Solving Oscillating Problems Using Modifying Runge-Kutta Methods

Zainab Khaled Ghazal   **Kasim Abbas Hussain**

Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq.

zkgzainab@gmail.com   kasimabbas@uomustansiriyah.edu.iq

Abstract

This paper develop conventional Runge-Kutta methods of order four and order five to solve ordinary differential equations with oscillating solutions. The new modified Runge-Kutta methods (MRK) contain the invalidation of phase lag, phase lag's derivatives, and amplification error. Numerical tests from their outcomes show the robustness and competence of the new methods compared to the well-known Runge-Kutta methods in the scientific literature.

Keywords: Explicit Runge-Kutta methods, oscillating problems Phase-lag properties

1. Introduction

This essay dealing with the system of special first-order ordinary differential equation in the following form:

$$u'(t) = f(t, u), \quad u(t_0) = u_0. \tag{1}$$

Such problems are often observed in various applied sciences, such as quantum chemistry, astronomy, quantum mechanics, electronics, elastics, and chemical physics (see [1,2]). Conventionally equation (1) is solved by using Runge-Kutta (RK) methods or two-step methods [3]. Several authors presented optimized numerical methods based on the phase lag characteristics (such as [4,5,6]). Exact and numerical solutions produce an angle between them is known as Phase-lag, and the distance of the numerical solution from the periodic solution is called amplification error. A five-stage four order Runge-Kutta method with phase-fitted and amplification-fitted proposed by Adel et al. [7]. Hussain et al. [8] proposed an optimized Runge-Kutta method to solve problem (1). The phase lag has the feature to develop more accurate numerical methods. This motivates us to propose the Runge-Kutta

methods by using phase lag properties. Based on the previous work, we present a modified approach for the Runge–Kutta methods by combining the annulment of phase lag, phase lag's derivative, and amplification factor. The new MRK methods have been used to solve oscillation problems. The paper is orderly as follows: In Section 2, the phase-lag properties of the MRK method are given. In Section 3, we present how the new methods are derived. In section 4, we offer numerical results to illustrate the efficiency of the new MRK method. Section 5 is dedicated to Conclusions.

2. Phase Lag Properties of MRK Methods

We are concerned with the numerical solution of the ODEs of the form (1) with an oscillating solution. The general form of explicit MRK method for solving (1) is defined as follows:

$$u_{n+1} = u_n + \sum_{i=1}^{s} b_i f(t_n + c_i, U_i) \tag{2}$$
$$U_i = d_i u_n + h \sum_{j=1}^{i-1} a_{ij} f(t_n + c_i h, U_j), \quad i = 1,2,\ldots,s. \tag{3}$$

The corresponding Butcher tableau for equations (2) – (3) is expressed as follows

Table 1. S-stage modified MRK method

0						
c_2	d_2	a_{21}				
c_3	d_3	a_{31}	a_{32}			
.	.	.	.	.		
.	.	.	.	.		
.	.	.	.	.		
c_s	d_s	a_{s1}	a_{s2}	...	a_{ss-1}	
		b_1	b_2	...	b_{ss-1}	b_s

Constructing the new method depends on phase-lag and dissipation analysis proposed in [9]. To this goal, we use the following test equation,

$$u' = iwu, \quad w \in R \tag{4}$$

An implementation of MRK method (2)-(3) to the test equation (4) we yield

$$u_n = q_*^n u_0, \quad q_*^n = S(z^2) + iz\, V(z^2) \tag{5}$$

where $z = wh$ and S, V are polynomials in z^2 entirely specified by the coefficients a_{ij}, c_i and b_i of MRK method (2)-(3).

The comparison between equations (5) and (4) produces the following definition.
Definition: [9] The following quantities in RK method (2)-(3)

1- $P(z) = z - \arg[q_*(z)] = z - \tan^{-1}\left(\frac{z.V(z^2)}{S(z^2)}\right),$

2- $D(z) = 1 - |q_*(z)| = 1 - \sqrt{(S^2(z^2) + z^2\, V^2(z^2))}.$

are respectively called phase lag or dispersion error and the amplification factor or dissipation error. If $P(z) = O(z^{r+1})$ and $D(z) = O(z^{p+1})$, then the method is said to be of dispersion order r and dissipation order p .

3. Derivation of new MRK methods

This section proceeds to derive the modified RK method by abolishing phase lag, phase lag's derivative, and amplification error.

3.1 Five-Stage Fourth-Order Method

A fourth-order MRK method of five-stage is given as follows [10]:

Table 2. Five-stage modified explicit Runge-Kutta method

0						
$\frac{1}{5}$	d_2	$\frac{1}{5}$				
$\frac{2}{5}$	d_3	0	$\frac{2}{5}$			
$\frac{4}{5}$	d_4	$\frac{6}{5}$	$-\frac{12}{5}$	2		
1	1	$-\frac{17}{8}$	5	$-\frac{5}{2}$	$\frac{5}{8}$	
		$\frac{13}{96}$	0	$\frac{25}{48}$	$\frac{25}{96}$	$\frac{1}{12}$

To construct a modified RK method, we set d_2, d_3, and d_4 as free parameters and the other parameters are the same as in Table 2. Motivated by the approach in [8], we obtain the phase lag, phase lag derivative and the amplification factor, which depend on d_2, d_3 and d_4 as follow:

$$P(z) = \tan(z)\left(\frac{1}{24} d_2 z^4 + \left(-\frac{5}{16} d_3 - \frac{5}{96} d_4 - \frac{13}{96}\right) z^2 + 1\right) - z\left(\frac{1}{120} z^4 + \right.$$

$$\left.\left(-\frac{1}{16} - \frac{5}{48} d_3\right) z^2 + \frac{7}{32} + \frac{25}{48} d_3 + \frac{25}{96} d_4\right), \qquad (6)$$

$$P'(z) = \left(1 + \tan^2(z)\right)\left(\frac{1}{24} d_2 z^4 + \left(-\frac{5}{16} d_3 - \frac{5}{96} d_4 - \frac{13}{96}\right) z^2 + 1\right) +$$

$$\tan(z)\left(\frac{1}{6} d_2 z^3 + 2\left(-\frac{5}{16} d_3 - \frac{5}{96} d_4 - \frac{13}{96}\right) z\right) - \frac{1}{120} z^4 -$$

$$\left(-\frac{1}{16} - \frac{5}{48} d_3\right) z^2 - \frac{7}{32} - \frac{25}{48} d_3 - \frac{25}{96} d_4 - z\left(\frac{1}{30} z^3 + 2\left(-\frac{1}{16} - \frac{5}{48} d_3\right) z\right), \quad (7)$$

$$D(z) = \frac{1}{14400} z^{10} + \left(-\frac{1}{960} + \frac{1}{576} d_2^2 - \frac{1}{576} d_3\right) z^8 + \left(\frac{5}{1152} d_4 + \frac{29}{3840} + \frac{25}{1152} d_3 + \right.$$

$$\frac{25}{2304} d_3^2 - \frac{5}{1152} d_2 d_4 - \frac{13}{1152} d_2 - \frac{5}{192} d_2 d_3\right) z^6 + \left(-\frac{25}{1152} d_3 d_4 + \frac{25}{9216} d_4^2 + \right.$$

$$\frac{1}{12} d_2 - \frac{25}{2304} d_3^2 - \frac{83}{9216} - \frac{85}{4608} d_4 - \frac{5}{192} d_3\right) z^4 + \left(-\frac{685}{3072} - \frac{305}{768} d_3 + \right.$$

$$\left.\frac{625}{9216} d_4^2 + \frac{625}{2304} d_3 d_4 + \frac{625}{2304} d_3^2 + \frac{5}{512} d_4\right) z^2. \quad (8)$$

Now, solving equations (6),(7), and (8), we get the values of free parameters in terms of z, and the expressions are too complicated, we use the following Taylor series expressions,

$$d_2 = 1 - \frac{29}{630} z^2 - \frac{311}{75600} z^4 - \frac{46369}{74844000} z^6 - \frac{120976607}{1225944720000} z^8 -$$

$$\frac{1756548533}{110335024800000} z^{10} - \frac{3475880890247}{1350500703552000000} z^{12} - \cdots, \qquad d_3 =$$

$$1 - \frac{4}{1575} z^4 - \frac{47}{94500} z^6 - \frac{1009}{13365000} z^8 \frac{1695769}{139311900000} z^{10} -$$

$$\frac{271269671}{137918781000000} z^{12} - \cdots, \qquad d_4 =$$

$$1 - \frac{8}{1575} z^4 - \frac{37}{47250} z^6 - \frac{437}{11694375} z^8 \frac{4558639}{766215450000} z^{10} -$$

$$\frac{32225033}{34479695250000} z^{12} - \cdots. \qquad (9)$$

It can observe that for $z \to 0$, the new method reduces to an original RK method given in Dormand [10].

3.2 Six-Stage Five-Order Method
Consider the following six-stage modified RK method which can be expressed in Butcher tableau [3].

Table 3. Five-stage modified explicit Runge-Kutta method

0							
$\frac{1}{4}$	1	$\frac{1}{4}$					
$\frac{1}{4}$	1	$\frac{1}{8}$	$\frac{1}{8}$				
$\frac{1}{2}$	d_4	0	0	$\frac{1}{2}$			
$\frac{3}{4}$	d_5	$\frac{3}{16}$	$-\frac{3}{8}$	$\frac{3}{8}$	$\frac{9}{16}$		
1	d_6	$-\frac{3}{7}$	$\frac{8}{7}$	$\frac{6}{7}$	$-\frac{12}{7}$	$\frac{8}{7}$	
		$\frac{7}{90}$	0	$\frac{16}{45}$	$\frac{2}{15}$	$\frac{16}{45}$	$\frac{7}{90}$

According to the method in **Table 3**, the phase lag, phase lag derivative, and amplification factor become

$$P(z) = \tan(z)\left(1 + \left(-\frac{31}{90} - \frac{1}{15} d_4 - \frac{4}{45} d_5\right) z^2 + \frac{1}{24} z^4 - \frac{1}{1280} z^6\right) - z\left(\frac{7}{90} d_6 + \right.$$
$$\left. + \frac{13}{30} + \frac{2}{15} d_4 + \frac{16}{45} d_5 + \left(-\frac{7}{90} - \frac{1}{20} d_4\right) z^2 + \frac{1}{120} z^4\right) \quad (10)$$

$$P'(z) = \left(1 + tan^2(z)\right)\left(1 + \left(-\frac{31}{90} - \frac{1}{15} d_4 - \frac{4}{45} d_5\right) z^2 + \frac{1}{24} z^4 - \frac{1}{1280} z^6\right) +$$

$$\tan(z)\left(2\left(-\frac{31}{90} - \frac{1}{15} d_4 - \frac{4}{45} d_5\right) z + \frac{1}{6} z^3 - \frac{3}{640} z^5\right) - \frac{7}{90} d_6 - \frac{13}{30} - \frac{2}{15} d_4 -$$

$$\frac{16}{45} d_5 - \left(-\frac{7}{60} - \frac{1}{20} d_4\right) z^2 - \frac{1}{120} z^4 - z\left(\frac{1}{30} z^3 + 2\left(-\frac{7}{60} - \frac{1}{20} d_4\right) z\right), \quad (11)$$

$$D(z) = \frac{1}{1638400} z^{12} + \frac{1}{230400} z^{10} + \left(\frac{19}{57600} - \frac{7}{9600} d_4 + \frac{1}{7200} d_5\right) z^8 + \left(\frac{1}{120} d_4 - \right.$$

$$\frac{1}{675} d_5 + \frac{1}{400} d_4^2 - \frac{163}{17280} + \frac{7}{5400} d_6\right) z^6 + \left(-\frac{77}{2700} d_4 - \frac{2}{225} d_4^2 - \frac{44}{2025} d_5 + \right.$$

$$\frac{817}{8100} - \frac{49}{2700} d_6 - \frac{7}{900} d_4 d_6 - \frac{16}{675} d_4 d_5 + \frac{16}{2025} d_5^2\right) z^4 + \left(\frac{88}{675} d_5 + \frac{4}{225} d_4^2 + \right.$$

$$\frac{49}{8100} d_6^2 + \frac{256}{2025} d_5^2 + \frac{64}{675} d_4 d_5 + \frac{112}{2025} d_5 d_6 + \frac{14}{675} d_4 d_6 + $$

$$\left. + \frac{91}{1350} d_6 - \frac{4}{225} d_4 - \frac{451}{900}\right) z^2 \quad (12)$$

Solving equations (10),(11), and (12), we find

$$d_4 = 1 - \frac{25}{2016} z^4 - \frac{31}{4536} z^6 - \frac{613}{221760} z^8 - \frac{116287}{103783680} z^{10} - \frac{5938297}{1307643680} z^{12} - \frac{3273113521}{17784371404800} z^{14} - \cdots,$$

$$d_5 = 1 + \frac{347}{21504} z^4 + \frac{67}{13824} z^6 + \frac{7367}{3548160} z^8 + \frac{2819}{3354624} z^{10} + \frac{1827169}{5364817920} z^{12} + \frac{13092454033}{94849980825600} z^{14} + \cdots,$$

$$d_6 = 1 - \frac{247}{4704} z^4 - \frac{1775}{84672} z^6 - \frac{14143}{1552320} z^8 - \frac{447763}{121080960} z^{10} - \frac{22865693}{15256200960} z^{12} - \frac{2520658027}{4149686661120} z^{14} - \cdots. \quad (13)$$

It can demonstrate that for $z \to 0$, the new method same as the original RK method given in Butcher [3].

4. Numerical Results

To evaluate the performance of the new modified Runge-Kutta methods suggested in this paper, we apply them to five oscillatory problems and then compared the numerical results with the several well-known efficient methods. We use the criteria of absolute error to measure the accuracy of the method, which is given by

Absolute error $= \max (|y(t_n) - y_n|)$. Where $y(t_n)$ is the true solution and y_n is the numerical solution. Figures 1-5 demonstrate the efficiency graphs of Log 10 (Max Error) versus step size h. Integration interval is [0, 1000] for all problems with step sizes $h = 0.1/2^i$, $i = 1,2,3,4$. The following numerical methods are used in the comparison.

(i)
- **MRK4:** modified five-stage fourth-order RK method presented in **Table 1** and **Eqs. (9)** in **Section 3** in this paper.
- **RK4D:** the classical five-stage RK method of order four given in [10].
- **RK4PF:** the phase-fitted and amplification-fitted RK method given in [7].
- **RK5S:** six-stage RK method of order five given in [11].
- **RK4MS:** the modified four-order RK method given in [9].

- **RK4B:** the five-stage RK method of fourth-order given in [3].

(ii)

- **MRK5:** modified six-stage fifth-order RK method presented in **Table 2** and **Eqs. (13)** in **Section 3** in this paper.
- **RK5B:** the classical six-stage RK method of order five given in [3].
- **ORK5T:** Optimized fifth-order RK method given in [4].
- **RK5M:** Optimized RK method of order five given in [12].
- **ORK5A:** Optimized RK method of order five given in [5].
- **RK5K:** Optimized fifth-order RK method proposed in [6].

Problem 1: [13]

$$u''(t) = 100\, u(t) + 99 \sin(t), \quad u(0) = 1, \quad u'(0) = 11.$$

Whose exact solution is $u(t) = \cos(10t) + \sin(10t) + \sin(t)$, and $w = 10$.

Problem 2: [14]

$$u''(t) = -64\, u(t), \quad u(0) = 1, \quad u'(0) = -2.$$

Exact solution is $u(t) = \frac{1}{4}\sin(8t) + \cos(8t)$, and $w = 8$.

Problem 3: [15]

$$u_1''(t) + u_1(t) = 0.001\ \cos(t), \quad u_1(0) = 1, \quad u_1'(0) = 0,$$

$$u_2''(t) + u_2(t) = 0.001\ \sin(t), \quad u_2(0) = 0, \quad u_2'(0) = 0.9995.$$

Exact solution is $u_1(t) = \cos(t) + 0.0005\, t \sin(t)$,
$u_2(t) = \sin(t) - 0.0005\, t \cos(t)$, and $w = 1$.

Problem 4: [16]

$$u''(t) + \begin{pmatrix} \frac{101}{2} & -\frac{99}{2} \\ -\frac{99}{2} & \frac{101}{2} \end{pmatrix} u(t) = \begin{pmatrix} \frac{93}{2}\cos(2t) - \frac{99}{2}\sin(2t) \\ \frac{93}{2}\sin(2t) - \frac{99}{2}\cos(2t) \end{pmatrix},$$

$$u(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u'(0) = \begin{pmatrix} -10 \\ 12 \end{pmatrix}.$$

Exact solution and frequency are

$$u(t) = \begin{pmatrix} -\cos(10t)(t) - \sin(10t) + \cos(2t) \\ \cos(10t) + \sin(10t) + \sin(2t) \end{pmatrix}, \quad w = 10.$$
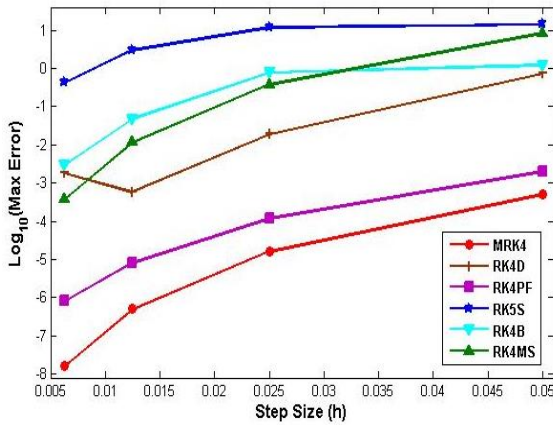
Problem 5: [17] Oscillatory system problem

$$u''(t) + \begin{pmatrix} 13 & -12 \\ -12 & 13 \end{pmatrix} u(t) = \begin{pmatrix} 9\cos(2t) - 12\sin(2t) \\ -12\cos(2t) + 9\sin(2t) \end{pmatrix},$$

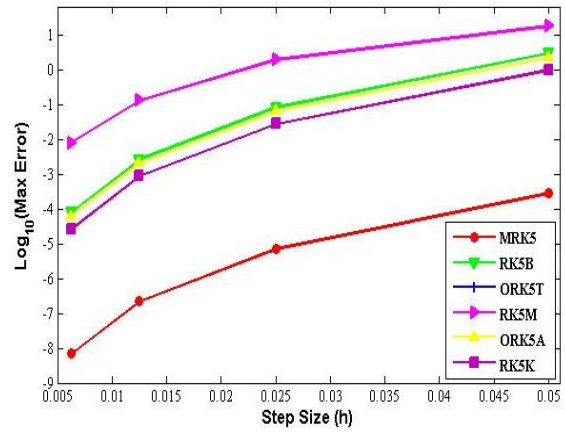$$u(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u'(0) = \begin{pmatrix} -4 \\ 8 \end{pmatrix}.$$

Exact solution and frequency are

$$u(t) = \begin{pmatrix} \sin(t) - \sin(5t) + \cos(2t) \\ \sin(t) + \sin(5t) + \sin(2t) \end{pmatrix}, \quad w = 5.$$
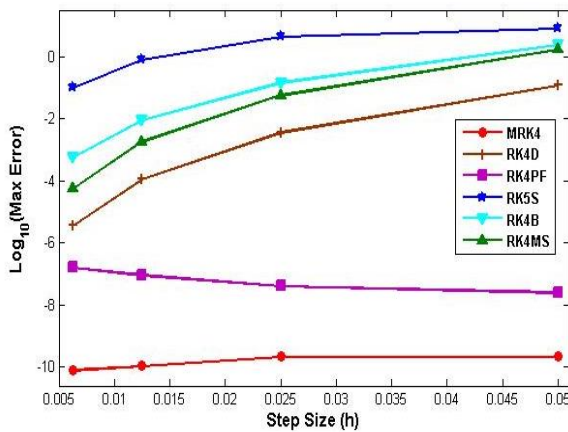

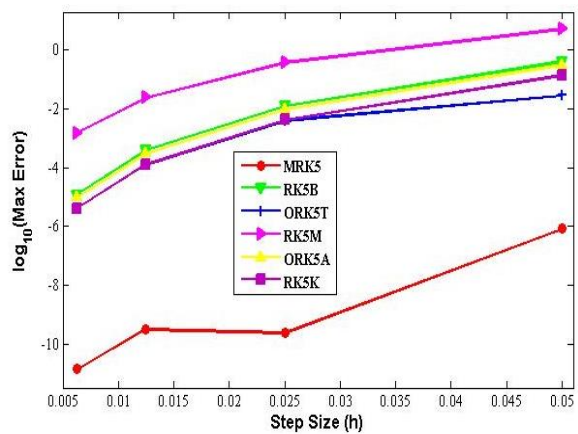**(a) Comparisons of the methods (i) in Section 4.**   **(b) Comparisons of the methods (ii) in Section 4.**
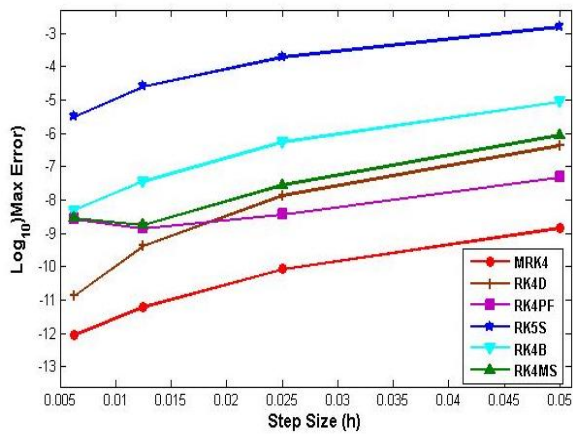
Figure 1. The competence curves for Problem 1.

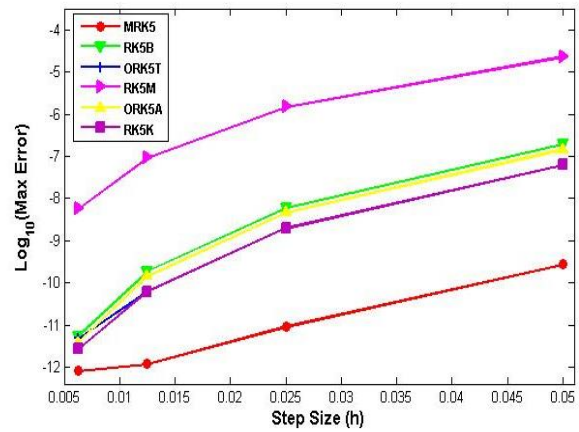**(a) Comparisons of the methods (i) in Section 4**   **(b) Comparisons of the methods (ii) in Section 4**.
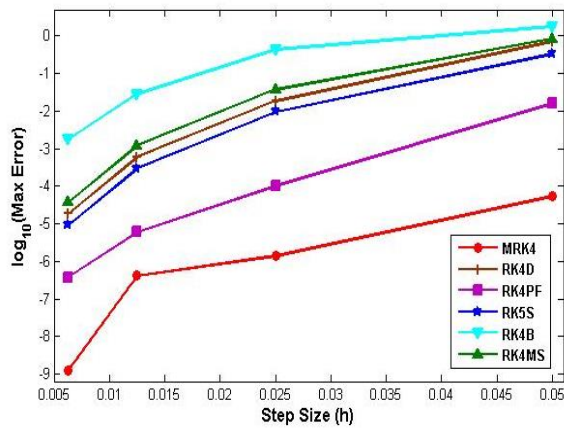
Figure 2. The competence curves for Problem 2.

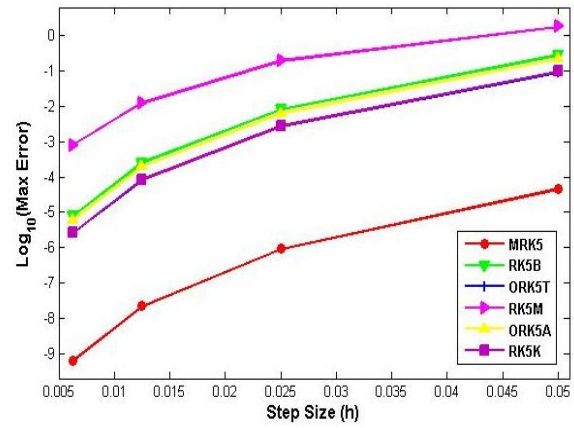(a) **Comparisons of the methods (i) in Section 4.**   (b) **Comparisons of the methods (ii) in Section 4.**

Figure 3. The competence curves for Problem 3.

(a) **Comparisons of the methods (i) in Section 4.**   (b) **Comparisons of the methods (ii) in Section 4.**

Figure 4. The competence curves for Problem 4.

(a) Comparisons of the methods (i) in Section 4.   (b) Comparisons of the methods (ii) in Section 4.

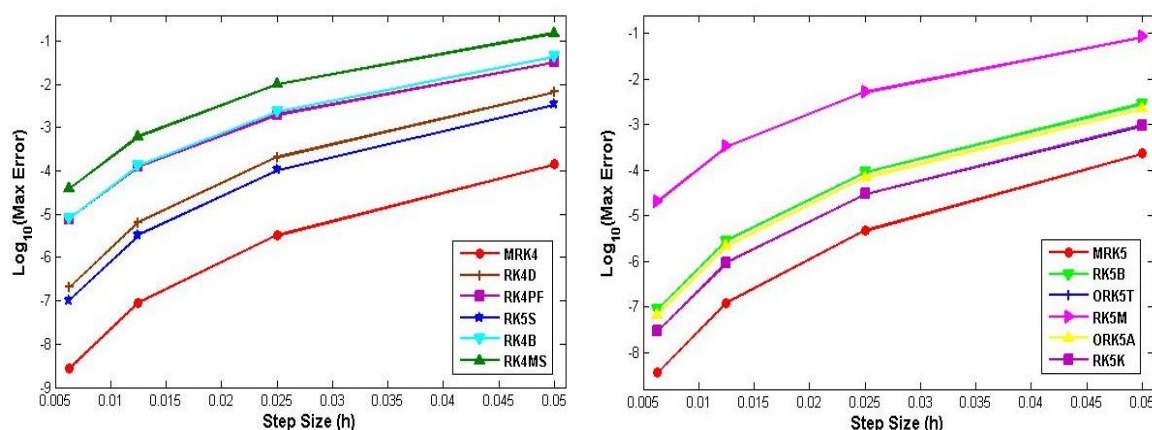**Figure 5:** The competence curves for Problem 5.

5. Conclusion

Classical Runge-Kutta methods are adjusted for solving the system of first-order ordinary differential equations whose solutions have a marked periodic style in this paper. The new methods are based on a fourth and fifth algebraic order given in [10] and [3], respectively. The proposed methods have nullified the phase-lag, amplification error, and phase lag's derivative. The results demonstrate the competence and effectiveness of the newly constructed methods, while they are compared to the standard methods and efficient RK methods. It is clear that the new MRK methods are the most accurate, and especially other than existing RK methods in the scientific literature.

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