Ibn Al Haitham Journal for Pure and Applied Science

Journal homepage: http://jih.uobaghdad.edu.iq/index.php/j/index

Zenali Iteration Method For Approximating Fixed Point of A $\delta\mathcal{ZA}$ $-$ Quasi Contractive mappings

Zena Hussein Maibed **Ali Qasem Thajil**

Department of Mathematics, College of Education for Pure Science (Ibn Al-Haitham), University of Baghdad, Iraq.

mrs_zena.hussein@yahoo.com Ali.Qasem1203a@ihcoedu.uobaghdad.edu.iq

Article history: Received 15 April 2021, Accepted 30 May 2021, Published in October 2021.

Doi: 10.30526/34.4.2705

Abstract.

This article will introduce a new iteration method called the zenali iteration method for the approximation of fixed points. We show that our iteration process is faster than the current leading iterations like Mann, Ishikawa, Noor, D- iterations, and $\mathcal{K}$*- iteration for new contraction mappings called $\delta\mathcal{ZA}$ $-$ quasi contraction mappings. And we proved that all these iterations (Mann, Ishikawa, Noor, D- iterations and $\mathcal{K}$*- iteration) equivalent to approximate fixed points of $\delta\mathcal{ZA}$ $-$ quasi contraction. We support our analytic proof by a numerical example, data dependence result for contraction mappings type $\delta\mathcal{ZA}$ by employing zenali iteration also discussed.

Keywords: Mann and $\mathcal{K}$*- iteration, $\delta\mathcal{ZA}$ $-$ quasi contraction mappings.

1. Introduction

The fixed point theory is one of the most important theories that play an important and fundamental role to solve many problems in various fields of science and knowledge such as Geometry, game theory, chemistry, etc. numerical calculation of fixed points for nonlinear operators is also an active research problem at present for nonlinear analysis due to its applications in balance problems, variable inequality, image coding, computer simulation and more. For that, many authors have created a large number of algorithms to approximate the fixed point for different types of applications for example see [1-9]. The well-known Banach contraction theorem uses the Picard iteration mechanism for fixed point approximation. This paper consists of three sections section one converges the zenali iteration with all these iterations. In section two rate of converge, section three equivalent, section four numerical example with real datasets. Many of the other well-known iterative methods are those of Mann [10], Ishikawa [11], D- iteration [12], Picard S iteration [13] , $\mathcal{K}$*-iteration [14] , Noor iteration [15].

Let $\mathcal{M}$ be a uniformly convex Banach space, $\emptyset \neq \mathcal{C}$ be a closed-convex subset of $\mathcal{M}$. We recall some definitions of those iterations as:

1- Let $< s_n >, < t_n >$ and $< u_n >$ are sequences lies in (0,1). The following iteration $\langle d_n \rangle$ is called D- iteration and defined as follows:

$d_0 \in \mathcal{C}, \quad s_n = (1 - u_n)d_n + u_n \mathcal{T}d_n,$

$t_n = (1 - t_n)\mathcal{T}d_n + t_n \mathcal{T}s_n, \; d_{n+1} = (1 - s_n)\mathcal{T}s_n + s_n \mathcal{T}t_n.$

2- Let $z_n \in \mathcal{C}$. The following iteration $\langle z_n \rangle$ is called *P*icard iteration and defined as follows:

$z_{n+1} = \mathcal{T}z_n, \; n \in N.$

3- Let $< s_n >$ be a sequence in (0,1). The following iteration is called Mann iteration and defined as follows:

$r_0 \in \mathcal{C}, \; r_{n+1} = (1 - s_n)r_n + s_n \mathcal{T}r_n, n \in N.$

4- Let $< s_n >, < t_n >$ and $< u_n >$ be real sequences in (0,1). The following iteration is called Ishikawa iteration and defined as follows
$w_0 \in \mathcal{C}, \quad w_{n+1} = (1 - s_n)w_n + s_n \mathcal{T}d_n,$

$d_n = (1 - t_n)w_n + t_n \mathcal{T}w_n, n \in.$

5- Let $< s_n >$ and $< t_n >$ are sequences lies in (0,1). The following iteration is called Picard $\mathcal{S}$-iteration and defined as follows:

$h_n \in \mathcal{C}, \; h_{n+1} = \mathcal{T}l_n,$

$l_n = (1 - s_n)\mathcal{T}h_n + s_n \mathcal{T}e_n, \; \check{e}_n = (1 - t_n)h_n + t_n \mathcal{T}h_n.$

6- Let $< s_n >, < t_n >$ and $< u_n >$ are sequences in (0,1). The following iteration $\langle q_n \rangle$ is called $\mathcal{K}^*$-iteration, and defined as follows: $q_0 \in \mathcal{C},$

$q_{n+1} = \mathcal{T}p_n, \; p_n = \mathcal{T}\big((1 - s_n)o_n + s_n \mathcal{T}o_n\big)$ and $o_n = (1 - t_n)q_n + t_n \mathcal{T}q_n.$

7- Let $< s_n >, < t_n >$ and $< u_n >$ are sequences in (0,1). The following iteration $\langle y_n \rangle$ is called Noor iteration and defined as follows:

$y_0 \in \mathcal{C}, y_{n+1} = (1 - s_n)y_n + s_n \mathcal{T}q_n,$

$q_n = (1 - s_n)y_n + s_n \mathcal{T}p_n,$

$p_n = (1 - u_n)y_n + u_n \mathcal{T}y_n, \; n \in N.$

Definition1.1 [16]: Let $< \mathcal{V}_n >, < \mathcal{U}_n >$ are sequences lies in R converge to $\mathcal{V}$ and $< \mathcal{U}_n >$ converge to $\mathcal{U}$, and Let $< \mathcal{V}_n >$ such that $\mathcal{Z} = \lim_{n\to\infty} \frac{|\mathcal{V}_n - \mathcal{V}|}{|\mathcal{U}_n - \mathcal{U}|}$

1. If $\mathcal{Z}$= 0. Then The sequence $< \mathcal{V}_n >$ is converge to $\mathcal{V}$ faster then $< \mathcal{U}_n >$ converge to $\mathcal{U}$.

2. If $0 \prec \mathcal{Z} \prec \infty \rightarrow$ $< x_n >$ and $< \mathcal{U}_n >$ have the same rate of convergence.

Lemma 1.2 [17]: Let $\mathcal{M}$ be a uniformly convex Banach space and $\langle \mathcal{I}_n \rangle_{n=0}^{\infty}$ be any sequence such that $0 < p \leq \mathcal{I}_n \leq r < 1$, for some $p, r \in R$ and for all $n \geq 1$, Let $\langle \mathcal{V}_n \rangle_{n=0}^{\infty}$ and $\langle \mathcal{U}_n \rangle_{n=0}^{\infty}$, be a nonnegative real sequences of $\mathcal{M}$ such that $\lim_{n\to\infty} sup \| \mathcal{V}_n \| \leq r$, $\lim_{n\to\infty} sup \| \mathcal{U}_n \| \leq r$ and

$\lim_{n\to\infty} \| \mathcal{I}_n \mathcal{V}_n - (1 - \mathcal{I}_n) \mathcal{U}_n \| = r$ for some $r \geq 0$. Then, $\lim_{n\to\infty} \| \mathcal{V}_n - \mathcal{U}_n \| = 0$.

Lemma 1.3 [18]: Let $\langle \mathcal{V}_n \rangle_{n=0}^{\infty}$ and $\langle \mathcal{U}_n \rangle_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following condition:

$\mathcal{V}_{n+1} \leq (1 - \mu_n)\mathcal{V}_n + \mathcal{U}_n$, where $\mu_n \in (0,1)$, for all $n \geq n_0$, $\sum_{n=1}^{\infty} \mu_n = \infty$

and $\frac{\mathcal{U}_n}{\mu_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} \mathcal{V}_n = 0$.

2. Main Results

In this section, we introduced a new iteration process known as Zenali Iteration

and new contraction mappings called a $\delta\mathcal{Z}\mathcal{A}-$ quasi contraction mappings.

Definition 2.1: Let $< s_n >$, $< t_n >$ and $< u_n >$ are sequences in $(0,1)$ and $\mathcal{T}: \mathcal{C} \to \mathcal{C}$. The following iteration is called Zenali iteration and defined as follows

$$x_0 \in \mathcal{C}, \qquad x_{n+1} = \mathcal{T} y_n$$

$y_n = \mathcal{T}((1 - s_n) z_n + s_n \mathcal{T} z_n),$

$z_n = \mathcal{T}((1 - t_n) x_n + t_n \mathcal{T} x_n).$

Definition 2.2: Let $\mathcal{T}$ be a self mapping on $\mathcal{C}$, then $\mathcal{T}$ called a $\delta\mathcal{Z}\mathcal{A}-$ quasi contraction for all $x, y \in \mathcal{C}$, if $|| \mathcal{T}x - \mathcal{T}y || \leq \delta || x - y || + \mathcal{Z}\mathcal{A}(nx, my)$ where

$\mathcal{A}(nx, my) = min\{n\|x - \mathcal{T}x\|, m\|y - \mathcal{T}y\|, nm\|x - \mathcal{T}y\|, mn\|y - \mathcal{T}x\|\}$

for some $0 < \delta \leq 1$, $\mathcal{Z} \geq 0$ and $n, m \geq 0$.

Lemma 2.3 : Let $\mathcal{C}$ be a nonempty convex and closed subset of a Banach space $\mathcal{M}$ and let $\mathcal{T}: \mathcal{C} \to \mathcal{C}$ a $\delta\mathcal{Z}\mathcal{A}-$ quasi contraction mapping. Suppose that $\langle x_n \rangle$ the Zenali iteration in $\mathcal{C}$.

If $\mathcal{F}(\mathcal{T}) \neq \emptyset$, then 1- $\|z_n - p\| \leq \|x_n - p\|$ and $\|y_n - p\| \leq \|x_n - p\|$.

2-$\lim_{n\to\infty} \|x_n - p\|$ exists, for all $n \in N$.

Proof. Let p be a fixed point of $\mathcal{T}$. Then the following inequalities hold

$\|z_n - p\| = \|\mathcal{T}[(1 - t_n)x_n - t_n \mathcal{T}x_n] - p\|$

$\leq \delta\|(1 - t_n)x_n - t_n \mathcal{T}x_n - p\| + \mathcal{Z}\mathcal{A}(n[(1 - t_n)x_n - t_n \mathcal{T}x_n], m\, p).$

Since $\|\mathcal{T}p - p\| \to 0$ as $n \to \infty$ then, $\mathcal{Z}\mathcal{A}(n[(1 - t_n)x_n - t_n \mathcal{T}x_n], m\, p) = 0$

$\|z_n - p\| \leq (1 - t_n)\|x_n - p\| + \delta t_n \|x_n - p\| + t_n \mathcal{Z}\mathcal{A}(nx_n, mp)$

$$\leq (1 - t_n(1-\delta))\|x_n - p\| + \mathcal{Z}\, min\{n\|x_n - \mathcal{T}x_n\|$$

$$, m\|p - \mathcal{T}p\|, nm\|x_n - \mathcal{T}p\|, mn\|p - \mathcal{T}x_n\|\}$$

$$\|z_n - p\| \leq \|x_n - p\| \tag{2.1}$$

And,

$$\|y_n - p\| = \|\mathcal{T}[(1 - s_n)z_n + s_n \mathcal{T}z_n] - p\|$$

$$\leq \delta\,\|(1 - s_n)z_n + s_n \mathcal{T}z_n - p\| + \mathcal{Z}\mathcal{A}\,(n[(1 - s_n)z_n + s_n \mathcal{T}z_n], mp)$$

Since $\|\mathcal{T}p - p\| \to 0$ as $n \to \infty$. Then, $\mathcal{Z}\mathcal{A}(n[(1 - s_n)z_n + s_n \mathcal{T}z_n], mp) = 0$

$$\|y_n - p\| \leq (1 - s_n)\|z_n - p\| + \delta s_n\|z_n - p\| + s_n\mathcal{Z}\mathcal{A}\,(nz_n, mp)$$

$$\leq (1 - s_n(1-\delta))\|z_n - p\| + \mathcal{Z}min\{n\|z_n - \mathcal{T}z_n\|$$

$$, m\|p - \mathcal{T}p\|, nm\|z_n - \mathcal{T}p\|, mn\|p - \mathcal{T}z_n\|\}$$

$$\|y_n - p\| \leq \|z_n - p\|. \tag{2.2}$$

Using inequality (2.1) in (2.2), it follows that:

$$\|y_n - p\| \leq \|x_n - p\| \tag{2.3}$$

And, $\|x_{n+1} - p\| = \|\mathcal{T}y_n - p\| \leq \delta\|y_n - p\| + \mathcal{Z}\mathcal{A}\,(ny_n, mp)$

$$\leq \|y_n - p\| \tag{2.4}$$

Using inequality (2.3), inequality (2.4) becomes

$$\|x_{n+1} - p\| \leq \|x_n - p\| \text{ for all } n \in N. \tag{2.5}$$

So, $\{\|x_n - p\|\}$ is decreasing, for each $p \in \mathcal{F}(\mathcal{T})$, this implies that the sequence

$\lim_{n\to\infty}\|x_n - p\|$ exists. ■

Theorem 2.4 : Let $\mathcal{M}$ be a uniformly convex Banach space, $\mathcal{C}$ be nonempty convex and closed subset of $\mathcal{M}$ and let $\mathcal{T}: \mathcal{C} \to \mathcal{C}$ a $\delta\mathcal{Z}\mathcal{A}$ − quasi contraction mapping. Suppose that $\langle x_n \rangle$ the Zenali iteration in $\mathcal{C}$. Then $F(\mathcal{T}) \neq \emptyset$ if and only if $\langle x_n \rangle$ is bounded and $\lim_{n\to\infty}\|x_n - \mathcal{T}x_n\| = 0$.

Proof. Since $p \in F(\mathcal{T})$, from (2.5) we get

$\|x_{n+1} - p\| \leq \|x_n - p\| \leq \cdots \leq \|x_0 - p\|$ for all $n \in N$.Thus, $\langle x_n \rangle$ is bounded set in $\mathcal{C}$.

Put, $$r = \lim_{n\to\infty}\|x_n - p\| \tag{2.6}$$

And, $\lim_{n\to\infty}\|\mathcal{T}x_n - p\| \leq \lim_{n\to\infty}(\delta\|x_n - p\| + \mathcal{Z}\mathcal{A}\,(nx_n, mp))$

$$\leq \lim_{n\to\infty}(\|x_n - p\| + \mathcal{Z}min\{n\|x_n - \mathcal{T}x_n\|$$

$$, m\|p - \mathcal{T}p\|, nm\|x_n - \mathcal{T}p\|, mn\|p - \mathcal{T}x_n\|\})$$

$$\lim_{n\to\infty} \|\mathcal{T}x_n - \wp\| \le \lim_{n\to\infty} \|x_n - \wp\| \quad (2.7)$$

Using inequality (2.6) in (2.7) becomes, $\lim_{n\to\infty} \|\mathcal{T}x_n - \wp\| \le r$ (2.8)

From Equations(2.3) and (2.6), we have

$$\liminf_{n\to\infty} \|y_n - \wp\| \le r. \quad (2.9)$$

Similarly by using (2.1) and (2.6),we have

$$\limsup_{n\to\infty} \|z_n - \wp\| \le r \quad (2.10)$$

Now, $r = \liminf_{n\to\infty} \|x_{n+1} - \wp\| = \liminf_{n\to\infty} \|\mathcal{T}y_n - \wp\|$

$$\le \liminf_{n\to\infty} (\delta\|y_n - \wp\| + \mathcal{Z}\mathcal{A}(ny_n, m\wp))$$

$$\le \liminf_{n\to\infty} (\|y_n - \wp\| + \mathcal{Z}min\{n\|y_n - \mathcal{T}y_n\|$$

$$, m\|\wp - \mathcal{T}\wp\|, nm\|y_n - \mathcal{T}\wp\|, mn\|\wp - \mathcal{T}y_n\|\})$$

Then, we get $r \le \liminf_{n\to\infty} \|y_n - \wp\|$. (2.11)

Having in mind (2.2), inequality (2.11) becomes, $r \le \liminf_{n\to\infty} \|z_n - \wp\|$ (2.12)

From Eqs (2.10) and (2.12), we obtain

$$r = \lim_{n\to\infty} \|z_n - \wp\| = \lim_{n\to\infty} (\|\mathcal{T}((1 - t_n)x_n - t_n\mathcal{T}x_n) - \wp\|)$$

$$\le \delta \lim_{n\to\infty} \|(1 - t_n)(x_n - \wp) - t_n(\mathcal{T}x_n - \wp)\|$$

$$+\mathcal{Z}\mathcal{A}(n[(1 - t_n)x_n - t_n\mathcal{T}x_n], m\wp).$$

Since $\|\mathcal{T}\wp - \wp\| \to 0$ as $n \to \infty$ then, $\mathcal{Z}\mathcal{A}(n[(1 - t_n)x_n - t_n\mathcal{T}x_n], m\wp) = 0$

$$\le \lim_{n\to\infty} \delta(1 - t_n(1-\delta))\|x_n - \wp\| + \mathcal{Z}\min\{n\|x_n - \mathcal{T}x_n\|$$

$$, m\|\wp - \mathcal{T}\wp\|, nm\|x_n - \mathcal{T}\wp\|, mn\|\wp - \mathcal{T}x_n\|\}$$

$$\le \lim_{n\to\infty} \|x_n - \wp\| = r$$

So, $r \le \|(1 - t_n)(x_n - \wp) - t_n(\mathcal{T}x_n - \wp)\| \le r$

Then $\|(1 - t_n)(x_n - \wp) - t_n(\mathcal{T}x_n - \wp)\| = r$ (2.13)

Thus From Eqs (2.6), (2.8), (2.13) and lemma(1.2) we obtain, $\lim_{n\to\infty} \|x_n - \mathcal{T}x_n\| = 0$.

Now, we prove that $F(\mathcal{T}) \neq \emptyset$

Let $\wp \in A(\mathcal{C}, \langle x_n \rangle) \Rightarrow r(\mathcal{C}, \langle x_n \rangle) = r(\wp, \langle x_n \rangle)$

$$r(\mathcal{T}\wp, \langle x_n \rangle) = \limsup_{n\to\infty} \|x_n - \mathcal{T}\wp\| = \limsup_{n\to\infty}[\|x_n - \mathcal{T}x_n\| + \|\mathcal{T}x_n - \mathcal{T}\wp\|]$$

$$\leq \limsup_{n\to\infty}[\delta\|x_n - \mathcal{T}x_n\| + \mathcal{ZA}(nx_n, m\wp)$$

$$\leq \limsup_{n\to\infty}\|x_n - \wp\| = r(\wp, \langle x_n \rangle) = r(\mathcal{C}, \langle x_n \rangle)$$

$\mathcal{T}\wp \in A(\mathcal{C}, \langle x_n \rangle)$ C a uniformly convex $\Rightarrow A(C, \langle x_n \rangle)$ is a singleton

$\Rightarrow \wp = \mathcal{T}\wp \Rightarrow \wp \in F(\mathcal{T}) \Rightarrow F(\mathcal{T}) \neq \emptyset$. ∎

Lemma 2.5: Let $\mathcal{C}$ be a nonempty convex and closed subset of a Banach space $\mathcal{M}$ and let $\mathcal{T}: \mathcal{C} \to \mathcal{C}$ a $\delta\mathcal{ZA} -$ quasi contraction mapping. Suppose that $\langle r_n \rangle$ the Mann iteration in $\mathcal{C}$ If $\mathcal{F}(\mathcal{T}) \neq \emptyset$, then $\lim_{n\to\infty}\|r_n - \wp\|$ exists.

Proof: Let $\wp$ be a c fixed point of. The following inequalities hold

$$\|r_{n+1} - \wp\| = \|(1 - a_n)r_n + a_n\mathcal{T}r_n - \wp\|$$

$$\leq (1 - a_n)\|r_n - \wp\| + a_n\|\mathcal{T}r_n - \wp\|$$

$$\leq (1 - a_n)\|r_n - \wp\| + \delta a_n\|r_n - \wp\| + a_n\mathcal{ZA}(nr_n, m\wp)$$

$$\leq (1 - a_n(1 - \delta)\|r_n - \wp\|$$

$$\|r_{n+1} - \wp\| \leq \|r_n - \wp\| \tag{2.14}$$

So, $\{\|r_n - \wp\|\}$ is decreasing, for each $\wp \in \mathcal{F}(\mathcal{T})$, this implies that the sequence $\lim_{n\to\infty}\|r_n - \wp\|$ exists. ∎

Theorem 2.6: Let $\mathcal{M}$ be a uniformly convex Banach space, $\mathcal{C}$ be nonempty convex and closed subset of $\mathcal{M}$ and let $\mathcal{T}: \mathcal{C} \to \mathcal{C}$ a $\delta\mathcal{ZA} -$ quasi contraction mapping. Suppose that $\langle r_n \rangle$ the Mann iteration in $\mathcal{C}$. Then $F(\mathcal{T}) \neq \emptyset$ if and only if $\langle r_n \rangle$ is bounded and $\lim_{n\to\infty}\|r_n - \mathcal{T}r_n\| = 0$.

Proof. Since $\wp \in F(\mathcal{T})$, from(2.14) we get :

$\|r_{n+1} - \wp\| \leq \|r_n - \wp\| \leq \cdots \leq \|r_0 - \wp\|$ for all n ∈ N.

Thus, $\langle r_n \rangle$ is bounded set in $\mathcal{C}$.

Put, $$r = \lim_{n\to\infty}\|r_n - \wp\| \tag{2.15}$$

$$\limsup_{n\to\infty}\|\mathcal{T}r_n - \wp\| \leq \limsup_{n\to\infty}(\delta\|r_n - \wp\| + \mathcal{ZA}(nr_n, m\wp))$$

$$\leq \limsup_{n\to\infty}\|r_n - \wp\| \tag{2.16}$$

Having in mind (2.15), inequality(2.16)becomes:

$$\limsup_{n\to\infty}\|\mathcal{T}r_n - \wp\| \leq r \tag{2.17}$$

Now, $r = \lim_{n\to\infty}\|r_{n+1} - \wp\| = \lim_{n\to\infty}\|(1 - s_n)r_n + s_n\mathcal{T}r_n - \wp\|$

$$= \lim_{n\to\infty} \| (1- s_n)(r_n - p) + s_n(\mathcal{T}r_n - p) \| \quad (2.18)$$

Thus From Eqs (2.15), (2.17), (2.18) and lemma(1.2) we obtain, $\lim_{n\to\infty} \|r_n - \mathcal{T}r_n\| = 0$.

Now, we prove that F($\mathcal{T}$) $\neq \emptyset$. By the same proof way of the previous theorem. ■

Now, we will study the equivalent between many of iterations by using a $\delta\mathcal{ZA}$ − quasi contraction mappings.

Theorem 2.7: Let $\mathcal{C}$ closed a nonempty convex and subset of a Banach space X, $\mathcal{T}$ be a $\delta\mathcal{ZA}$ − quasi contraction mapping on $\mathcal{C}$ and has a unique fixed point p. Consider the Zenali iteration and Mann iteration with real sequences. Then the following a assertions are equivalent:

The Mann iteration converges to p .

The Zenali iteration converges to p.

Proof. We show that (i) → (ii) that is, if the Mann iteration converges, then the Zenali iteration does too. Since, the Mann iteration converges to p

$\Longrightarrow \|r_n - p\| \to 0 \; as \; n \longrightarrow \infty$.

Now, consider Mann and the Zenali iterations, we have:

$$\|r_{n+1} - x_{n+1}\| = \|(1- s_n)r_n + s_n\mathcal{T}r_n - \mathcal{T}y_n\|$$
$$\leq (1- s_n)\|r_n - \mathcal{T}y_n\| + \delta s_n\|r_n - y_n\| + s_n\mathcal{ZA}(nr_n, my_n)$$
$$\leq (1- s_n)\|r_n - \mathcal{T}r_n\| + \delta(1- s_n)\|r_n - y_n\|$$
$$+(1- s_n)Lm(r_n, y_n) + \delta s_n\|r_n - y_n\| + s_n\mathcal{ZA}(nr_n, my_n)$$
$$= (1- s_n)\|r_n - \mathcal{T}r_n\| + \delta\|r_n - y_n\| + \mathcal{ZA}(nr_n, my_n) \quad (2.19)$$

$$\|r_n - y_n\| = \|r_n - \mathcal{T}[(1- s_n)z_n - s_n\mathcal{T}z_n]\|$$
$$\leq \|r_n - \mathcal{T}r_n\| + \|\mathcal{T}r_n - \mathcal{T}[(1- s_n)z_n - s_n\mathcal{T}z_n]\|$$
$$\leq \|r_n - \mathcal{T}r_n\| + \delta\|r_n - (1- s_n)z_n - s_n\mathcal{T}z_n\|$$
$$+\mathcal{ZA}(nr_n, m[(1- s_n)z_n + s_n\mathcal{T}z_n])$$
$$\leq \|r_n - \mathcal{T}r_n\| + \delta(1- s_n)\|r_n - z_n\| + \delta s_n\|r_n - \mathcal{T}r_n\|$$
$$+\delta^2 s_n\|r_n - z_n\| + \delta s_n\mathcal{ZA}(nr_n, mz_n) + \mathcal{ZA}(nr_n, m[(1- s_n)z_n + s_n\mathcal{T}z_n])$$
$$= (1 + \delta s_n)\|r_n - \mathcal{T}r_n\| + (\delta(1- s_n) + \delta^2 s_n)\|r_n - z_n\|$$
$$+\delta s_n\mathcal{ZA}(nr_n, mz_n) + \mathcal{ZA}(nr_n, m[(1- s_n)z_n + s_n\mathcal{T}z_n]) \quad (2.20)$$

$$\|r_n - z_n\| = \|r_n - \mathcal{T}[(1- t_n)x_n - t_n\mathcal{T}x_n]\|$$
$$\leq \|r_n - \mathcal{T}r_n\| + \|\mathcal{T}r_n - \mathcal{T}[(1- t_n)x_n - t_n\mathcal{T}x_n]\|$$

$$\leq \|r_n - \mathcal{T}r_n\| + \delta\|r_n - (1 - t_n)x_n - t_n \mathcal{T}x_n\| + \mathcal{Z}\mathcal{A}(nr_n, mx_n)$$

$$\leq \|r_n - \mathcal{T}r_n\| + \delta(1 - t_n)\|r_n - x_n\| + \delta t_n\|r_n - \mathcal{T}x_n\| + \mathcal{Z}\mathcal{A}(nr_n, mx_n)$$

$$\leq \|r_n - \mathcal{T}r_n\| + \delta(1 - t_n)\|r_n - x_n\| + \delta t_n\|r_n - \mathcal{T}r_n\|$$

$$+\delta^2 t_n\|r_n - x_n\| + \delta t_n \mathcal{Z}\mathcal{A}(nr_n, mx_n) + \mathcal{Z}\mathcal{A}(nr_n, mx_n)$$

$$= (1 + \delta t_n)\|r_n - \mathcal{T}r_n\| + (\delta(1 - t_n) + \delta^2 t_n)\|r_n - x_n\|$$

$$+(1 + \delta t_n)\mathcal{Z}\mathcal{A}(nr_n, mx_n) \tag{2.21}$$

Substituting (2.21) in (2.20), we obtain

$$\|r_n - y_n\| \leq (1 + \delta s_n)\|r_n - \mathcal{T}r_n\| + (\delta(1 - s_n) + \delta^2 s_n)(1 + \delta t_n)\|r_n - \mathcal{T}r_n\|$$

$$+(\delta(1 - s_n) + \delta^2 a_n)(\delta(1 - t_n) + \delta^2 t_n)\|r_n - x_n\|$$

$$+(\delta(1 - s_n) + \delta^2 s_n)(1 + \delta t_n)\mathcal{Z}\mathcal{A}(nr_n, mx_n)$$

$$+(1 + \delta s_n)\mathcal{Z}\mathcal{A}(nr_n, mz_n) \tag{2.22}$$

Substituting (2.22) in (2.19), we obtain

$$\|r_{n+1} - x_{n+1}\| \leq (1 - s_n)\|r_n - \mathcal{T}r_n\| + \delta(1 + \delta s_n)\|r_n - \mathcal{T}r_n\|$$

$$+\delta(\delta(1 - s_n) + \delta^2 s_n)(1 + \delta t_n)\|r_n - \mathcal{T}r_n\|$$

$$+\delta^2(1 - s_n(1 - \delta))(1 - t_n(1 - \delta))\|r_n - x_n\|$$

$$+\delta(\delta(1 - s_n) + \delta^2 s_n)(1 + \delta t_n)\mathcal{Z}\mathcal{A}(nr_n, mx_n)$$

$$+\delta(1 + \delta s_n)\mathcal{Z}\mathcal{A}(nr_n, mz_n) + \mathcal{Z}\mathcal{A}(nr_n, my_n)$$

$$\leq [(1 - s_n) + \delta(1 + \delta s_n) + \delta(\delta(1 - s_n) + \delta^2 s_n)(1 + \delta t_n)]\|r_n - \mathcal{T}r_n\|$$

$$+((1 - s_n(1 - \delta)\|r_n - x_n\| + \delta(\delta(1 - s_n) + \delta^2 s_n)(1 + \delta t_n)$$

$$\mathcal{Z}\min\{n\|r_n - \mathcal{T}r_n\|, m\|x_n - \mathcal{T}x_n\|, nm\|r_n - \mathcal{T}x_n\|, mn\|x_n - \mathcal{T}r_n\|\}$$

$$+\delta(1 + \delta s_n)\mathcal{Z}\min\{n\|r_n - \mathcal{T}r_n\|, m\|z_n - \mathcal{T}z_n\|, nm\|r_n - \mathcal{T}z_n\|$$

$$, mn\|z_n - \mathcal{T}r_n\|\} + \mathcal{Z}\min\{n\|r_n - \mathcal{T}r_n\|, m\|y_n - \mathcal{T}y_n\|$$

$$, nm\|r_n - \mathcal{T}y_n\|, mn\|y_n - \mathcal{T}r_n\|\}$$

Let $\mu_n = s_n(1 - \delta) \epsilon (0, 1)$, $\mathcal{V}_n = \|r_n - x_n\|$

$$\mathcal{U}_n = [(1 - s_n) + \delta(1 + \delta s_n) + \delta(\delta(1 - s_n) + \delta^2 s_n)(1 + \delta t_n)]\|r_n - \mathcal{T}r_n\|$$

$$+\delta(\delta(1 - s_n) + \delta^2 s_n)(1 + \delta t_n)\mathcal{Z}\min\{n\|r_n - \mathcal{T}r_n\|, m\|x_n - \mathcal{T}x_n\|$$

$$, nm\|r_n - \mathcal{T}x_n\|, mn\|x_n - \mathcal{T}r_n\|\} + \delta(1 + \delta s_n)\mathcal{Z}\min\{n\|r_n - \mathcal{T}r_n\|$$

$$, m\|z_n - \mathcal{T}z_n\|, nm\|r_n - \mathcal{T}z_n\|, mn\|z_n - \mathcal{T}r_n\|\}$$

$$+\mathcal{Z}\, min\{n\|r_n - \mathcal{T}r_n\|, m\|y_n - \mathcal{T}y_n\|, nm\|r_n - \mathcal{T}y_n\|, mn\|y_n - \mathcal{T}r_n\|\}$$

Furthermore, using $\mathcal{T}p = p$ and $\|r_n - p\| \to 0$, we have

$$\|r_n - \mathcal{T}r_n\| = \|r_n - p + \mathcal{T}p - \mathcal{T}r_n\|$$

$$\leq \|r_n - p\| + \delta\|r_n - p\| + \mathcal{Z}\mathcal{A}\,(nr_n, mp)$$

$$= (1+\delta)\|r_n - p\| + \mathcal{Z} min\{n\|r_n - \mathcal{T}r_n\|$$

$$, m\|p - \mathcal{T}p\|, nm\|r_n - \mathcal{T}p\|, mn\|p - \mathcal{T}r_n\|\}$$

Then, $\|r_n - \mathcal{T}r_n\| \to 0$.

Now, because of these results, we get $\mathcal{U}_n \to 0$

By applying lemma(1.3), we obtain $\mathcal{V}_n = \|r_n - x_n\| \to 0$ as $n \to 0$.

Consequently , $\|r_{n+1} - x_{n+1}\| \to 0$ as $n \to 0$. Therefore,

$\|x_n - p\| \leq \|r_n - x_n\| + \|r_n - p\| \to 0$ as $n \to \infty$

Now, we show that, (ii) ⇒ (i).

Since, the Zenali iteration converges to $p \Longrightarrow \|x_n - p\| \to 0\ as\ n \longrightarrow \infty$.

Now, consider the following

$$\|x_{n+1} - r_{n+1}\| \leq \|x_{n+1} - p\| + \|r_{n+1} - p\|$$

$$= \|\mathcal{T}y_n - p\| + \|(1 - s_n)r_n + s_n\mathcal{T}r_n - p\|$$

$$\leq \delta\|y_n - p\| + \mathcal{Z}\mathcal{A}\,(ny_n, mp) + (1 - s_n)\|r_n - p\|$$

$$+\delta s_n\|r_n - p\| + s_n\mathcal{Z}\mathcal{A}\,(nr_n, mp)$$

$$= \delta\|y_n - p\| + \mathcal{Z}\mathcal{A}\,(ny_n, mp) + \big(1 - s_n(1-\delta)\big)\|r_n - p\|$$

$$+s_n Lm\,(r_n, p) \qquad (2.23)$$

$$\|y_n - p\| = \|\mathcal{T}[(1 - s_n)z_n + s_n\mathcal{T}z_n] - p\|$$

$$\leq \delta\|(1 - s_n)z_n + s_n\mathcal{T}z_n - p\| + \mathcal{Z}\mathcal{A}\,(nz_n, mp)$$

$$\leq \delta(1 - s_n)\|z_n - p\| + \delta^2 s_n\|z_n - p\| + \delta s_n\mathcal{Z}\mathcal{A}\,(nz_n, mp) + \mathcal{Z}\mathcal{A}\,(nz_n, mp)$$

$$\leq \delta\big(1 - s_n(1-\delta)\big)\|z_n - p\| + (\delta s_n + 1)\mathcal{Z}\mathcal{A}\,(nz_n, mp) \qquad (2.24)$$

$$\|z_n - p\| = \|\mathcal{T}[(1 - t_n)x_n + t_n\mathcal{T}x_n] - p\|$$

$$\leq \delta\|(1 - t_n)x_n + t_n\mathcal{T}x_n - p\| + \mathcal{Z}\mathcal{A}\,(n((1 - t_n)x_n + t_n\mathcal{T}x_n), mp)$$

$$\leq \delta(1 - t_n)\|x_n - p\| + \delta^2 t_n\|x_n - p\| + \delta t_n\mathcal{Z}\mathcal{A}\,(nx_n, mp)$$

$$\leq \delta\big(1 - t_n(1-\delta)\big)\|x_n - p\| + (\delta t_n + 1) + \delta t_n\mathcal{Z}\mathcal{A}\,(nx_n, mp) \qquad (2.25)$$

Substituting (2.25) in (2.24), we obtain

$$\|y_n - p\| \leq \delta^2 \left(1 - s_n(1-\delta)\right)\left(1 - t_n(1-\delta)\right)\|x_n - p\| + \delta\left(1 - s_n(1-\delta)\right)$$

$$(\delta t_n + 1)\mathcal{ZA}\,(nx_n, mp) + (\delta s_n + 1)\mathcal{ZA}\,(nz_n, mp) \qquad (2.26)$$

Substituting (2.26) in (2.23), we obtain

$$\|x_{n+1} - r_{n+1}\| \leq \delta^3 \left(1 - s_n(1-\delta)\right)\left(1 - t_n(1-\delta)\right)\|x_n - p\| + \left(1 - s_n(1-\delta)\right)$$

$$\|r_n - x_n + x_n - p\| + \delta^2 \left(1 - s_n(1-\delta)\right)(\delta t_n + 1)\mathcal{ZA}\,(nx_n, mp)$$

$$+\mathcal{ZA}\,(nz_n, mp) + \delta(\delta s_n + 1)\mathcal{ZA}\,(nz_n, mp) + s_n\mathcal{ZA}\,(nr_n, mp)$$

$$\leq \left[\left(1 - s_n(1-\delta)\right) + \left(1 - s_n(1-\delta)\right)\right]\|x_n - p\|$$

$$+ \left(1 - s_n(1-\delta)\right)\|x_n - r_n\| + \delta^2 \left(1 - s_n(1-\delta)\right)(\delta t_n + 1)$$

$$\mathcal{Z}min\{n\|x_n - \mathcal{T}x_n\|, m\|p - \mathcal{T}p\|, nm\|y_n - \mathcal{T}p\|, mn\|p - \mathcal{T}y_n\|\}$$

$$+\mathcal{Z}\min\{n\|y_n - \mathcal{T}y_n\|, m\|p - \mathcal{T}p\|, nm\|x_n - \mathcal{T}p\|, mn\|p - \mathcal{T}x_n\|\}$$

$$+\delta(\delta s_n + 1)\mathcal{Z}\min\{n\|z_n - \mathcal{T}z_n\|, m\|p - \mathcal{T}p\|, nm\|z_n - \mathcal{T}p\|$$

$$, mn\|p - \mathcal{T}z_n\|\} + s_n\mathcal{Z}min\{n\|r_n - \mathcal{T}r_n\|, m\|p - \mathcal{T}p\|$$

$$, nm\|r_n - \mathcal{T}p\|, mn\|p - \mathcal{T}r_n\|\}$$

Let $\mu_n = s_n(1-\delta) \epsilon\, (0, 1)$, $\quad \mathcal{V}_n = \|x_n - r_n\|$

$$\mathcal{U}_n = \left[\left(1 - s_n(1-\delta)\right) + \left(1 - s_n(1-\delta)\right)\right]\|x_n - p\| + \delta^2 \left(1 - s_n(1-\delta)\right)(\delta t_n + 1)$$

$$\mathcal{Z}min\{n\|x_n - \mathcal{T}x_n\|, m\|p - \mathcal{T}p\|, nm\|x_n - \mathcal{T}p\|, mn\|p - \mathcal{T}x_n\|\}$$

$$+\mathcal{Z}min\{n\|y_n - \mathcal{T}y_n\|, m\|p - \mathcal{T}p\|, nm\|y_n - \mathcal{T}p\|, mn\|p - \mathcal{T}y_n\|\}$$

$$+\delta(\delta s_n + 1)\mathcal{Z}min\{n\|z_n - \mathcal{T}z_n\|, m\|p - \mathcal{T}p\|, nm\|z_n - \mathcal{T}p\|, mn\|p - \mathcal{T}z_n\|\}$$

$$+s_n\mathcal{Z}min\{n\|r_n - \mathcal{T}r_n\|, m\|p - \mathcal{T}p\|, nm\|r_n - \mathcal{T}p\|, mn\|p - \mathcal{T}r_n\|\}$$

Since, $\|x_n - p\| \to 0$ as $n \to \infty$. So, we get $\mathcal{U}_n \to 0$, thus from lemma (1.3), we get

$\mathcal{V}_n = \|x_n - r_n\| \to 0$ as $n \to 0$. Consequently; $\|x_{n+1} - r_{n+1}\| \to 0$ as $n \to 0$

Therefore, $\|r_n - p\| \leq \|r_n - x_n\| + \|x_n - p\| \to 0$ as $n \to \infty$. ■

Now, we will prove that our new iteration is faster than many know iterations By using new contraction mappings.

Theorem 2.8: Let $\mathcal{T}$ be a $\delta\mathcal{ZA} -$ quasi contraction mapping on $\mathcal{C}$. Suppose that the iterations Zenali iteration, Ishikawa iteration and Mann iteration converge to $p \in \mathcal{F}(\mathcal{T})$ where $0 < v \leq u_n$, $s_n, t_n < 1$, $\forall n \in N$. Then the Zenali iteration converges faster than of Mann iteration and Ihikawa iteration.

Proof. Consider Zenali iteration, we obtain

$$\|\tilde{x}_{n+1} - p\| = \|\mathcal{T}y_n - p\| \le \delta\, \|y_n - p\| + \mathcal{ZA}\,(ny_n, mp)$$

$$= \delta\, \|\mathcal{T}\,((1 - s_n)z_n + s_n\mathcal{T}z_n) - p\|$$

$$\le \delta^2 \|\,((1 - s_n)z_n + s_n\mathcal{T}z_n) - p\,\| + \mathcal{ZA}\,(n(1 - s_n)z_n + s_n\mathcal{T}z_n), mp)$$

Since $\|\mathcal{T}p - p\| \to 0$ as $n \to \infty$ then, $\mathcal{ZA}\,(n(1 - s_n)z_n + s_n\mathcal{T}z_n), m\,p) = 0$

$$= \delta^2 \|\,((1 - s_n)(z_n - p) + s_n(\mathcal{T}z_n - p)\,\|$$

$$\le \delta^2\,[\,(1 - s_n)\|z_n - p\| + \delta\, s_n\|z_n - p\| + \mathcal{ZA}\,(nz_n, mp)]$$

$$= \delta^2\,[\,((1 - s_n) + \delta\, s_n]\,\|z_n - p\|$$

$$= \delta^2\,[\,(1 - s_n(1 - \delta)\,]\,\|\mathcal{T}\,((1 - t_n)x_n + t_n\mathcal{T}x_n) - p\|$$

$$\le \delta^2[\,(1 - s_n(1 - \delta)\,]\,\delta\|\,((1 - t_n)(x_n - p) + t_n(\mathcal{T}x_n - p)\|$$

$$+\mathcal{ZA}\,(n(1 - t_n)x_n + t_n\mathcal{T}x_n), mp)$$

$$\le \delta^3\,[\,(1 - \alpha_n(1 - \delta)]\,[\,((1 - t_n)\|x_n - p\| + \delta t_n\|x_n - p\| + \mathcal{ZA}\,(nx_n, mp)]$$

$$= \delta^3\,[\,(1 - s_n(1 - \delta)\,]\,[\,((1 - t_n) + \delta t_n]\,\|x_n - p\|$$

$$\le \delta^3\,[\,(1 - v(1 - \delta)\,]^2\,\|x_n - p\|$$

$$\vdots$$

$$\le (\delta^3\,[\,1 - v(1 - \delta)\,]^2)^n\,\|x_0 - p\|$$

Suppose that $ZA_n = (\delta^3\,[1 - v(1 - \delta)\,]^2)^n\,\|x_0 - p\|$

Consider the Mann iteration, we have

$$\|r_{n+1} - p\| = \|(1 - s_n)r_n + s_n\,\mathcal{T}r_n - p\|$$

$$= \|\,((1 - s_n)(r_n - p) + s_n(\mathcal{T}r_n - p)\,\|$$

$$\le (1 - s_n)\|r_n - p\| + s_n\|\mathcal{T}r_n - p\|$$

$$\le (1 - s_n)\|r_n - p\| + \delta s_n\|r_n - p\| + \mathcal{ZA}\,(nr_n, mp)$$

$$\le [1 - v(1 - \delta)]\,\|r_n - p\| + \mathcal{Z}\min\{n\|r_n - \mathcal{T}r_n\|, m\|p - \mathcal{T}p\|$$

$$, nm\|r_n - \mathcal{T}p\|, mn\|p - \mathcal{T}r_n\|\}$$

$$= [\,1 - v(1 - \delta)]\,\|r_n - p\|$$

$$\vdots$$

$$\le [\,1 - v(1 - \delta)]^n\,\|r_0 - p\|$$

Suppose that $\mathcal{S}_n = [\,1 - v(1 - \delta)]^n\,\|r_0 - p\|$

Here, after simple compute, we have

$\frac{ZA_n}{\mathcal{M}_n} = \frac{(\delta^3\ [1-v(1-\delta)\ ]^2)^n \|x_0-p\|}{[\ 1-v(1-\delta))]^n \|w_0-p\|} \rightarrow 0$ as $n \rightarrow \infty$.

Then, the Zenali iteration converges to p faster than Ishikawa iteration and Mann iteration. ∎

Theorem 2.9 : Let $\mathcal{T}$ be a $\delta\mathcal{ZA}$ − quasi contraction self- mapping on $\mathcal{C}$. Suppose that the Zenali iteration and D - iteration converge to the same fixed point p of $\mathcal{T}$ where $0 < v \leq u_n, s_n, t_n < 1$, $\forall n \in N$. Then, the Zenali iteration converges faster than D - iteration.

Proof. Form D - iteration, we obtain

$$\begin{aligned}
\|d_{n+1} - p\| &= \|(1- s_n)\mathcal{T}s_n + s_n\mathcal{T}t_n - p\| \\
&\leq \delta(1- s_n)\|s_n - p\| + (1- s_n)\mathcal{ZA}(ns_n, mp) \\
&\quad +\delta s_n\|t_n - p\| + s_n\mathcal{ZA}(nt_n, mp) \\
&= \delta[(1- s_n)\|s_n - p\| + s_n\|(1- t_n)\mathcal{T}d_n + t_n\mathcal{T}s_n - p\|] \\
&\leq \delta[(1- s_n)\|s_n - p\| + \delta s_n((1-t_n)\|d_n - p\| \\
&\quad + \delta s_n t_n\|s_n - p\|)] + \mathcal{ZA}(ns_n, mp) + \mathcal{ZA}(nd_n, mp) \\
&\leq \delta[(1- s_n) + \delta s_n t_n)\|s_n - p\| + \delta s_n(1-t_n)\|d_n - p\|] \\
&= \delta\big[((1 - s_n(1-\delta t_n))\|(1- u_n)d_n + u_n\mathcal{T}d_n - p\| \\
&\quad +\delta s_n(1-t_n)\|d_n - p\|] \\
&\leq \delta\big[((1- s_n(1-\delta t_n))((1- u_n)\|d_n - p\| + \delta u_n\|d_n - p\|) \\
&\quad +\delta s_n(1-t_n)\|d_n - p\| + \mathcal{ZA}(nd_n, mp)] \\
&\leq \delta\big[((1- s_n(1-\delta t_n))((1- u_n) + u_n))\|d_n - p\| \\
&\quad + \delta s_n(1-t_n)\|d_n - p\|] \\
&\leq \delta\big[((1- s_n(1-\delta t_n))\|d_n - p\| + \delta s_n(1-t_n)\|d_n - p\|] \\
&= \delta\big[((1- s_n(1-\delta))\|d_n - p\|] \\
&\leq \delta((1- v(1-\delta))\|d_n - p\|] \\
&\ \ \vdots \\
&\leq [\delta((1- v(1-\delta))]^n\|d_0 - p\|
\end{aligned}$$

Let $D_n = [\delta((1- v(1-\delta))]^n\|d_0 - p\|$

Form Zenali-Iteration, we have, $ZA_n = (\delta^3\ [1 - v(1-\delta)\ ]^2)^n\|x_0 - p\|$

$\frac{ZA_n}{D_n} = \frac{(\delta^3\ [1-v(1-\delta)\ ]^2)^n \|x_0-p\|}{[\ \delta((1-v(1-\delta))]^n \|d_0-p\|} \rightarrow 0$ as $n \rightarrow \infty$. Thus $< x_n >$ converges to p faster than $< d_n >$.
So, the Zenali-Iteration converges faster than D - iteration. ∎

We proof other iterations by the same proof way of the previous Theorem.

Example 2.10: Let $\mathcal{N} = R$ and $\mathcal{C} = [0,100]$. and $\mathcal{T}$ be a mapping on $\mathcal{C}$ defined by $\tilde{x} = \sqrt{\tilde{x}^2 - 9\tilde{x} + 54}$, for all $x \epsilon \mathcal{C}$, such that $\mathcal{T}$ is a $\delta\mathcal{Z}\mathcal{A}$ $-$ quasi contraction $\mathcal{m}$apping and unique fixed point say $\mathcal{p} = 6$.

Take $< \mathcal{s}_n > = < \mathcal{t}_n > = < \mathcal{u}_n > = \frac{3}{4}, v = \frac{1}{2}$ with initial value 30.

Table 1. Comparison speed of convergence among various iteration methods.

n	Zenali	K*	D-	Ishikawa	Mann
1	30	30	30	30	30
2	13.9156	17.1404	21.1334	25.0120	27.1150
3	6.1717	7.9203	13.2989	20.2548	24.2908
4	6.0006	6.0388	7.8776	15.8509	21.5421
5	6.0000	6.0004	6.1725	12.0133	18.8893
6		6.0000	6.0087	9.0688	16.3607
7			6.0007	7.2820	13.9954
8			6.0000	6.4668	11.8476
9				6.1601	9.8476
10				6.0537	8.4901
11				6.0179	7.4083
12				6.0060	6.7247
13				6.0020	6.3468
14				6.0007	6.1587
15				6.0002	6.0709
16				6.0001	6.0313
17				6.0000	6.0137
18					6.0011
19					6.0005
20					6.0001
21					6.0000

3.Conclusion

In this section, a new iteration method for approximation of fixed points and a new contraction mappings called $\delta\mathcal{ZA}$ − quasi contraction mappings are introduced. Also, we proved that our iteration process is faster than the existing leading iterations like Mann, Ishikawa, Noor, D-iterations and $\mathcal{K}$*- iteration and proved that all these iterations are equivalent to approximate fixed points of $\delta\mathcal{ZA}$ − quasi contraction.

References

1. Maibed, Z. H.Generalized Tupled Common Fixed Point Theorems for Weakly Compatibile Mappings in Fuzze Metric Space,(IJCIET) **2019**,*10*,255-273.
2. Maibed, Z. H. Strong Convergence of Iteration Processes for Infinite Family of General Extended Mappings, IOP Conf. Series: *Journal of Physics*, **2018**. Conf. Series 1003, 012042 doi :10.1088/1742-6596/1003/1/012042.
3. Maibed, Z. H., Some Generalized n-Tuplet Coincidence Point Theorems for Nonlinear Contraction Mappings, *Journal of Engineering and Applied Sciences*, **2018**,*13*, 10375-10379.
4. Maibed, Z. H. Common Fixed Point problem for Classes of Nonlinear Maps in Hilbert Space , IOP Conf. Series: *Journal of Physics: Conf. Series 871*, **2020**, 871 012037.
5. Maibed, Z.M, A New Iterative Methods For a Family of Asymptotically Severe Accretive Mappings, *IOP Conf. Series: Journal of Physics: Conf. Series* 1591, **2020**, 012087.
6. Thajil ,A Q ; Maibed. Z H,2021. The Convergence of Iterative Methods for Quasi δ-Contraction Mappings ,*Journal of Physics: Conference Series* 1804 ,**2021**, 012017.
7. Hussein, S. S.; Maibed, Z. H..2021. "Iterative Methods for Approximation of Fixed Points Via Like Contraction Mappings, *Journal of Physics: Conference Series* 1818 ,**2021**, 012152.
8. Maibed,Z. H.; Hussein, S. S. Some Theorems of Fixed Point Approximations By Iteration Processes, *Journal of Physics: Conference Series* 1818, *2021*, 012153
9. Berinde, V. Iterative Approximation of Fixed Points. Springer, Berlin. **2007**.
10. Mann W. R., Mean Value Methods in Iteration, Proc, Am.Math.soc. **1953**, *4*, 506-510.
11. Ishikawa S.Fixed Point By a New Iteration Method ,Proceedings of the American, Soc.44,**1974**.147-150.
12. Daengsaen, J, ; Khemphet ,A. On The Rate of Convergence of P-Iteration, SP-Iteration, and D-Iteration Methods for Continuous Nondecreasing Functions on Closed Intervals. *Hindawi Abstract and Applied Analysis*. **2018** ,6.
13. Picard, E. Memoire Sur la Theorie des Equations Aux Derivees Partielles Etla Methode des Approximations Successives, J. *Math. Pures Appl. 6*: **1890**.145-210.
14. Ullah, K. ; Arshad, M. New Three-Step Iteration Process and Fxed Point Approximation in Banach Spaces. *Linear. Topological. Algebra*. **2018**. 87-100.
15. Noor, M. A. New approximation schemes for general variational inequalities , *Journal of Mathematical Analysis and Applications* 251: **2000**,217-229.
16. Berinde, V.picard iteration converges faster than Mann iteration for a class of quasi contractive operators,*Fixed Point Theory and Appl2*, **2004**,97-105.
17. Schu, J. Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, B. Aust. Math. Soc., **1991**,*43*, 153–159.
18. Weng, X., Fixed point iteration for local strictly pseudocontractive mapping, *Proc. Amer. Math. Soc.*, **1991**, *113*, 727.