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# Weakly Approximaitly Quasi-Prime Submodules And Related Concepts

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**Abstract**

Let R be commutative Ring , and let T be unitary left R − module .In this paper ,WAPP-quasi prime submodules are introduced as new generalization of Weakly quasi prime submodules , where proper submodule C of an R-module T is called WAPP –quasi prime submodule of T, if whenever 0≠rst϶C, for r, s ϵR , t ϵT, implies that either r tϵ C +soc(T) or s tϵC +soc(T) .Many examples of characterizations and basic properties are given . Furthermore several characterizations of WAPP-quasi prime submodules in the class of multiplication modules are established.

**Keywords**: Weakly quasi prime submodules ,WAPP-quasi prime submodules , Socle of modules , Z-Regular modules , Projective modules .

## 1. Introduction

Throughout this paper , all rings are commutative with identity , and all modules are left unitary R-modules . Weakly quasi prim submodules was first introduced and studied in 2013 by [1] as a generalization of a weakly prime submodule , where proper submodule C of R-module T was called weakly prime submodule of C , if whenever 0≠atϵC ,for a ϵR , t ϵT, implies that either tϵC or aT⊆ C [2] , and a proper submodule C of R-module T is called weakly quasi prime submodule of T , if whenever 0≠abtϵC , for a ,b ϵR t ϵT, implies that either at ϵC or bt ϵC . Recently many generalization of weakly quasi prime submodules were introduced see [3, 4, 5] . In this research we introduced another generalization of weakly quasi prime submodule , where proper submodule C of R-module T is called WAPP-quasi prime submodule of T , if whenever 0≠abtϵC for a, b ϵR , tϵT implies that either atϵC+soc(T) or bt ϵC +soc(T). Soc(T) is the socle of a module T, defined by the intersection of all essential submodule of T [6] , where a nonzero submodule A of an R-module T is called essential if A∩ B ≠ (0)for each nonzero submodule B of T [6] . Recall that R-module T is multiplication if every submodule C of T is of the form IT for some ideal I of R , in particular C=[C:$_R$ T] T [7]. Let A and B be a submodule of multiplication module T with A=IM and B=JT for some ideals I ,J of R , then AB=IJT=IB . In particular AT=ITT=IT=A. Also for any t ϵT , At=A$< t >$= It [8]. Recall that an R-module T is faithful , if ann(T)= (0) [7] .A R-module T is a projective if for any epimorphism f from R-module X into X' and for any homomorphism g from Tin to X' there exists a homomorphism h from T in to X such that f o h=g [7].Recall that an R-module T is a Z-regular , if for each tϵT there exists fϵ$T^*$=Hom(T,R) such that t=f(t)t [10]

## 2.Basic Properties of WAPP-Quasi Prime Submodule

In this section, we introduced the definition of WAPP-quasi prime submodules and established some of its basic properties , characterization and examples.

**Definition(1)**

A proper submodule C of an R − module T is called Weakly approximaitly quasi prime submodule of T (for short WAPP-quasi prime submodule) , if whenever 0≠abt ϵC , for a, bϵR, tϵT, implies that either atϵ C+Soc(T) or btϵC+Soc(T).

And an ideal J of ring R is called WAPP-quasi prime ideal of R if J is WAPP- quasi prime submodule of R-module T.

**Examples and Remarks(2)**

1. The submodule C=$<\overline{12}>$ of the Z-module $Z_{24}$ is a WAPP-quasi prime submodule of $Z_{24}$, , since Soc($Z_{24}$)=$<\overline{4}>$ , and for 0≠abtϵ$<\overline{12}>$ ={$\overline{0}$ , $\overline{12}$} for a,b ϵZ ,tϵ$Z_{24}$, implies that either atϵ$<\overline{12}>$+Soc($Z_{24}$) or bt ϵ$<\overline{12}>$+Soc($Z_{24}$) . That is either atϵ$<\overline{12}>$+Soc($Z_{24}$)=$<\overline{4}>$ or btϵ$<12>$+Soc($Z_{24}$)=$<\overline{4}>$ thus 0≠2.3. $\overline{2}$ϵ$<\overline{12}>$ for 2,3,ϵZ, $\overline{2}$ϵ$Z_{24}$ implies that 2. $\overline{2}$=$\overline{4}$ϵ $<\overline{12}>$+$<\overline{4}>$=$<\overline{4}>$={$\overline{0}$, $\overline{4}$, $\overline{8}$, $\overline{12}$, $\overline{16}$, $\overline{20}$} .
2. The submodule 12Z of the Z-module Z is not WAPP-quasi prime submodule , since Soc(Z)=(0) and whenever 0≠3.4.1ϵ 12Z , for 3,4,1ϵZ , implies that 3.1∉12Z+Soc(Z) and 4.1∉12Z+Soc(Z)
3. It is clear that every weakly quasi prime submodule of an R-module T is WAPP-quasi prime but not conversely .

The following example explains that :

Consider the Z-module $Z_{24}$, and the submodule C=$<\overline{6}>$ ={$\overline{0}$, $\overline{6}$, $\overline{12}$, $\overline{18}$} , C is not weakly quasi prime submodule of $Z_{24}$ since 2.3.$\overline{1}$ϵC =$<\overline{6}>$ , for 2,3 ϵZ , $\overline{1}$ϵ$Z_{24}$ , implies that 2. $\overline{1}$=$\overline{2}$≠$<\overline{6}>$ and 3. $\overline{1}$=$\overline{3}$∉ $<\overline{6}>$ . Bat C is a WAPP-quasi prime submodule of $Z_{24}$ , since Soc($Z_{24}$)=$<\overline{4}>$ , and whenever 0≠ abt ∈ C = $<\overline{6}>$ = {$\overline{0}$, $\overline{6}$, $\overline{12}$, $\overline{18}$} for a, b ∈ Z , t ∈$Z_{24}$ implies that either at∈ C + Soc($Z_{24}$)= $<\overline{6}>$ +$<\overline{4}>$=$<\overline{2}>$ or bt∈ C + Soc($Z_{24}$)= $<\overline{6}>$ +$<\overline{4}>$=$<\overline{2}>$ .

That is 0≠ 2.3. $\overline{1}$ ∈ C, for 2,3 ∈ Z , $\overline{1}$ ∈$Z_{24}$ , implies that 2.$\overline{1}$ ∈ C + Soc($Z_{24}$)=$<\overline{2}>$.

4. **I**t is clear that ever weakly prime submodule of an R-module T is a WAAP-quasi prime but not conversely .

The following example explains that :

Consider the Z-module $Z_{24}$ and the submodule C=$< \overline{12} >=\{\overline{0}, \overline{12}\}$. From (1), C is WAPP-quasi prime submodule of $Z_{24}$. But C is not weakly prime submodule of $Z_{24}$ . Since if $0\neq 3.\overline{4} \in C$ , for $3\epsilon Z$ , $\overline{4}\epsilon Z_{24}$ , but $\overline{4} \notin C$ and $3\notin[C:Z_{24}]=6Z$

5. The residual of WAPP-quasi prime submodule C of an R-module T needs not to be WAPP-quasi prime ideal of R .

The following example explains that :

We have seen in(1) that the submodule C=$< \overline{12} >$ of the $Z-$ module $Z_{24}$ is a WAPP-quasi prime but $[C:_Z Z_{24}]=[< \overline{12} >:_Z Z_{24}]=12Z$ is not WAPP-quasi prime ideal by (2).

6. The submodules PZ of a Z-module Z is a WAPP-quasi prime if and only if P is prime number
7. The intersection of two WAPP-quasi prime submodule of R-module, T need, not to be WAPP-quasi prime submodule of T for example:

The submodule 2Z and 5Z of the Z-module Z are WAPP-quasi prime submodule by (6).

But $2Z\cap 5Z=10Z$ is not WAPP-quasi prime submodule of the $Z-$ module Z , since $0\neq 2.5.1\epsilon 10Z$, for $2,5,1\epsilon Z$ but $2.1=2\notin 10Z+Soc(Z)$ and $5.1=5\notin 10Z+Soc(Z)$

The following proposition are characterizations of WAPP-quasi prime submodules .

**Proposition(3)**

Let $T$ be an $R-$ modul and C be proper submodul of T , then C is WAPP $-$ quasi prime sub modul of T if and only if , whenever $0\neq rsB\subseteq C$ , for r, s$\epsilon$ R, B is submodul of T , implies that either r $B\subseteq C$ +Soc(T) or s $B\subseteq C$ +Soc(T).

**Proof:**

($\Rightarrow$) Assum that C is AWPP-quasi prime submodule of T and $0\neq rsB\subseteq C$ . For r, s $\epsilon$R , B is a submodule of T , with r $B\not\subset$ C+ Soc(T) and s $B\not\subset$ C +Soc(T), that is there exists a nonzero elements $b_1$ ,$b_2$ $\epsilon$B such that $rb_1 \notin C$ +Soc(T) and s $b_2 \notin$ C +Soc(T).Now $0\neq rsb_1 \epsilon C$ , and C is WAPP-quasi prime submodule and r $b_1 \notin C$ +Soc(T) , implies that s $b_1 \epsilon C$ +Soc(T). Also $0\neq rsb_2 \epsilon C$ , and C is a WAPP-quasi prime submodule of T, and $sb_2 \notin C$ +Soc(T) .,implies that $rb_2\epsilon C$+Soc(T). Again since $0\neq rs(b_1+b_2)\epsilon C$ and C is WAPP-quasi prime submodule of T , implies that either $r(b_1+b_2)\epsilon C$ +Soc(T) or $s(b_1+b_2)\epsilon C$ +Soc(T) . If $r(b_1+b_2)\epsilon C$ +Soc(T) ,that is $rb_1+rb_2\epsilon C$+Soc(T) ,and since $rb_2\epsilon C$+Soc(T), it follows that $rb_1\epsilon C$+Soc(T) which is contradiction . If $s(b_1+b_2)\epsilon C$ +Soc(T) , that is $sb_1+sb_2\epsilon C$+Soc(T) and since $sb_1\epsilon C$+Soc(T), it follows that $sb_2\epsilon C$+Soc(T) which is contradiction. Hence r $B\subseteq C$ +Soc(T) or s $B \subseteq C$ +Soc(T).

**($\Leftarrow$)**

Let $0\neq rst\epsilon C$ , for r, s $\epsilon$R , t $\epsilon$T , it follows that $0\neq rs<t>\subseteq C$ , so by hypothesis either $r<t>\subseteq C$+ Soc(T) or $s<t>\subseteq C$+ Soc(T) . That is either r t$\epsilon$ C+ Soc(T) or s t$\epsilon$ C+ Soc(T) .Hence C is WAPP $-$ quasi prime submodul of T.

**Proposition(4)**

Let $T$ be R − module and C be proper submodule of T . Then C is WAPP − quasi prime submodul of T if and only if whenever 0≠IJB ⊆C , for I,J are ideals of R and B is a submodule of T , implies that either IB⊆C +Soc(T) or JB⊆C +Soc(T).

**Proof:**

(⇒)Assume that 0≠IJB⊆C . For I,J are ideal of R , B is a submodule of T , with IB⊄C+Soc(T) and J B⊄C +Soc(T), so there exists a nonzero elements $b_1$ ,$b_2$ ϵ B and a nonzero elements r ϵI , s ϵJ such that $rb_1$ ∉C +Soc(T) and $sb_2$∉ C +Soc(T).Now 0≠$rsb_1$ ϵC , and C is a WAPP-quasi prime submodule and $rb_1$∉ C+Soc(T) , implies that $sb_1$ϵ C+Soc(T). Also 0≠rsb2ϵC , and C is a WAPP-quasi prime submodule of T, and $sb_2$ ∉C+Soc(T) .,implies that $rb_2$ϵC+Soc(T). Again 0≠rs($b_1$+$b_2$)ϵC and C is WAPP-quasi prime submodule of T , implies that either r($b_1$+$b_2$)ϵC +Soc(T) or s($b_1$+$b_2$)ϵC +Soc(T) . If r($b_1$+$b_2$)ϵC +Soc(T) ,that is $rb_1$+$rb_2$ϵC+Soc(T) ,and $rb_2$ϵC+Soc(T), implies that $rb_1$ϵC+Soc(T) contradiction . If s($b_1$+$b_2$)ϵC +Soc(T) , that is $sb_1$+$sb_2$ϵC+Soc(T) and $sb_1$ϵC+Soc(T), implies that $sb_2$ϵC+Soc(T) which is contradiction. Hence IB⊆C+Soc(T) or JB⊆C+Soc(T).

**(**⇐)

Suppose that 0≠rstϵ C , for r, s ϵR , tϵT , that is 0≠<r><s><t>⊆C , so by our assumption either (r)(t)⊆ C+ Soc(T) or (r)(t) ⊆ C+ Soc(T) . That is either rt∈ C + Soc(T) or st∈ C + Soc(T) .Hence C is WAPP-quasi prime submodule of T.

As a direct consequence of the above propositions, we get the following corollaries.

**Corollary(5)**

Let $T$ be R − module and C be proper submodule of T . Then C is WAPP − quasi prime submodule of T iff whenever 0≠rIt ⊆C , for r ϵR, I is an ideals of R and t∈ T, implies that either r t ϵC +Soc(T) or I t ⊆C +Soc(T).

**Corollary(6)**

Let $T$ be R − module and C be proper submodule of T . Then C is WAPP − quasi prime submodule of T iff whenever 0≠IJt ⊆C , for J, , I is an ideals of R and tϵ T , implies that either J t⊆C +Soc(T) or It⊆C +Soc(T).

**Corollary(7)**

Let $T$ be R − module and C be proper submodule of T . Then C is WAPP − quasi prime submodul of T if and only if, for each r ϵR and every ideal I of R and every submodule B of T, with 0≠rIB ⊆C , implies that either rB⊆C +Soc(T) or IB⊆C +Soc(T).

**Proposition(8)**

Let $T$ be R − module and C be proper submodule of T . Then C is WAPP − quasi prime submodul of T if and only if for each r,s ϵR ,[C:rs]⊆[0:$_T$ rs]∪[C+ Soc(T)::$_T$ r]∪[C+Soc(T):$_T$ s] .

**Proof:**

**(⇒)**Let tϵ[C:$_T$ rs] , implies that rstϵC .If rst=0 , then tϵ[0:$_T$ rs], and hence tϵ[0:$_T$ rs]∪[C+ Soc(T)::$_T$ r]∪[C+ Soc(T)::$_T$ s] .Suppose that 0≠rstϵC and since C is WAPP-quasi prime submodule of T , it follows that either rt ϵC +Soc(T) or st ϵC +Soc(T) , implies that either tϵ[C+Soc(T):$_T$ r] or tϵ[C+Soc(T):$_T$ s] .That is tϵ[0:$_T$ rs]∪[C+ Soc(T)::$_T$ r]∪[C+ Soc(T)::$_T$ s] .Hence, [C:$_T$ rs]⊆[0:$_T$ rs]∪[C+ Soc(T)::$_T$ r]∪[C+ Soc(T):$_T$ s] .

**(⇐)**Assume that 0≠rstϵC , for r,sϵR , tϵT , implies that tϵ,[C:$_T$ rs]⊆[ [0:$_T$ rs]∪ [C+ Soc(T):+:$_T$ r]∪[C+ Soc(T):$_T$ s] . But 0≠rst , then t ∉[0:$_T$ rs] , hence tϵ [C+ Soc(T):$_T$ r]∪[C+ Soc(T):$_T$ s] , it follows that rtϵC+Soc(T) or stϵC+Soc(T).Hence, C is WAPP-quasi prime submodule of T.

**Proposition(9)**

Let $T$ be R − modul and C be proper submodul of T . Then C is WAPP − quase priem submodul of T iff for every rϵR , and tϵT with rt∉C +Soc(T) ,[C:$_R$ rt]⊆[0:$_R$ rt]∪[C+Soc(T):$_R$ t]

**Proof:**

(⇒)Suppose that C is WAPP-quase ,and let sϵ[C:$_R$ rt], implies that rstϵC .If rst=0 then sϵ[0:$_R$ r], hence sϵ[0:$_R$ rt]∪[C+Soc(T):$_R$ t]. If 0≠rstϵC and C is a WAPP-quasi prime submodule of T and rt∉C +Soc(T) ,then stϵC+ Soc(T) that is sϵ[C+ Soc(T):$_R$ t] . Hence sϵ[0:$_R$ rt]∪[C+ Soc(T):$_R$ t] . Thus ,[C:$_R$ rt]⊆[0:$_R$ rt]∪[C+ Soc(T):$_R$ t].

As a direct consequence of proposition (9) and proposition (3),we get the following corollary:

**Corollary(10)**

Let $T$ be R − modul and C be proper submodul of T . Then C is WAPP − quase prim submodul of T iff for every rϵR , and any submodule B of T with rB⊄C +Soc(T) ,[C:$_R$ rB]⊆[0:$_R$ rB]∪[C+Soc(T):$_R$ B]

As a direct consequence of proposition (9) and proposition (4)we get the following corollary.

**Corollary(11)**

Let $T$ be R − module and C be proper submodule of T . Then C is WAPP − quasi prime submodule of T iff for every ideal I of R , and every submodule B of T with IB⊄C +Soc(T) , ,[C:$_R$ IB]⊆[0:$_R$ IB]∪[C+Soc(T):$_R$ B].

**Proposition(12)**

Let $T$ be R − module and C be proper submodule of T . Then for every s, r∈R , and t∈T ,$[C:_R rst]\subseteq[0:_R rst]\cup[C+Soc(T):_R r t]\cup[C+Soc(T):_R s t]$.

**Proof:**

Suppose that $e\in[C:_R rst]$ ,implies that $rs(et)\in C$ .If $rs(et)=0$, implies that $e\in[0:_R rst]$ and hence $e\in[0:_R rst]\cup[C+Soc(T):_R r t]\cup[C+Soc(T):_R s t]$.If $rs(et)\neq0$ ,and C is a WAPP-quasi prime submodule of T , then either $r(et)\in C+Soc(T)$ or $s(et)\in C+Soc(T)$. That is either $e\in[C+Soc(T):_R rt]$ or $e\in[C+Soc(T):_R st]$ thus $e\in[0:_R rst]\cup[C+Soc(T):_R r t]\cup[C+Soc(T):_R s t]$.Therefore ,$[C:_R rst]\subseteq[0:_R rst]\cup[C+Soc(T):_R r t]\cup[C+Soc(T):_R s t]$.

The following are characterizations in the multiplication module **.**

**Proposition(13)**

Let T be multiplcation R_module and C be proper submodule of T . Then C is a WAPP − quasi prime submodule of T iff $0\neq K_1K_2t\subseteq C$ , for some submodules $K_1$ ,$K_2$ of T , and t∈T implies that either $K_1t\subseteq C+Soc(T)$ or $K_2t\subseteq C+Soc(T)$.

**Proof:**

(⇒) Suppos that C is WAPP − quasi prime submodul of T , and $0\neq K_1K_2t\subseteq C$ for some submodules $K_1$ ,$K_2$ of T , and t∈T . Since T is a multiplication , then $K_1=IT$ and $K_2=JT$ for some ideals I,J of R .Thus $0\neq K_1K_2t=IJt\subseteq C$. Since C is a WAPP-quasi prime submodule of T then by corollary (6) either $It\subseteq C+Soc(T)$ or $Jt\subseteq C+Soc(T)$. Hence either $K_1t\subseteq C+Soc(T)$ or $K_2t\subseteq C+Soc(T)$.

(⇐) Assume that $0\neq IJt\subseteq C$ , for some ideals I,J of R ,t∈T .That is $0\neq K_1K_2t\subseteq C$ for $K_1=IT$ and $K_2=JT$ . It follows that either $K_1t\subseteq C+Soc(T)$ or $K_2t\subseteq C+Soc(T)$; that is $It\subseteq C+Soc(T)$ or $Jt\subseteq C+Soc(T)$.Hence C is a WAPP-quasi prime submodule of T by corollary(6).

**Proposition(14)**

Let T be multiplcation R_module and C be proper submodule of T . Then C is WAPP − quasi prime submodule of T iff $0\neq K_1K_2H\subseteq C$ , for some submodules $K_1$ ,$K_2$ and H of T , implies that either $K_1H\subseteq C+Soc(T)$ or $K_2H\subseteq C+Soc(T)$.

**Proof:**

(⇒) Assume that $0\neq K_1K_2H\subseteq C$ for some submodules $K_1$ ,$K_2$ and H of T . Since T is a multiplication , then $K_1=IT$ ,$K_2=JT$ for some ideals I,J of R hence $0\neq K_1K_2H=IJH\subseteq C$. But C is WAPP-quasi prime submodule of T then by proposition (4) either $IH\subseteq C+Soc(T)$. or $JH\subseteq C+Soc(T)$.. Hence either $K_1H\subseteq C+Soc(T)$. or $K_2H\subseteq C+Soc(T)$..

(⇐) Let $0\neq IJH\subseteq C$ , where I, J are ideals of R , and H is a submodule of T .Since T is multiplication , then $0\neq IJH=K_1K_2H\subseteq C$ , hence by assumption either $K_1H\subseteq C+Soc(T)$ or $K_2H\subseteq C+Soc(T)$.That is either $IH\subseteq C+Soc(T)$ or $JH\subseteq C+Soc(T)$. Thus by proposition (4)C is WAPP-quasi prime submodul of T .

It is well − knwon that if T is Z − regular R − module , then Soc(T)=Soc(R)T [11;prop.(3-25)] .

**Proposition(15)**

Let T be Z_regular multiplcation R_module and C be proper submodule of T . Then C is WAPP − quasi prime submodul of T iff $[C:_R T]$ is WAPP- quasi prime ideal of R.

proof:

(⇒)Suppose that C is WAPP − quasi prime submodule of T and let $0\neq abI\subseteq[C:_R T]$ ,for a,b∈R , I is an ideal of R .it follows that $0\neq ab(IT)\subseteq C$ . Since C is WAPP- quasi prime submodule of T , then by proposition(3) either $aIT\subseteq C+Soc(T)$ or $bIT\subseteq C+Soc(T)$.But T is a Z –regular module , then Soc(T)=Soc(R)T ,and since T is multiplication , then $C=[C:_R T]T$ . Hence either $aIT\subseteq[C:_R T]T+Soc(R)T$ or $bIT\subseteq[C:_R T]T+Soc(R)T$. Thus either $aI\subseteq[C:_R T]+Soc(R)$ or $bI\subseteq[C:_R T]+Soc(R)$ . Hence by proposition $[C:_R T]$ is a WAPP- quase priem ideal of R.

**(⇐)**Suppose that $[C:_R T]$ is a WAPP-quasi prime ideal of R , and $0\neq rsB\subseteq C$ , for r,s∈R , and B is a submodule of T . Since T is a multiplication , then B=IT ,for some ideal I of R ,that is $0\neq rsI T\subseteq C$, it follows that$0\neq rsI\subseteq[C:_R T]$ . For $[C:_R T]$ is a WAPP-quasi prime ideal , then by proposition(3) either $rI\subseteq[C:_R T]+Soc(R)$ or $sI\subseteq[C:_R T]+Soc(R)$ , it follows that either $rIT\subseteq[C:_R T]T+Soc(R)T$ or $sIT\subseteq[C:_R T]T+Soc(R)T$. But T is a Z-regular Soc(T)=Soc(R)T and since T is a multiplication , then $[C:_R T]T=C$ .Thus either $rB\subseteq C+Soc(T)$ or $sB\subseteq C+Soc(T)$.Hence by proposition (3) C is a WAPP-quasi prime submodule of T.

It is well-known that if an R-module T is projective , then Soc(T)=Soc(R)T [11;prop.(3-24)]

**Proposition(16)**

Let T be a projective multiplication R-module and C be a proper submodule of T . Then C is WAPP-quasi prime submodule of T if and only if $[C:_R T]$ is a WAPP- quasi prime ideal of R.

**Proof:**

**(⇒)**Let $0\neq rIJ\subseteq[C:_R T]$ ,for r∈R , I ,J are ideal of R .then $0\neq r\, I(JT)\subseteq C$ . Since C is WAPP- quasi prime submodule of T , then by corollary(7) either $r(JT)\subseteq C+Soc(T)$ or $I(JT)\subseteq C+Soc(T)$.Now since T is a projective module , then Soc(T)=Soc(R)T ,and since T is multiplication , then $C=[C:_R T]T$ . Hence either $r(JT)\subseteq[C:_R T]T+Soc(R)T$ or $I(JT)\subseteq[C:_R T]T+Soc(R)T$. It follows that , either $rJ\subseteq[C:_R T]+Soc(R)$ or $IJ\subseteq[C:_R T]+Soc(R)$ . Hence by corollary(7) $[C:_R T]$ is a WAPP-quasi prime ideal of R.

**(⇐)**Let T $0\neq rIB\subseteq C$ , for r ∈R ,I is an ideal in R, and B is submodule of. Since T is a multiplication , then B=JT ,for some ideal J of R .Thus $0\neq rIJ\, T\subseteq C$, implies that $0\neq rIJ\subseteq[C:_R T]$ . But $[c:_R T]$ is a WAPP-quasi prime ideal , then by corollary(7) either $rJ\subseteq[C:_R T]+Soc(R)$ or $IJ\subseteq[C:_R T]+Soc(R)$ , that is either $rJT\subseteq[C:_R T]T+Soc(R)T$ or $IJT\subseteq[C:_R T]T+Soc(R)T$. Since T is a projective then Soc(T)=Soc(R)T and since T is a multiplication , then $[C:_R T]T=C$ .Thus either $rB\subseteq C+Soc(T)$ or $IB\subseteq C+Soc(T)$.Hence by corollary (7) C is a WAPP-quasi prime submodule of T.

We need to recall the following lemma before we introduce the next proposition .

**Lemma(17)[12, coro, of theo, (9)]**

Let T be a finitely generated multiplication R-module and I ,J are ideals of R . Then $IT\subseteq JT$ if and only if $I\subseteq J+ann_R(T)$.

**Proposition(18)**

Let T be a finitely generated multiplcation Z_regular R_module and I is WAPP − quasi prime ideal of R with annR (T)$\subseteq$ I . Then IT is an WAPP-quasi prime submodule of T.

**Proof:**

Let $0\neq I_1 I_2 B\subseteq IT$ , for $I_1,I_2$ are is ideals of R,and B is submodul of T. Since T is a multiplication then B=J T for some ideal J of R. That is Let $0\neq I_1 I_2 (J T)\subseteq IT$, it follows by lemma (17) $0\neq I_1 I_2 J\subseteq I+ann_R(T)$. But $ann_R(T)\subseteq I$ ,implies that $I+ann_R(T)=I$. That is $0\neq I_1 I_2 J\subseteq I$. But I is a WAPP-quasi prime ideal of R , then by proposition (4) either$0\neq I_1 J\subseteq I+Soc(R)$ or $0\neq I_2 J\subseteq I+Soc(R)$. It follows that either$0\neq I_1 J T\subseteq IT+Soc(R)T$ or$0\neq I_2 J T\subseteq IT+Soc(R)T$. But T is a Z-regular then soc(R)T=Soc(T). Hence either $0\neq I_1 B\subseteq IT+Soc(T)$ or$0\neq I_2 B\subseteq IT+Soc(T)$. Thus by proposition (4) IT is WAPP-quasi prime submodule of T .

**Proposition(19)**

Let T be a finitely generated multiplication projective R-module and I is a WAPP-quasi prime ideal of R with $ann_R(T)\subseteq I$ . Then IT is WAPP-quasi prime submodule of T.

**Proof:**

Let $0\neq rI_1 B\subseteq IT$ , for $r\in R$ ,$I_1$ is an ideal of R, and B is submodule of T. Since T is multiplication then B=JT for some ideal J of R. That is Let $0\neq rI_1 (J T)\subseteq IT$, it follows by lemma (17) $0\neq rI_1 J\subseteq I+ann_R(T)$. But $ann_R(T)\subseteq I$ ,implies that $I+ ann_R(T)=I$. Hence $0\neq rI_1 J\subseteq I$, and since I is WAPP-quasi prime ideal of R , then by corollary (7) either$0\neq I_1 J\subseteq I+Soc(R)$ or$0\neq r J\subseteq I+Soc(R)$.That is either $0\neq I_1 J T\subseteq IT+Soc(R)T$ or$0\neq r J T\subseteq IT+Soc(R)T$. But T is a projective then soc(R)T=Soc(T). Thus either $0\neq I_1 B\subseteq IT+Soc(T)$ or$0\neq r B\subseteq IT+Soc(T)$. Hence by corollary (7) IT is WAPP-quasi prime submodule of T .

It is well-known that cyclic R-module is multiplication [13], and since cyclic R-module is a finitely generated, we get the following corollaries:

**Corollary(20)**

Let T be a cyclic Z-regular R-module and I is WAPP-quasi prime ideal of R with $ann_R(T)\subseteq I$ . Then IT is an WAPP-quasi prime submodule of T.

**Corollary(21)**

Let T be a cyclic projective R-module and I is an WAPP-quasi prime ideal of R with $ann_R$ (T)⊆ I . Then IT is an WAPP-quasi prim submodule of T.

It is well-known that if a submodule C of an R-module T is essential in T, then Soc(C)=Soc(T) [6, P.29].

**Proposition(22)**

Let T be R-module ,and A,B are submodules of T with A⊄ B and B is an essential in T. If A is an WAPP-quasi prime submodule of T , then A is a WAPP-quasi prime submodule of B.

**Proof:**

Let 0≠rst϶A ,for r,s϶R ,t϶B, that is t϶T . Since A is a WAPP-quasi prime submodule of T , then either rt϶A +Soc(T) or st϶ A +Soc(T). But B is essential in T , then soc(B)=Soc(T). That is either rt϶A+Soc(B) or st϶A+Soc(B).Hence A is an WAPP-quasi prime submodule of B.

**Corollary(23)**

Let T be R-module ,and A,B are submodules of T with A⊄ B and Soc(T)⊆ Soc(B). Then A is a WAPP-quasi prime submodule of B.

It well-known that if A is a submodule of an R-module T , then Soc(A)=A∩Soc(T) [9,lema 2.3.15]

**Proposition(24)**

Let T be $R-$ module , and A, B are submodules of T with B not contain in A ,and Soc(T)⊆ B. If A is a WAPP-quasi prime submodule of T , then A∩ B is a WAPP-quasi prime submodule of B.

**Proof:**

It is clear that A∩ B is an proper submodule of B .Now ,let 0≠rst϶ A∩ B, for r,s϶R ,t϶B, implies that 0≠rst϶ A, since A is a WAPP-quasi prime submodule of T ,then either rt϶A+Soc(T) or st϶A+Soc(T) ,hence either rt϶(A+Soc(T) )∩ Bor st϶(A+Soc(T))∩ B. Since Soc(T)⊆B, then by module law either rt϶(A∩ B)+ (B∩ Soc(T)) or st϶(A∩ B)+(B∩ Soc(T)). That is either rt϶(A∩ B)+ Soc(B) or st϶(A∩ B)+Soc(B).Thus A∩ B is a WAPP-quasi prime submodule of B.

It well-known that for each submodule A of an R-module T , then Soc(A)=A, then A⊆Soc(T)[9,theo.(9.1.4)(c)].

**Proposition(25)**

Let T be an $R-$ module , and A, B are submodules of T with B not contain in A, with Soc(A)=A and soc(B)=B. Then A∩ B is a WAPP-quasi prime sub module of T.

**Proof:**

Let $0 \neq rsL \subseteq A \cap B$, for $r,s \in R$ ,L is submodule of T, then $0 \neq rs\, L \subseteq A$ ,and $0 \neq rs\, L \subseteq B$. But both A ,B are WAPP-quasi prime submodule of T, then either $rL \subseteq A+Soc(T)$ or $sL \subseteq A+Soc(T)$, and $rL \subseteq B+Soc(T)$ or $sL \subseteq B+Soc(T)$. But Soc(A)=A and soc(B)=B , then $A \subseteq Soc(T)$ and $B \subseteq Soc(T)$, hence A+Soc(T)=Soc(T) and B+Soc(T)=Soc(T) ,$A \cap B \subseteq Soc(T)$ , implies that $A \cap B + Soc(T) = Soc(T)$ ,so either $rL \subseteq Soc(T) = A \cap B + Soc(T)$ or $sL \subseteq Soc(T) = A \cap B + Soc(T)$. Hence $A \cap B$ is WAPP-quasi prime submodule of T

**Proposition(26)**

Let $f:T \rightarrow T'$ be an R-epimorphism , and C be an WAPP-quasi prime submodule of T with $kerf \subseteq C$ . Then f(C) is WAPP-quasi prime submodule of $T'$.

**Proof:**

Let $f:T \rightarrow T'$ be an R-epimorphism , and C be an WAPP-quasi prime submodule of T with $kerf \subseteq C$ ,let $0 \neq rst' \in f(C)$ , for $r,s \in R$ ,$t' \in T'$ .Since f is onto , then f(t)= t′ , for some $t \in T$ , it follows that $0 \neq rsf(t) \in f(C)$, $0 \neq f(rst) \in f(C)$ , so there exists a nonzero $x \in C$ such that, $0 \neq f(rst) = f(x)$. That is f(rst-x) =0, implies that $rst-x \in kerf \subseteq C$, implies that $0 \neq rst \in C$. But C is a WAPP-quasi prime submodule of T , then either $rt \in C+Soc(T)$ or $st \in C+Soc(T)$. That is either $r\,f(t) \in f(C)+f(Soc(T)) \subseteq f(C)+Soc(T')$ or $sf(t) \in f(C)+f(Soc(T)) \subseteq f(C)+Soc(T')$ . Thus either $rt' \in f(C)+Soc(T')$ or $st' \in f(C)+Soc(T')$ . Hence f(C) is an WAPP-quasi prime submodule of $T'$ .

**Proposition(27)**

Let $f:T \rightarrow T'$ be an R-epimorphism , and C be WAPP-quasi prime submodule of $T'$ . Then $f^{-1}(C)$ is an WAPP-quasi prime submodule of T .

**Prove:**

It is clearly that $f^{-1}(C)$ is proper submodule of T. Let $0 \neq rst \in f^{-1}(C)$ , for $r,s \in R$ ,$t \in T$ , it follows that then $0 \neq rsf(t) \in C$, but C is a WAPP-quasi prime submodule of $T'$ , then either $r\,f(t) \in C+Soc(T)$ or $sf(t) \in C+Soc(T')$. Thus either $r\,t \in f^{-1}(C)+f^{-1}(Soc(T')) \subseteq f^{-1}(C)+Soc(T)$ or $s\,t \in f^{-1}(C)+f^{-1}(Soc(T')) \subseteq f^{-1}(C)+Soc(T)$ .Hence $f^{-1}(C)$ is WAPP-quasi prime submodule of T .

**Proposition(28)**

Let T be a Z-regular finitely generated multiplication $R-$module , and C be a proper submodule of T . Then the following statements are equivalent :

1. C is WAPP-quasi prime submodule of T .
2. $[C:_R T]$ is WAPP-quasi prime ideal of R .
3. C=IT for some WAPP-quasi prime ideal I of R with $ann_R(T) \leq I$ .

**Poof:**

$(1) \Rightarrow (2)$ Follows by proposition [15]

(2) $\Rightarrow$ (3) Follows directly .

(3) $\Rightarrow$ (2) Suppose that C=IT for some a some WAPP-quasi prime ideal of R. Since T is multiplication , then C=$[C:_R T]$T=IT and since M is finitely generated multiplication , then .$[C:_R T]$= I+$ann_R(T)$. But $ann_R(T)\subseteq I$ it follows that I+$ann_R(T)$=I. Thus $[C:_R T]$=I is a WAPP-quasi prime ideal of R. Hence $[C:_R T]$ is WAPP-quasi prime ideal of R.

The following corollary is a direct consequence of proposition (28)

**Corollary(29)**

Let T be a cyclic Z-regular R-module , and C be proper submodule of T . Then the following statements are equipollent :

1. C is WAPP-quasi prime submodule of T .
2. $[C:_R T]$ is WAPP-quasi prime ideal of R .
3. C=IT for some WAPP-quasi prime ideal I of R with $ann_R(T)\subseteq I$ .

**Proposition(30)**

Let T be a finitely generated multiplication projective R-module , and C be a proper submodule of T . Then the following statements are equipollent :

1. C is a WAPP-quasi prime submodule of T .
2. $[C:_R T]$ is WAPP-quasi prime ideal of R .
3. C=IT for some WAPP-quasi prime ideal I of R with $ann_R(T)\subseteq I$ .

**Proof:**

(1) $\Rightarrow$ (2) Follows by proposition (16)

(2) $\Rightarrow$ (3) Follows directly.

(3) $\Rightarrow$ (2) Follows as in proposition(28).

As a direct consequence of proposition (30), we get the following corollary :

**Corollary(31)**

Let T be cyclic projctive $R-$module , and C be proper submodule of T , and C be a proper submodule of T . Then the following statements are equipollent :

1. C is WAPP-quasi prime submodule of T .
2. $[C:_R T]$ is WAPP-quasi prime ideal of R .
3. C=IT for some WAPP-quasi prime ideal I of R with $ann_R(T)\subseteq I$ .

It is well-known that if T is faithful multiplication$R-$module , then Soc(T)=Soc(R)T [7,CORO.(2.14)(1)].

**Proposition(32)**

Let T be a faithful multiplication $R-$module and C be a proper submodule of T . Then C is a WAPP-quasi prime submodule of T iff $[C:_R T]$ is a WAPP- quasi prime ideal of R.

**Proof:**

**(⇒)**Let $0\neq IJk\subseteq[C:_R T]$ ,where I , J and k are ideals of R .then $0\neq IJ(kT)\subseteq C$ . Since C is WAPP- quasi prime submodule of T , then by proposition(4) either $J(kT)\subseteq C+Soc(T)$ or $J(kT)\subseteq C+Soc(T)$.But T is a faithful multiplication ,it follows that $C=[C:_R T]T$ and $Soc(T)=Soc(R)T$ . Thus either $I(KT)\subseteq[C:_R T]T+Soc(R)T$ or $J(KT)\subseteq[C:_R T]T+Soc(R)T$. Hence either $I K\subseteq[C:_R T]+Soc(R)$ or $JK\subseteq[C:_R T]+Soc(R)$ . Thus by proposition(4) $[C:_R T]$ is WAPP-quasi prime ideal of R.

**(⇐)**Let T $0\neq abB\subseteq C$ , for a,b ϵR , and B is submodule of T. Since T is multiplication , then B=JT ,for some ideal J of R . Thus $0\neq abJT\subseteq C$, it follows that $0\neq abJ\subseteq[C:_R T]$ . But $[C:_R T]$ is WAPP-quasi prime ideal of R , then by proposition(3) either $aJ\subseteq[C:_R T]+Soc(R)$ or $bJ\subseteq[C:_R T]+Soc(R)$ , it follows that either $aJT\subseteq[C:_R T]T+Soc(R)T$ or $bJT\subseteq[C:_R T]T+Soc(R)T$. But T is a faithful multiplication R-module then either $aB\subseteq C+Soc(T)$ or $bB\subseteq C+Soc(T)$.Thus by proposition (3) C is a WAPP-quasi prime submodule of T.

The following corollary is a direct consequence of proposition(32)

**Corollary(33)**

Let T be a faithful cyclic $R-$module and C be a proper submodule of T . Then C is WAPP-quasi prime submodule of T if and only if $[C:_R T]$ is a WAPP- quasi prime ideal of R.

## 3.Conclusion

In this proper, we introduced and studied the concept WAPP-quasi prime submodule , and we established several examples , characterizations and basic properties of this concept . WAPP-quasi prime submodule is generalization of a Weakly quasi prime submodule so we give example for converse .

Among C, the main results of this paper are the following:

1. Proper submoduel C of R-module T is WAPP-quasi prime submodule of T iff whenever $(0)\neq rsB\subseteq C$ , for r,sϵR, B is a submodule of T ,implies that either $rB\subseteq C+Soc(T)$ or $sB\subseteq C+Soc(T)$

2. Proper submodule C of R-module T is WAPP-quasi prime submodule of T iff whenever $(0)\neq IJB\subseteq C$ , for I,J are ideals of R, and B is submodule of T ,implies that either $IB\subseteq C+Soc(T)$. or $JB\subseteq C+Soc(T)$.

3. Proper submodule C of R-module T is WAPP-quasi prime submodule of T iff for all r,sϵR , $[c:_T rs]\subseteq[0:_T rs]\cup[C:d_T r]\cup[C:_T s]$

4. Proper submoduel C of R-module T is WAPP-quasi prime submodule of T iff for all r$\epsilon$R ,t$\epsilon$T with rt$\not\in C + Soc(T)$ ,$[c:_T rt]\subseteq[0:_T rt]\cup[C + Soc(T):_T t]$.

5. Proper submodule C of multiplication R-module T is WAPP-quasi prime submodule of T iff whenever $(0)\neq K_1K_2t\subseteq C$ , for some submodules $K_1$,$k_2$ of T and t$\epsilon$T, implies that either $K_1t\subseteq C+Soc(T)$ or $K_2t\subseteq C+Soc(T)$

6. Proper submodule C of Z-regular multiplication R-module T is a WAPP-quasi prime submodule of T iff $[C:_R T]$ is WAPP-quasi prime ideal of R.

7. Proper submodule C of projective multiplication R-module T is WAPP-quasi prim submodule of T iff $[C:_R T]$ is WAPP-quasi prime ideal of R.

8. If T is a cyclic a Z-regular R-module and I is WAPP-quasi prime ideal of R with $ann_R(T)\subseteq I$. Then IT is WAPP-quasi submodule of T.

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