \beta, \beta) = D_p(\gamma, \gamma, \gamma)$

$since$ $D_p(\alpha, \alpha, \alpha) = D_p(\beta, \beta, \beta) = D_p(\gamma, \gamma, \gamma) = 0$

$\Rightarrow D_p(\alpha, \beta, \gamma) = 0$

$\Rightarrow p(\alpha, \beta) + p(\beta, \gamma) + p(\alpha, \gamma) - p(\alpha, \alpha) - p(\beta, \beta) - p(\gamma, \gamma) = 0$

$\Rightarrow p(\alpha, \beta) - p(\alpha, \alpha) = 0 \Rightarrow p(\alpha, \beta) = p(\alpha, \alpha)$ ... 1,

$\Rightarrow p(\alpha, \gamma) - p(\gamma, \gamma) = 0 \Rightarrow p(\alpha, \gamma) = p(\gamma, \gamma)$ ... 2,

$and\ p(\beta,\gamma) - p(\beta,\beta) = 0 \Rightarrow p(\beta,\gamma) = p(\beta,\beta) \dots 3$

$From\ 1\ p(\alpha,\alpha) = p(\alpha,\beta)\ and\ by\ definition\ p(\alpha,\alpha) = p(\alpha,\beta) \le p(\alpha,\gamma) + p(\gamma,\beta) - p(\gamma,\gamma)$

$Since\ p(\alpha,\gamma) = p(\gamma,\gamma)\ \&\ p(\beta,\gamma) = p(\beta,\beta)\ we\ get\ p(\alpha,\alpha) = p(\beta,\beta)$

$From\ 2\ p(\beta,\beta) = p(\beta,\gamma)\ and\ by\ definition\ p(\beta,\beta) = p(\beta,\gamma) \le p(\beta,\alpha) + p(\alpha,\gamma) - p(\alpha,\alpha)$

$b\beta\ 1\&2\ we\ get\ p(\beta,\beta) = (\gamma,\gamma)$

$From\ 3\ p(\gamma,\gamma) = p(\alpha,\gamma) and\ by\ definition\ p(\gamma,\gamma) = p(\alpha,\gamma) \le p(\alpha,\beta) + p(\beta,\gamma) - p(\beta,\beta)$

$by\ 1\&3\ we\ get\ p(\gamma,\gamma) = p(\alpha,\alpha)$

$Hence\ p(\alpha,\alpha) = p(\beta,\beta) = p(\gamma,\gamma)$

$Thus\ p(\alpha,\alpha) = p(\alpha,\beta) = p(\beta,\beta)\ by\ definition\ \alpha = \beta, and\ p(\beta,\beta) = p(\beta,\gamma)$
$= p(\gamma,\gamma)\ by\ definition\ \beta = \gamma\ then\ we\ get\ \alpha = \beta = \gamma.$

3) $Trivial$

4) $since\ p(\mu,\gamma) - p(\gamma,\gamma) \ge 0, p(\mu,\beta) - p(\mu,\mu) \ge 0, p(\alpha,\mu) - p(\alpha,\alpha) \ge 0,$
$p(\mu,\gamma) - p(\mu,\mu) \ge 0, p(\beta,\mu) - p(\beta,\beta) \ge 0,$
$p(\alpha,\mu) - p(\mu,\mu) \ge 0$

When combined, it is more than and equal to zero and when added these values

$p(\alpha,\beta), p(\beta,\gamma), p(\alpha,\gamma), -p(\alpha,\alpha), -p(\beta,\beta), -p(\gamma,\gamma)\ to\ both\ said, we\ get$

$$p(\alpha,\beta) + p(\beta,\gamma) + p(\alpha,\gamma) - p(\alpha,\alpha) - p(\beta,\beta) - p(\gamma,\gamma) \le p(\alpha,\beta) + p(\beta,\gamma) + p(\alpha,\gamma) - p(\alpha,\alpha) - p(\beta,\beta) - p(\gamma,\gamma) + p(\mu,\gamma) - p(\gamma,\gamma) + p(\alpha,\mu) - p(\alpha,\alpha) + p(\beta,\mu) - p(\beta,\beta) + p(\mu,\beta) - p(\mu,\mu) + p(\mu,\gamma) - p(\mu,\mu) + p(\alpha,\mu) - p(\mu,\mu).$$

$$\Rightarrow D_p(\alpha,\beta,\gamma) \le [p(\mu,\beta) + p(\mu,\gamma) + p(\gamma,\beta) - p(\mu,\mu) - p(\beta,\beta) - p(\gamma,\gamma)] + [p(\alpha,\mu) + p(\alpha,\gamma) + p(\mu,\gamma) - p(\alpha,\alpha) - p(\mu,\mu) - p(\gamma,\gamma)] + [p(\alpha,\beta) + p(\alpha,\mu) + p(\beta,\mu) - p(\alpha,\alpha) - p(\beta,\beta) - p(\mu,\mu)]$$

$$D_p(\alpha,\beta,\gamma) \le D_p(\mu,\beta,\gamma) + D_p(\alpha,\mu,\gamma) + D_p(\alpha,\beta,\mu) - D_p(\mu,\mu,\mu) \qquad \square$$

Proposition 20

Let (Y, D_p) be a general partial metric space and

$$D_p(\alpha,\beta,\beta) \le D_p(\alpha,\gamma,\gamma) + D_p(\gamma,\beta,\beta) - D_p(\gamma,\gamma,\gamma) \qquad (1.4)$$

holds then the function $p: Y^2 \to [0,\infty)$ which is defied by $p(\alpha,\beta) = D_p(\alpha,\beta,\beta)$ is a partial metric on Y .

Proof

1) $since\ p(\alpha,\beta) = p(\alpha,\alpha) = p(\beta,\beta) \Leftrightarrow D_p(\alpha,\beta,\beta) = D_p(\alpha,\alpha,\alpha) = D_p(\beta,\beta,\beta) \Leftrightarrow \alpha = \beta.$

2) $since\ D_p(\alpha,\alpha,\alpha) \le D_p(\alpha,\beta,\beta)\ then\ p(\alpha,\alpha) \le p(\alpha,\beta)\ \ \forall\ \alpha,\beta \in \alpha$

3) Trivial

4) $p(\alpha,\beta) = D_p(\alpha,\beta,\beta) \le D_p(\alpha,\gamma,\gamma) + D_p(\gamma,\beta,\beta) - D_p(\gamma,\gamma,\gamma)\ from\ (1.3)$

$$= p(\alpha,\gamma) + p(\gamma,\beta) - p(\gamma,\gamma).$$ □

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