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Bayesian Estimation for Two Parameters of Weibull Distribution under Generalized Weighted Loss Function

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Abstract

In this paper, Bayes estimators for the shape and scale parameters of Weibull distribution have been obtained using the generalized weighted loss function, based on Exponential priors. Lindley's approximation has been used effectively in Bayesian estimation. Based on theMonte Carlo simulation method, those estimators are compared depending on the mean squared errors (MSE's).

Keywords: Weibull Distribution; Bayesian estimation; Exponential prior; Generalized weighted loss function; Lindley's approximation.

1.Introduction

The Weibull Distribution is a continuous, it is one of the best-known lifetime distributions. It is widely used in reliability, Quality Control, weather forecasting and used to describe different types of observed failures of components and phenomena [1]. This distribution is named after Wal Oddi Weibull who described it in details in 1951.

There are some recent works and literature of Weibull distribution: Xie et al. (2002) suggested the three parameters of modified Weibull distribution with a hazard function fashioned like a bathtub [2-3] recommend the Maximum likelihood method to estimate the unknown parameters of Weibull distribution, and [4] presented Bayesian estimation for Weibull distribution under asymmetric and symmetric loss function and that of maximum likelihood estimation.

The probability density function (pdf) of the two parameters Weibull distribution is defined as:

$$f(x \mid \lambda, \vartheta) = \frac{\lambda}{\vartheta}\left(\frac{x}{\vartheta}\right)^{\lambda-1} e^{-(\frac{x}{\vartheta})^{\lambda}} \quad ; \quad x > 0\ , \quad \lambda, \vartheta > 0 \qquad (1)$$

The corresponding cumulative distribution function (CDF) is given by:

$$F(x \mid \lambda, \vartheta) = \int_0^x \frac{\lambda}{\vartheta^{\lambda}}\, t^{\lambda-1} e^{-\left(\frac{t}{\vartheta}\right)^{\lambda}} dx = 1 - e^{-\left(\frac{x}{\vartheta}\right)^{\lambda}}$$

The Reliability function is defined:

$$\mathrm{R}(\mathrm{X}\,; \lambda, \vartheta) = 1 - \mathrm{F}(X\,; \lambda, \vartheta) = e^{-\left(\frac{x}{\vartheta}\right)^{\lambda}}$$

2. Bayesian Estimators

Bayesian estimation is an estimation that aims to minimize the posterior expected value of a loss function. [5] In this suction, Bayesian estimators are obtained based on a new loss function, which is generalized weighted loss function. The Bayesian estimators are derived with assuming the exponential prior for each of λ and ϑ, i.e.:

$\lambda \sim Exponential\ (1/\delta)$ with the following pdf

$$g_1(\lambda) = \delta e^{-\delta\lambda} \qquad \delta > 0$$

$\vartheta \sim Exponential\ (1/\gamma)$ with the pdf defined as

$$g_2(\vartheta) = \gamma e^{-\gamma\vartheta} \qquad \gamma > 0$$

λ and ϑ are independent of each other. Therefore, the joint exponential prior is:

$$g(\lambda, \vartheta) = g_1(\lambda)\ g_2(\vartheta) = \delta\, e^{-\delta\lambda}\ \gamma e^{-\gamma\vartheta}$$

Hence, the posterior distribution of ϑ and λ is given by

$$\pi(\lambda, \vartheta; \underline{X}) = \frac{e^{-(\gamma\vartheta+\delta\lambda)} \left(\frac{\lambda}{\vartheta^{\lambda}}\right)^n \prod_{i=1}^n x_i^{\lambda-1} \exp\left(-\frac{\sum_{i=1}^n x_i^{\lambda}}{\vartheta^{\lambda}}\right)}{\int_0^\infty \int_0^\infty e^{-(\gamma\vartheta+\delta\lambda)} \left(\frac{\lambda}{\vartheta^{\lambda}}\right)^n \prod_{i=1}^n x_i^{\lambda-1} \exp\left(-\frac{\sum_{i=1}^n x_i^{\lambda}}{\vartheta^{\lambda}}\right) d\lambda d\vartheta}$$

3. Bayesian Estimator under Generalized Weighted Loss Function

[6] suggested a new loss function in estimating the scale parameter for Laplace distribution, which is called generalized weighted loss function and introduced as follows:

$$L(\hat{\theta}, \theta) = \frac{\left(\sum_{j=0}^k a_j \theta^j\right)(\hat{\theta} - \theta)^2}{\theta^{\tau}}$$

$$a_j > 0, \qquad \theta > 0, \qquad j = 1,2, \dots, k$$

Where k, τ are constants

The risk function $R_{Gw}(\hat{\theta}, \theta)$ can be derived as

$$R_{Gw}(\hat{\theta}, \theta) = \mathrm{E}\,[\mathrm{L}(\hat{\theta}, \theta)] = \int_0^\infty \left[\frac{1}{\theta^{\tau}}\left(\sum_{j=0}^k a_j \theta^j\right)(\hat{\theta} - \theta)^2\right] \pi(\theta \mid \underline{x}) d\theta$$

Then, Bayesian estimator under the generalized weighted error loss function minimizes the risk function, as follows:

$$\hat{\theta}_{Gw} = \frac{a_0 E\left(\frac{1}{\theta^{\tau-1}} \mid x\right) + a_1 E\left(\frac{1}{\theta^{\tau-2}} \mid \underline{x}\right) + \cdots + a_K E\left(\frac{1}{\theta^{\tau-(K+1)}} \mid \underline{x}\right)}{a_0 E\left(\frac{1}{\theta^{\tau}} \mid \underline{x}\right) + a_1 E\left(\frac{1}{\theta^{\tau-1}} \mid \underline{x}\right) + \cdots + a_K E\left(\frac{1}{\theta^{\tau-K}} \mid \underline{x}\right)} \tag{2}$$

Whereas $a_0, a_1, \ldots, a_k$ are constants.

a-Bayesian Estimator for ϑ under Generalized Weighted Loss Function

Bayesian estimation for ϑ under generalized weighted loss function can be obtained as follows:

By suppose that, k=1 and $\tau = 0$ then,

$$\hat{\vartheta}_{10} = \frac{a_0 E(\vartheta \mid \underline{X}) + a_1 E(\vartheta^2 \mid \underline{x})}{a_0 + a_1 E(\vartheta \mid \underline{x})} \tag{3}$$

Assumed that w (λ, ϑ) be any function for λ, ϑ. Therefore:

$$E[w(\lambda,\vartheta)] = \int_0^\infty \int_0^\infty w(\lambda,\vartheta)\, \pi(\lambda,\vartheta)\, d\lambda\, d\vartheta$$

$$= \int_0^\infty \int_0^\infty w(\lambda,\vartheta) \frac{L(x_1, x_2, \ldots, x_n; \lambda, \vartheta)\pi(\lambda,\vartheta)\, d\lambda\, d\vartheta}{\int_0^\infty \int_0^\infty L(x_1, x_2, \ldots, x_n; \lambda, \vartheta)\pi(\lambda,\vartheta)\, d\lambda\, d\vartheta}$$

$$= \frac{\int_0^\infty \int_0^\infty w(\lambda,\vartheta) L(x_1, x_2, \ldots, x_n; \lambda, \vartheta)\pi(\lambda,\vartheta)\, d\lambda\, d\vartheta}{\int_0^\infty \int_0^\infty L(x_1, x_2, \ldots, x_n; \lambda, \vartheta)\pi(\lambda,\vartheta)\, d\lambda\, d\vartheta}$$

$$E[\vartheta \mid \underline{X}] = \frac{\int_0^\infty \int_0^\infty \vartheta \;\; e^{-(\gamma\vartheta+\delta\lambda)} \left(\frac{\lambda}{\vartheta^\lambda}\right)^n \prod_{i=1}^n x_i^{\lambda-1} \exp\left(-\frac{\sum_{i=1}^n x_i^\lambda}{\vartheta^\lambda}\right) d\lambda d\vartheta}{\int_0^\infty \int_0^\infty e^{-(\gamma\vartheta+\delta\lambda)} \left(\frac{\lambda}{\vartheta^\lambda}\right)^n \prod_{i=1}^n x_i^{\lambda-1} \exp\left(-\frac{\sum_{i=1}^n x_i^\lambda}{\vartheta^\lambda}\right) d\lambda\, d\vartheta}$$

Observed that, it is difficult to obtain the solution of the ratio of two integrals. Hence, the solution will be approximately by using Lindley's approximation [7], as follows:

$$E[\vartheta \mid \underline{X}] \approx \hat{\vartheta} + p_1 w_1 \sigma_{11} + \frac{1}{2}(L_{30} w_1 \sigma_{11}^2) + \frac{1}{2}(L_{12} w_1 \sigma_{11} \sigma_{22}) \tag{4}$$

Where,

Assuming that
w (λ, ϑ) = ϑ
Thus,

$$w_1 = \frac{\partial w(\lambda,\vartheta)}{\partial\vartheta} = \frac{\partial}{\partial\vartheta}(\vartheta) = 1 \tag{5}$$

$$L_{ij} = \frac{\partial^{i+j}}{\partial\vartheta^i\, \partial\lambda^j} \ln L(\lambda,\vartheta) \qquad\qquad i,j = 0,1,2,3$$

$$= \frac{\partial^{i+j}}{\partial\vartheta^i\, \partial\lambda^j}\left[n\ln(\lambda) - n\,\lambda\ln(\vartheta) + (\lambda-1)\sum_{i=1}^n \ln(x_i) - \sum_{i=1}^n \left(\frac{x_i}{\vartheta}\right)^\lambda\right]$$

$$L_{12} = \frac{\partial^3 lnL(\lambda,\vartheta)}{\partial\vartheta\, \partial\lambda^2} = \frac{\lambda}{\vartheta}\sum_{i=1}^n \left(\frac{x}{\vartheta}\right)^\lambda \left(\ln\left(\frac{x_i}{\vartheta}\right)\right)^2 + \frac{2}{\vartheta}\sum_{i=1}^n \left(\frac{x_i}{\vartheta}\right)^\lambda \ln\left(\frac{x_i}{\vartheta}\right) \tag{6}$$

$$L_{20} = \frac{\partial^2 lnL(\lambda,\vartheta)}{\partial\vartheta^2} = \frac{n\,\lambda}{\vartheta^2} - \frac{\lambda(\lambda+1)}{\vartheta^2}\sum_{i=1}^n \left(\frac{x_i}{\vartheta}\right)^\lambda \quad , \quad L_{02} = \frac{\partial^2 lnL(\lambda,\vartheta)}{\partial\lambda^2} = -\frac{n}{\lambda^2} - \sum_{i=1}^n \left(\frac{x_i}{\vartheta}\right)^\lambda \left(\ln\left(\frac{x_i}{\vartheta}\right)\right)^2$$

$$L_{30}=\frac{\partial^3 \ln L(\lambda,\vartheta)}{\partial\vartheta^3}=-\frac{2n\lambda}{\vartheta^3}+\frac{\lambda(\lambda+1)(\lambda+2)}{\vartheta^3}\sum_{i=1}^{n}\left(\frac{x_i}{\vartheta}\right)^{\lambda} \quad (7)$$

$$\sigma_{11}=-\frac{1}{L_{20}}=\frac{-\vartheta^2}{n\lambda-\lambda(\lambda+1)\sum_{i=1}^{n}\left(\frac{x_i}{\vartheta}\right)^{\lambda}} \quad (8)$$

$$\sigma_{22}=-\frac{1}{\mathrm{L}_{02}}=\frac{\lambda^2}{n+\lambda^2\sum_{i=1}^{n}\left(\frac{x_i}{\vartheta}\right)^{\lambda}\left(\ln\left(\frac{x_i}{\lambda}\right)\right)^2} \quad (9)$$

We have,

$g(\lambda, \vartheta)=\gamma\,\delta\; e^{-(\gamma\vartheta+\delta\lambda)}$

$P=\ln g(\lambda,\vartheta)=\ln\gamma+\ln\delta-\gamma\,\vartheta-\delta\,\lambda$

$$\mathrm{p}_1=\frac{\partial \mathrm{p}}{\partial\vartheta}=-\gamma \quad (10)$$

Substituting (5), (6), (7), (8), (9) and (10) into (4), yields:

$$\mathrm{E}\,[\vartheta|\underline{X}]\approx\hat{\vartheta}-\gamma\sigma_{11}+\frac{1}{2}(\mathrm{L}_{30}\sigma_{11}^2)+\frac{1}{2}(\mathrm{L}_{12}\sigma_{11}\sigma_{22}) \quad (11)$$

Similarly, Lindley's approximation for $\mathrm{E}\left[\vartheta^2\,\middle|\,\underline{X}\right]$ is given by:

$$\mathrm{E}\left[\vartheta^2\,\middle|\,\underline{X}\right]\approx\hat{\vartheta}^2+\frac{1}{2}(\mathrm{w}_{11}\sigma_{11})+\mathrm{p}_1\mathrm{w}_1\sigma_{11}+\frac{1}{2}(\mathrm{L}_{30}\mathrm{w}_1\sigma_{11}^2)+\frac{1}{2}(\mathrm{L}_{12}\mathrm{w}_1\sigma_{11}\sigma_{22}) \quad (12)$$

Assuming that, $\mathrm{w}(\lambda,\vartheta)=\vartheta^2$

$$\mathrm{w}_1=\frac{\partial\mathrm{w}(\lambda,\vartheta)}{\partial\vartheta}=2\vartheta \quad (13)$$

$$\mathrm{w}_{11}=\frac{\partial^2\mathrm{w}(\lambda,\vartheta)}{\partial\vartheta^2}=2 \quad (14)$$

Substituting (6), (7), (8), (9), (10), (13) and (14) into (12), yields:

$$\mathrm{E}\left[\vartheta^2\,\middle|\,\underline{X}\right]\approx\hat{\vartheta}^2+(1-2\gamma\hat{\vartheta})\,\sigma_{11}+\hat{\vartheta}[\,(\mathrm{L}_{30}\sigma_{11}^2)+(\mathrm{L}_{12}\sigma_{11}\sigma_{22})\,] \quad (15)$$

After substituting (11) and (15) into (2), yields:

$$\hat{\vartheta}_{10}=\frac{(\mathrm{a}_0\hat{\vartheta}+\mathrm{a}_1\hat{\vartheta}^2)+[-\mathrm{a}_0\gamma+\mathrm{a}_1(1-2\hat{\vartheta}\gamma)]\sigma_{11}+(\frac{\mathrm{a}_0}{2}+\mathrm{a}_1\hat{\vartheta})[\mathrm{L}_{30}\sigma_{11}^2+\mathrm{L}_{12}\sigma_{11}\sigma_{22}]}{\mathrm{a}_0+\mathrm{a}_1[\hat{\vartheta}-\gamma\sigma_{11}+\frac{1}{2}(\mathrm{L}_{30}\sigma_{11}^2)+\frac{1}{2}(\mathrm{L}_{12}\sigma_{11}\sigma_{22})\;]} \quad (16)$$

Now, another estimator under generalized weighted loss function will be derived, when letting k=1 and τ =1, gives:

$$\hat{\vartheta}_{11}=\frac{\mathrm{a}_0+\mathrm{a}_1\mathrm{E}(\vartheta|\underline{\mathrm{x}})}{\mathrm{a}_0\mathrm{E}\left(\frac{1}{\vartheta}|\underline{\mathrm{x}}\right)+\mathrm{a}_1}$$

Similarly, the Lindley's approximation for $\mathrm{E}\left[\frac{1}{\vartheta}\,\middle|\,\underline{X}\right]$ is given by:

$$\mathrm{E}\left[\frac{1}{\vartheta}\,\middle|\,\underline{X}\right]\approx\frac{1}{\hat{\vartheta}}+\frac{1}{2}(\mathrm{w}_{11}\sigma_{11})+\mathrm{p}_1\mathrm{w}_1\sigma_{11}+\frac{1}{2}(\mathrm{L}_{30}\mathrm{w}_1\sigma_{11}^2)+\frac{1}{2}(\mathrm{L}_{12}\mathrm{w}_1\sigma_{11}\sigma_{22}) \quad (17)$$

Assuming that, $\mathrm{w}(\lambda,\vartheta)=\frac{1}{\vartheta}$

$$\mathrm{w}_1=\frac{\partial\mathrm{w}(\lambda,\vartheta)}{\partial\vartheta}=\frac{-1}{\vartheta^2} \quad (18)$$

$$w_{11} = \frac{\partial^2 w(\lambda,\vartheta)}{\partial\vartheta^2} = \frac{2}{\vartheta^3} \quad (19)$$

Substituting (6), (7), (8), (9), (10), (18) and (19) into (17), yields:

$$E\left[\frac{1}{\vartheta} \mid \underline{X}\right] \approx \frac{1}{\hat{\vartheta}} + \left(\frac{1}{\hat{\vartheta}^3} + \frac{\gamma}{\hat{\vartheta}^2}\right)\sigma_{11} - \frac{1}{2\hat{\vartheta}^2}\left[(L_{30}\sigma_{11}^2) + (L_{12}\sigma_{11}\sigma_{22})\right] \quad (20)$$

After substituting (11) and (20) into (2) give up,

$$\hat{\vartheta}_{11} = \frac{a_0 + a_1\left[\hat{\vartheta} - \gamma\sigma_{11} + \frac{1}{2}(L_{30}\sigma_{11}^2) + \frac{1}{2}(L_{12}\sigma_{11}\sigma_{22})\right]}{a_1 + a_0\left[\frac{1}{\hat{\vartheta}} + \left(\frac{1}{\hat{\vartheta}^3} + \frac{\gamma}{\hat{\vartheta}^2}\right)\sigma_{11} - \frac{1}{2\hat{\vartheta}^2}\left[(L_{30}\sigma_{11}^2) + (L_{12}\sigma_{11}\sigma_{22})\right]\right]} \quad (21)$$

Another estimator under generalized weighted loss function will be derived, when letting k=1 and τ =2, gives:

$$\hat{\vartheta}_{12} \approx \frac{a_0 E\left(\frac{1}{\vartheta} | \underline{x}\right) + a_1}{a_0 E\left(\frac{1}{\vartheta^2} | \underline{x}\right) + a_1 E\left(\frac{1}{\vartheta} | \underline{x}\right)}$$

Similarly, the Lindley's approximation for E $[\frac{1}{\vartheta^2} | \underline{X}]$ is given by:

$$E\left[\frac{1}{\vartheta^2} | \underline{X}\right] \approx \frac{1}{\hat{\vartheta}^2} + \frac{1}{2}(w_{11}\sigma_{11}) + p_1 w_1 \sigma_{11} + \frac{1}{2}(L_{30} w_1 \sigma_{11}^2) + \frac{1}{2}(L_{12} w_1 \sigma_{11}\sigma_{22}) \quad (22)$$

Assuming that, w $(\lambda, \vartheta) = \frac{1}{\vartheta^2}$

$$w_1 = \frac{\partial w(\lambda,\vartheta)}{\partial\vartheta} = \frac{-2}{\vartheta^3} \quad (23)$$

$$w_{11} = \frac{\partial^2 w(\lambda,\vartheta)}{\partial\vartheta^2} = \frac{6}{\vartheta^4} \quad (24)$$

Substituting (6), (7), (8), (9), (10), (23) and (24) into (22), yields:

$$E\left[\frac{1}{\vartheta^2} | \underline{X}\right] \approx \frac{1}{\hat{\vartheta}^2} + \left(\frac{3}{\hat{\vartheta}^4} + \frac{2\gamma}{\hat{\vartheta}^3}\right)\sigma_{11} - \frac{1}{\hat{\vartheta}^3}\left[(L_{30}\sigma_{11}^2) + (L_{12}\sigma_{11}\sigma_{22})\right] \quad (25)$$

After substituting (20) and (25) into (2) give up,

$$\hat{\vartheta}_{12} \approx \frac{a_1 + a_0\left[\frac{1}{\hat{\vartheta}} + \left(\frac{1}{\hat{\vartheta}^3} + \frac{\gamma}{\hat{\vartheta}^2}\right)\sigma_{11} - \frac{1}{2\hat{\vartheta}^2}\left[(L_{30}\sigma_{11}^2) + (L_{12}\sigma_{11}\sigma_{22})\right]\right]}{\left(\frac{a_0}{\hat{\vartheta}^2} + \frac{a_1}{\hat{\vartheta}}\right) + z_1\sigma_{11} - \left(\frac{a_0}{\hat{\vartheta}^3} + \frac{a_1}{2\hat{\vartheta}^2}\right)\left[(L_{30}\sigma_{11}^2) + (L_{12}\sigma_{11}\sigma_{22})\right]} \quad (26)$$

Whereas, $z_1 = a_0\left(\frac{3}{\hat{\vartheta}^4} + \frac{2\gamma}{\hat{\vartheta}^3}\right) + a_1\left(\frac{1}{\hat{\vartheta}^3} + \frac{\gamma}{\hat{\vartheta}^2}\right)$.

Another estimator under generalized weighted loss function will be derived, when letting k=1 and τ =0, gives:

$$\hat{\vartheta}_{20} = \frac{a_0 E(\vartheta|\underline{x}) + a_1 E(\vartheta^2|\underline{x}) + a_2 E(\vartheta^3|\underline{x})}{a_0 + a_1 E(\vartheta|\underline{x}) + a_2 E(\vartheta^2|\underline{x})}$$

Similarly, the Lindley's approximation for $E[\vartheta^3|\underline{X}]$ is given by:

$$E[\vartheta^3|\underline{X}] \approx \hat{\vartheta}^3 + \frac{1}{2}(w_{11}\sigma_{11}) + p_1 w_1 \sigma_{11} + \frac{1}{2}(L_{30} w_1 \sigma_{11}^2) + \frac{1}{2}(L_{12} w_1 \sigma_{11}\sigma_{22}) \quad (27)$$

Assuming that, $w(\lambda, \vartheta) = \vartheta^3$

$$w_1 = \frac{\partial w(\lambda,\vartheta)}{\partial\vartheta} = 3\vartheta^2 \tag{28}$$

$$w_{11} = \frac{\partial^2 w(\lambda,\vartheta)}{\partial\vartheta^2} = 6\vartheta \tag{29}$$

Substituting (6), (7), (8), (9), (10), (28) and (29) into (27), yields:

$$E[\vartheta^3|\underline{X}] \approx \hat{\vartheta}^3 + \left(3\hat{\vartheta} - 3\gamma\hat{\vartheta}^2\right)\sigma_{11} + \frac{3\hat{\vartheta}^2}{2}[(L_{30}\sigma_{11}^2) + (L_{12}\sigma_{11}\sigma_{22})] \tag{30}$$

After substituting (11), (15) and (30) into (2) give up,

$$\hat{\vartheta}_{20} = \frac{z_4 + z_3\sigma_{11} + z_2[(L_{30}\sigma_{11}^2) + (L_{12}\sigma_{11}\sigma_{22})]}{z_5 + z_6\,\sigma_{11} + \left(\frac{a_1}{2} + a_2\hat{\vartheta}\right)[(L_{30}\sigma_{11}^2) + (L_{12}\sigma_{11}\sigma_{22})]} \tag{31}$$

Where,

$z_2 = \frac{a_0}{2} + a_1\hat{\vartheta} - \frac{3a_2\hat{\vartheta}^2}{2}$, $z_3 = a_0\gamma + a_1(1 - 2\gamma) + a_2(3\hat{\vartheta} - 3\gamma\hat{\vartheta}^2)$

$z_4 = a_0\hat{\vartheta} + a_1\hat{\vartheta}^2 + a_2\hat{\vartheta}^3$, $z_5 = a_0 + a_1\hat{\vartheta} + a_2\,\hat{\vartheta}^2$, $z_6 = -a_1\gamma + a_2(1 - 2\gamma\hat{\vartheta})$

Now, when k=2 and τ =1, gives:

$$\hat{\vartheta}_{21} = \frac{a_0 + a_1 E(\vartheta|\underline{x}) + a_2 E(\vartheta^2|\underline{x})}{a_0 E\left(\frac{1}{\vartheta}|\underline{x}\right) + a_1 + a_2 E(\vartheta|\underline{x})}$$

After substituting (11), (15) and (20) into (2), yields:

$$\hat{\vartheta}_{21} = \frac{z_5 + z_6\sigma_{11} + \left(\frac{a_1}{2} + a_2\hat{\vartheta}\right)[(L_{30}\sigma_{11}^2) + (L_{12}\sigma_{11}\sigma_{22})]}{\left(\frac{a_0}{\hat{\vartheta}} + a_1 + a_2\hat{\vartheta}\right) + \left(\frac{a_0}{\hat{\vartheta}^3} + \frac{a_0\gamma}{\hat{\vartheta}^2} - a_2\gamma\right)\sigma_{11} + \left(\frac{a_0}{2\hat{\vartheta}^2} + \frac{a_2}{2}\right)[(L_{30}\sigma_{11}^2) + (L_{12}\sigma_{11}\sigma_{22})]} \tag{32}$$

Where, $z_5 = a_0 + a_1\hat{\vartheta} + a_2\,\hat{\vartheta}^2$, $z_6 = -a_1\gamma + a_2(1 - 2\gamma\hat{\vartheta})$

When k=2 and τ =2, gives:

$$\hat{\vartheta}_{22} = \frac{a_0 E\left(\frac{1}{\vartheta}|\underline{x}\right) + a_1 + a_2 E(\vartheta|\underline{x})}{a_0 E\left(\frac{1}{\vartheta^2}|\underline{x}\right) + a_1 E\left(\frac{1}{\vartheta}|\underline{x}\right) + a_2}$$

After substituting (11), (20) and (25) into (2), yields:

$$\hat{\vartheta}_{22} = \frac{\left(\frac{a_0}{\hat{\vartheta}} + a_1 + a_2\hat{\vartheta}\right) + \left(\frac{a_0}{\hat{\vartheta}^3} + \frac{a_0\gamma}{\hat{\vartheta}^2} - a_2\gamma\right)\sigma_{11} + \left(\frac{a_0}{2\hat{\vartheta}^2} + \frac{a_2}{2}\right)[(L_{30}\sigma_{11}^2) + (L_{12}\sigma_{11}\sigma_{22})]}{\left(\frac{a_0}{\hat{\vartheta}^2} + \frac{a_1}{\hat{\vartheta}} + a_2\right) + y_1\sigma_{11} + \left(\frac{-a_0}{\hat{\vartheta}^3} + \frac{a_1}{2\hat{\vartheta}^2}\right)[(L_{30}\sigma_{11}^2) + (L_{12}\sigma_{11}\sigma_{22})]} \tag{33}$$

Where, $y_1 = a_0\left(\frac{3}{\hat{\vartheta}^4} + \frac{a_1}{\hat{\vartheta}^3}\right) + a_1\left(\frac{1}{\hat{\vartheta}^3} + \frac{\gamma}{\hat{\vartheta}^2}\right)$.

b- Bayesian Estimation for λ under Generalized Weighted Loss Function

We can obtain Bayesian estimation for λ under generalized weighted loss function, when letting, k=1 and τ = 0, gives:

$$\hat{\lambda}_{10} = \frac{a_0 E(\lambda|\underline{x}) + a_1 E(\lambda^2|\underline{x})}{a_0 + a_1 E(\lambda|\underline{x})}$$

The Lindley's approximation for $E[\lambda \mid \underline{X}]$ is given by:

$$E[\lambda \mid \underline{X}] \approx \hat{\lambda} + p_2 w_2 \sigma_{22} + \frac{1}{2}(L_{03} w_2 \sigma_{22}^2) + \frac{1}{2}(L_{21} w_2 \sigma_{11} \sigma_{22}) \quad (34)$$

Where,

Assume that, $W(\lambda, \vartheta) = \lambda$

$$w_1 = \frac{\partial w(\lambda, \vartheta)}{\partial \vartheta} = \frac{\partial}{\partial \vartheta}(\lambda)$$

$$= 0 = w_{11}$$

$$w_2 = \frac{\partial w(\lambda, \vartheta)}{\partial \lambda} = 1 \quad (35)$$

$$L_{03} = \frac{\partial^3 L(\lambda, \vartheta)}{\partial \lambda^3} = \frac{2n}{\lambda^3} - \sum_{i=1}^{n} \left(\frac{x_i}{\vartheta}\right)^{\lambda} \left(\ln\left(\frac{x_i}{\vartheta}\right)\right)^3 \quad (36)$$

$$L_{21} = \frac{\partial^3 L(\lambda, \vartheta)}{\partial \vartheta^2 \partial \lambda} = \frac{n}{\vartheta^2} - \frac{2\lambda+1}{\vartheta^2} \sum_{i=1}^{n} \left(\frac{x_i}{\vartheta}\right)^{\lambda} - \frac{\lambda(\lambda+1)}{\vartheta^2} \sum_{i=1}^{n} \left(\frac{x_i}{\vartheta}\right)^{\lambda} \ln\left(\frac{x_i}{\vartheta}\right) \quad (37)$$

$P = \ln g(\lambda, \vartheta) = \ln(\gamma) + \ln(\delta) - \gamma\vartheta - \delta\lambda$

$$p_2 = \frac{\partial p}{\partial \lambda} = -\delta \quad (38)$$

Substituting (8), (9), (35), (36), (37) and (38) into (34), gives us:

$$E[\lambda \mid \underline{X}] \approx \hat{\lambda} - \delta \sigma_{22} + \frac{1}{2}[(L_{03} \sigma_{22}^2) + (L_{21} \sigma_{11} \sigma_{22})] \quad (39)$$

Similarly, the Lindley's approximation for $E[\lambda^2 \mid \underline{X}]$ is given by:

$$E[\lambda^2 \mid \underline{X}] \approx \hat{\lambda}^2 + \frac{1}{2}(w_{22}\sigma_{22}) + p_2 w_2 \sigma_{22} + \frac{1}{2}(L_{03} w_2 \sigma_{22}^2) + \frac{1}{2}(L_{21} w_2 \sigma_{11} \sigma_{22}) \quad (40)$$

Assume that, $W(\lambda, \vartheta) = \lambda^2$

$$w_2 = \frac{\partial w(\lambda, \vartheta)}{\partial \lambda} = 2\lambda \quad (41)$$

$$w_{22} = \frac{\partial^2 w(\lambda, \vartheta)}{\partial \lambda^2} = 2 \quad (42)$$

Substituting (8), (9), (36), (37), (38), (41) and (42) into (40), yields:

$$E[\lambda^2 \mid \underline{X}] \approx \hat{\lambda}^2 + (1 - 2\delta\hat{\lambda})\sigma_{22} + \hat{\lambda}[(L_{03}\sigma_{22}^2) + (L_{21}\sigma_{11}\sigma_{22})] \quad (43)$$

After substituting (39) and (43) into (2) give up,

$$\hat{\lambda}_{10} = \frac{(a_0\hat{\lambda} + a_1\hat{\lambda}^2) + [a_1(1 - 2\delta\hat{\lambda}) - a_0\delta]\sigma_{22} + (\frac{a_0}{2} + a_1\hat{\lambda})[(L_{03}\sigma_{22}^2) + (L_{21}\sigma_{11}\sigma_{22})\ ]}{(a_0 + a_1\hat{\lambda}) - a_1\delta\,\sigma_{22} + \frac{a_1}{2}[(L_{03}\sigma_{22}^2) + (L_{21}\sigma_{11}\sigma_{22})]} \quad (44)$$

Now, another estimator under generalized weighted loss function will be derived by letting, k=1 and τ =1, gives:

$$\hat{\lambda}_{11}=\frac{a_0+a_1E(\lambda|\underline{x})}{a_0E\left(\frac{1}{\lambda}|\underline{x}\right)+a_1}$$

Similarly, the Lindley's approximation for $E\left[\frac{1}{\lambda}\mid \underline{X}\right]$ is given by:

$$E\left[\frac{1}{\lambda}\mid \underline{X}\right]\approx\frac{1}{\hat{\lambda}}+\frac{1}{2}(w_{22}\sigma_{22})+p_2w_2\sigma_{22}+\frac{1}{2}(L_{03}w_2\sigma_{22}^2)+\frac{1}{2}(L_{21}w_2\sigma_{11}\sigma_{22}) \tag{45}$$

Assume that, W (λ, ϑ) $=\frac{1}{\lambda}$

$$w_2=\frac{\partial w(\lambda,\vartheta)}{\partial\lambda}=\frac{-1}{\lambda^2} \tag{46}$$

$$w_{22}=\frac{\partial^2 w(\lambda,\vartheta)}{\partial\lambda^2}=\frac{2}{\lambda^3} \tag{47}$$

Substituting (8), (9), (36), (37), (38), (46) and (47) into (45), yields:

$$E\left[\frac{1}{\lambda}\mid \underline{X}\right]\approx\frac{1}{\hat{\lambda}}+\left(\frac{1}{\hat{\lambda}^3}+\frac{\delta}{\hat{\lambda}^2}\right)\sigma_{22}-\frac{1}{2\hat{\lambda}^2}[(L_{03}\sigma_{22}^2)+(L_{21}\sigma_{11}\sigma_{22})] \tag{48}$$

After substituting (39) and (48) into (2) give up,

$$\hat{\lambda}_{11}=\frac{(a_0+a_1\hat{\lambda})-a_1\delta\,\sigma_{22}+\frac{a_1}{2}[(L_{03}\sigma_{22}^2)+(L_{21}\sigma_{11}\sigma_{22})]}{a_1+\frac{a_0}{\hat{\lambda}}+a_0\left(\frac{1}{\hat{\lambda}^3}+\frac{\delta}{\hat{\lambda}^2}\right)\sigma_{22}-\frac{a_0}{2\hat{\lambda}^2}[(L_{03}\sigma_{22}^2)+(L_{21}\sigma_{11}\sigma_{22})]} \tag{49}$$

Another estimator under generalized weighted loss function will be derived, when letting k=1 and τ =2, gives:

$$\hat{\lambda}_{12}=\frac{a_0E\left(\frac{1}{\lambda}|\underline{x}\right)+a_1}{a_0E\left(\frac{1}{\lambda^2}|\underline{x}\right)+a_1E\left(\frac{1}{\lambda}|\underline{x}\right)}$$

Similarly, the Lindley's approximation for $E\left[\frac{1}{\lambda^2}\mid \underline{X}\right]$ is given by:

$$E\left[\frac{1}{\lambda^2}\mid \underline{X}\right]\approx\frac{1}{\hat{\lambda}^2}+\frac{1}{2}(w_{22}\sigma_{22})+p_2w_2\sigma_{22}+\frac{1}{2}(L_{03}w_2\sigma_{22}^2)+\frac{1}{2}(L_{21}w_2\sigma_{11}\sigma_{22}) \tag{50}$$

Assume that, W (λ, ϑ) = $\frac{1}{\lambda^2}$

$$w_2=\frac{\partial w(\lambda,\vartheta)}{\partial\lambda}=\frac{-2}{\lambda^3} \tag{51}$$

$$w_{22}=\frac{\partial^2 w(\lambda,\vartheta)}{\partial\lambda^2}=\frac{6}{\lambda^4} \tag{52}$$

Substituting (8), (9), (36), (37), (38), (51) and (52) into (50), yields:

$$E\left[\frac{1}{\lambda^2}\mid \underline{X}\right]\approx\frac{1}{\hat{\lambda}^2}+\left(\frac{3}{\hat{\lambda}^4}+\frac{2\delta}{\hat{\lambda}^3}\right)\sigma_{22}-\frac{1}{\hat{\lambda}^3}[(L_{03}\sigma_{22}^2)+(L_{21}\sigma_{11}\sigma_{22})] \tag{53}$$

After substituting (48) and (53) into (2) give up,

$$\hat{\lambda}_{12}=\frac{a_1+\frac{a_0}{\hat{\lambda}}+a_0\left(\frac{1}{\hat{\lambda}^3}+\frac{\delta}{\hat{\lambda}^2}\right)\sigma_{22}+\frac{a_0}{2\hat{\lambda}^2}[(L_{03}\sigma_{22}^2)+(L_{21}\sigma_{11}\sigma_{22})]}{\left(\frac{a_0}{\hat{\lambda}^2}+\frac{a_1}{\hat{\lambda}}\right)+y_2\sigma_{22}-\left(\frac{a_0}{\hat{\lambda}^3}+\frac{a_1}{2\hat{\lambda}^2}\right)[(L_{03}\sigma_{22}^2)+(L_{21}\sigma_{11}\sigma_{22})]} \tag{54}$$

Where, $y_2 = a_0\left(\frac{3}{\hat{\lambda}^4}+\frac{2\delta}{\hat{\lambda}^3}\right)+a_1(\frac{1}{\hat{\lambda}^3}+\frac{\delta}{\hat{\lambda}^2})$.

Now, when K=1 and τ =0, gives:

$$\hat{\lambda}_{20} = \frac{a_0E(\lambda|\underline{x})+a_1E(\lambda^2|\underline{x})+a_2E(\lambda^3|\underline{x})}{a_0+a_1E(\lambda|\underline{x})+a_2E(\lambda^2|\underline{x})}$$

Similarly, the Lindley's approximation for $E\left[\lambda^3 \mid \underline{X}\right]$ is given by:

$$E\left[\lambda^3 \mid \underline{X}\right] \approx \hat{\lambda}^3 + \frac{1}{2}(w_{22}\sigma_{22}) + p_2 w_2\sigma_{22} + \frac{1}{2}(L_{03}w_2\sigma_{22}^2) + \frac{1}{2}(L_{21}w_2\sigma_{11}\sigma_{22}) \quad (55)$$

Assume that, w (λ, ϑ) = λ^3

$$w_2 = \frac{\partial w(\lambda,\vartheta)}{\partial\lambda} = 3\lambda^2 \quad (56)$$

$$w_{22} = \frac{\partial^2 w(\lambda,\vartheta)}{\partial\lambda^2} = 6\lambda \quad (57)$$

Substituting (8), (9), (36), (37), (38), (56) and (57) into (55), gives us:

$$E\left[\lambda^3 \mid \underline{X}\right] \approx \hat{\lambda}^3 + \left(3\hat{\lambda} - 3\delta\hat{\lambda}^2\right)\sigma_{22} + \frac{3\hat{\lambda}^2}{2}[(L_{03}\sigma_{22}^2) + (L_{21}\sigma_{11}\sigma_{22})] \quad (58)$$

After substituting (39), (43) and (58) into (2) give up,

$$\hat{\lambda}_{20} = \frac{y_3+y_4\sigma_{22}+\left(\frac{a_0}{2}+a_1\hat{\lambda}+\frac{3a_2}{2}\hat{\lambda}^2\right)[(L_{03}\sigma_{22}^2)+(L_{21}\sigma_{11}\sigma_{22})]}{y_5+[-a_1\delta+a_2(1-2\delta\hat{\lambda})]\sigma_{22}+(\frac{a_1}{2}+a_2\hat{\lambda})[(L_{03}\sigma_{22}^2)+(L_{21}\sigma_{11}\sigma_{22})]} \quad (59)$$

Where, $y_3 = (a_0\hat{\lambda} + a_1\hat{\lambda}^2 + a_2\hat{\lambda}^3)$, $y_5 = (a_0 + a_1\hat{\lambda} + a_2\hat{\lambda}^2)$, $y_4 = -a_0\delta + a_1(1-2\delta\hat{\lambda}) + a_2(3\hat{\lambda} - 3\delta\hat{\lambda}^2$.

Another estimator under generalized weighted loss function will be derived, when letting

k=2 and τ =1, gives:

$$\hat{\lambda}_{21} = \frac{a_0 + a_1E(\lambda|\underline{x}) + a_2E(\lambda^2|\underline{x})}{a_0E\left(\frac{1}{\lambda}|\underline{x}\right) + a_1 + a_2E(\lambda|\underline{x})}$$

After substituting (39), (43) and (48) into (2) give up,

$$\hat{\lambda}_{21} = \frac{y_5+[-a_1\delta+a_2(1-2\delta\hat{\lambda})]\sigma_{22}+(\frac{a_1}{2}+a_2\hat{\lambda})[(L_{03}\sigma_{22}^2)+(L_{21}\sigma_{11}\sigma_{22})]}{\left(\frac{a_0}{\hat{\lambda}}+a_1+a_2\hat{\lambda}\right)+\left[a_0\left(\frac{1}{\hat{\lambda}^3}+\frac{\delta}{\hat{\lambda}^2}\right)-a_2\delta\right]\sigma_{22}+(\frac{-a_0}{2\hat{\lambda}^2}+\frac{a_2}{2})[(L_{03}\sigma_{22}^2)+(L_{21}\sigma_{11}\sigma_{22})]} \quad (60)$$

Where, $y_5 = \left(a_0 + a_1\hat{\lambda} + a_2\hat{\lambda}^2\right)$.

Now, when k=2 and τ =2, gives:

$$\hat{\lambda}_{22} = \frac{a_0E\left(\frac{1}{\lambda}|\underline{x}\right)+a_1+a_2E(\lambda|\underline{x})}{a_0E\left(\frac{1}{\lambda^2}|\underline{x}\right)+a_1E\left(\frac{1}{\lambda}|\underline{x}\right)+a_2}$$

After substituting (48) and (53) into (2) give up,

$$\hat{\lambda}_{22} = \frac{\left(\frac{a_1}{\hat{\lambda}}+a_1+a_2\hat{\lambda}\right)+\left[a_0\left(\frac{1}{\hat{\lambda}^3}+\frac{\delta}{\hat{\lambda}^2}\right)-a_2a\right]\sigma_{22}+(\frac{-a_0}{2\hat{\lambda}^2}+\frac{a_2}{2})[(L_{03}\sigma_{22}^2)+(L_{21}\sigma_{11}\sigma_{22})]}{\left(\frac{a_0}{\hat{\lambda}^2}+\frac{a_1}{\hat{\lambda}}+a_2\right)+\boldsymbol{y_6}\sigma_{22}-\left(\frac{a_0}{\hat{\lambda}^3}+\frac{a_1}{2\hat{\lambda}^2}\right))[(L_{03}\sigma_{22}^2)+(L_{21}\sigma_{11}\sigma_{22})]} \quad (61)$$

Where, $y_6 = a_0\left(\frac{3}{\hat{\lambda}^4}+\frac{2\delta}{\hat{\lambda}^3}\right)+a_1(\frac{1}{\hat{\lambda}^3}+\frac{\delta}{\hat{\lambda}^2})$

4. Simulation Study

In this section, we employed the Monte – Carlo simulation to compare the performance of different estimates (Bayes Estimators under generalized weighted loss function) for unknown shape and scale parameters of WD based on the mean squared errors, which can be written as:

$$\text{MSE}(\hat{\theta}) = \frac{\sum_{i=1}^{I}\left(\hat{\theta}_i-\theta\right)^2}{I}$$

Where, I is the number of replications.

We generated I = 5000 samples from two parameters WD with different sizes (n = 20, 50, and 100). With assuming (ϑ = 0.7, 1.4) and (λ = 0.5, 1.5).

The values of the prior's parameter of ϑ was chosen as δ = 0.5, 1.5, and for λ ' s prior parameter γ = 0.5, 1.5.

5. Discussion and Conclusion

The expected values and (MSE's) for estimating ϑ and λ are tabulated in **Tables (1-8).**

The following points can summarize the results of the tables:

1. The results of the two parameters of Weibull distribution shows that the expected values for different estimates are close to the real values for all sizes of samples.

2. The best estimator for λ is Bayesian estimation under generalized weighted loss function when (k=2, τ=0) ($\hat{\lambda}_{20}$) with ($\gamma = \delta = 1.5$) when n= 100 for different cases and with all sample sizes, from table (4).

3. The best estimator for ϑ is Bayesian estimation under generalized weighted loss function when (k=1, τ=2) ($\hat{\vartheta}_{20}$) with ($\gamma, \delta = 1.5$) when n=20 for different cases and with all sample sizes, from table (8).

4. It is clear that the results for ϑ, λ (expected values and MSE' s) at γ, δ = 1.5 are the best as the corresponding result when (γ, δ = 0.5).

5. It is observed that MSE's of all shape parameter estimators increase with the increase of the value of the shape parameter. Also, MSE values for all scale parameter estimates are increasing with the scale parameter value in all cases.

Table 1: Expected values and MSE's for $\hat{\lambda}$ when $\lambda = 0.5$ and $\vartheta = 0.7$

Estimate	Criterion	n = 20		n = 50		n = 100	
		γ =0.5 δ=0.5	γ =1.5 δ =1.5	γ =0.5 δ=0.5	γ =1.5 δ =1.5	γ =0.5 δ=0.5	γ =1.5 δ =1.5
$\hat{\lambda}_{10}$	Mean	0.503227	0.495727	0.507006	0.504095	0.508469	0.507031
	MSE	0.000025	0.000028	0.000050	0.000018	0.000072	0.000050
$\hat{\lambda}_{11}$	Mean	0.489750	0.482754	0.501506	0.498679	0.505701	0.504283
	MSE	0.000113	0.000310	0.000003	0.000003	0.000033	0.000019
$\hat{\lambda}_{12}$	Mean	0.478153	0.472111	0.496335	0.493675	0.503017	0.501642
	MSE	0.000494	0.000804	0.000015	0.000043	0.000009	0.000003
$\hat{\lambda}_{20}$	Mean	0.518424	0.511074	0.512979	0.510084	0.511433	0.509998
	MSE	0.000380	0.000150	0.000171	0.000104	0.000131	0.000100
$\hat{\lambda}_{21}$	Mean	0.504007	0.496489	0.507316	0.504403	0.508624	0.507185
	MSE	0.000032	0.000023	0.000055	0.000020	0.000075	0.000052
$\hat{\lambda}_{22}$	Mean	0.490440	0.483398	0.501803	0.498966	0.505852	0.504432
	MSE	0.000100	0.000288	0.000004	0.000002	0.000034	0.000020

Table 2: Expected values and MSE's for $\hat{\lambda}$ when $\lambda = 1.5$ and $\vartheta = 0.7$

Estimate	Criterion	n = 20		n = 50		n = 100	
		γ =0.5 δ=0.5	γ =1.5 δ =1.5	γ =0.5 δ=0.5	γ =1.5 δ =1.5	γ =0.5 δ=0.5	γ =1.5 δ =1.5
$\hat{\lambda}_{10}$	Mean	1.484634	1.411570	1.499238	1.472202	1.504442	1.491329
	MSE	0.000332	0.007883	0.000008	0.000776	0.000021	0.000075
$\hat{\lambda}_{11}$	Mean	1.442376	1.376958	1.482284	1.456403	1.496002	1.483166
	MSE	0.003336	0.015360	0.000315	0.001917	0.000016	0.000286
$\hat{\lambda}_{12}$	Mean	1.406637	1.352321	1.466431	1.442446	1.487826	1.475477
	MSE	0.008800	0.022182	0.001133	0.003348	0.000149	0.000607
$\hat{\lambda}_{20}$	Mean	1.531194	1.456616	1.517092	1.489821	1.513201	1.500028
	MSE	0.001342	0.001922	0.000321	0.000106	0.000178	0.000000
$\hat{\lambda}_{21}$	Mean	1.485492	1.412314	1.499574	1.472524	1.504610	1.491494
	MSE	0.000310	0.007750	0.000008	0.000758	0.000022	0.000073
$\hat{\lambda}_{22}$	Mean	1.443126	1.377496	1.482602	1.456688	1.496156	1.483315
	MSE	0.003250	0.015224	0.000303	0.001892	0.000015	0.000280

Table 3: Expected values and MSE's for $\hat{\lambda}$ when $\lambda = 0.5$ and $\vartheta = 1.4$

Estimate	Criterion	n = 20		n = 50		n = 100	
		γ =0.5 δ=0.5	γ =1.5 δ =1.5	γ =0.5 δ=0.5	γ =1.5 δ =1.5	γ =0.5 δ=0.5	γ =1.5 δ =1.5
$\hat{\lambda}_{10}$	Mean	0.503216	0.495738	0.507010	0.504107	0.508471	0.507037
	MSE	0.000025	0.000028	0.000050	0.000018	0.000072	0.000050
$\hat{\lambda}_{11}$	Mean	0.489778	0.482803	0.501525	0.498704	0.505712	0.504299
	MSE	0.000113	0.000309	0.000003	0.000003	0.000033	0.000019
$\hat{\lambda}_{12}$	Mean	0.478211	0.472186	0.496365	0.493712	0.503036	0.501666
	MSE	0.000492	0.000800	0.000015	0.000042	0.000010	0.000003
$\hat{\lambda}_{20}$	Mean	0.518368	0.511040	0.512969	0.510081	0.511426	0.509996
	MSE	0.000378	0.000150	0.000171	0.000104	0.000131	0.000100
$\hat{\lambda}_{21}$	Mean	0.503993	0.496498	0.507320	0.504413	0.508626	0.507192
	MSE	0.000031	0.000023	0.000055	0.000020	0.000075	0.000052
$\hat{\lambda}_{22}$	Mean	0.490466	0.483443	0.501818	0.498989	0.505862	0.504448
	MSE	0.000099	0.000287	0.000004	0.000002	0.000034	0.000020

Table 4: Expected values and MSE's for $\hat{\lambda}$ when λ = 1.5 and ϑ = 1.4

Estimate	Criterion	n = 20		n = 50		n = 100	
		γ =0.5 δ=0.5	γ =1.5 δ =1.5	γ =0.5 δ=0.5	γ =1.5 δ =1.5	γ =0.5 δ=0.5	γ =1.5 δ =1.5
$\hat{\lambda}_{10}$	Mean	1.484249	1.411831	1.499103	1.472299	1.504371	1.491378
	MSE	0.000341	0.007840	0.000008	0.000770	0.000020	0.000075
$\hat{\lambda}_{11}$	Mean	1.442389	1.377537	1.482297	1.456638	1.496009	1.483286
	MSE	0.003334	0.015225	0.000314	0.001897	0.000016	0.000282
$\hat{\lambda}_{12}$	Mean	1.406981	1.353100	1.466585	1.442799	1.487910	1.475667
	MSE	0.008740	0.021964	0.001124	0.003309	0.000147	0.000598
$\hat{\lambda}_{20}$	Mean	1.530402	1.456433	1.516805	1.489758	1.513049	1.499994
	MSE	0.001288	0.001935	0.000311	0.000107	0.000174	0.000000
$\hat{\lambda}_{21}$	Mean	1.485102	1.412570	1.499438	1.472615	1.504535	1.491541
	MSE	0.000318	0.007709	0.000008	0.000753	0.000022	0.000072
$\hat{\lambda}_{22}$	Mean	1.443130	1.378073	1.482612	1.456920	1.496159	1.483440
	MSE	0.003249	0.015092	0.000303	0.001872	0.000015	0.000276

Table 5: Expected values and MSE's for $\hat{\vartheta}$ when ϑ = 0.7 and λ =0.5

Estimate	Criterion	n = 20		n = 50		n = 100	
		γ =0.5 δ=0.5	γ =1.5 δ =1.5	γ =0.5 δ=0.5	γ =1.5 δ =1.5	γ =0.5 δ=0.5	γ =1.5 δ =1.5
$\hat{\vartheta}_{10}$	Mean	0.974984	0.928047	0.845343	0.815308	0.782665	0.764920
	MSE	0.086157	0.061646	0.023881	0.015392	0.007475	0.004655
$\hat{\vartheta}_{11}$	Mean	0.944438	0.845109	0.809432	0.767619	0.759095	0.738223
	MSE	0.080694	0.030476	0.014643	0.005823	0.003931	0.001666
$\hat{\vartheta}_{12}$	Mean	0.840278	0.709600	0.754818	0.709395	0.730536	0.708963
	MSE	0.043190	0.001755	0.004297	0.000292	0.001097	0.000110
$\hat{\vartheta}_{20}$	Mean	0.968277	0.955044	0.863843	0.846656	0.800397	0.787121
	MSE	0.076434	0.070536	0.028897	0.023547	0.010764	0.008180
$\hat{\vartheta}_{21}$	Mean	0.976019	0.930808	0.846594	0.817055	0.783544	0.765949
	MSE	0.086503	0.062885	0.024243	0.015826	0.007628	0.004799
$\hat{\vartheta}_{22}$	Mean	0.947983	0.850438	0.811464	0.769948	0.760205	0.739402
	MSE	0.082355	0.032423	0.015141	0.006200	0.004076	0.001768

Table 6: Expected values and MSE's for $\hat{\vartheta}$ when ϑ = 0.7 and λ =1.5

Estimate	Criterion	n = 20		n = 50		n = 100	
		γ =0.5 δ=0.5	γ =1.5 δ =1.5	γ =0.5 δ=0.5	γ =1.5 δ =1.5	γ =0.5 δ=0.5	γ =1.5 δ =1.5
$\hat{\vartheta}_{10}$	Mean	0.767471	0.756788	0.733466	0.728896	0.721592	0.719300
	MSE	0.004935	0.003493	0.001154	0.000857	0.000470	0.000375
$\hat{\vartheta}_{11}$	Mean	0.754188	0.741865	0.727286	0.722462	0.718422	0.716069
	MSE	0.003213	0.001904	0.000764	0.000515	0.000342	0.000259
$\hat{\vartheta}_{12}$	Mean	0.738110	0.724932	0.720667	0.715726	0.715148	0.712769
	MSE	0.001593	0.000669	0.000435	0.000250	0.000230	0.000163
$\hat{\vartheta}_{20}$	Mean	0.777992	0.769400	0.739284	0.735089	0.724739	0.722537
	MSE	0.006512	0.005171	0.001592	0.001268	0.000618	0.000513
$\hat{\vartheta}_{21}$	Mean	0.767971	0.757360	0.733709	0.729151	0.721721	0.719429
	MSE	0.005006	0.003563	0.001171	0.000873	0.000476	0.000380
$\hat{\vartheta}_{22}$	Mean	0.754805	0.742533	0.727550	0.722733	0.718554	0.716203
	MSE	0.003286	0.001965	0.000779	0.000528	0.000346	0.000264

Table 7: Expected values and MSE's for $\hat{\vartheta}$ when $\vartheta = 1.4$ and $\lambda = 0.5$

Estimate	Criterion	n = 20		n = 50		n = 100	
		γ =0.5 δ=0.5	γ =1.5 δ =1.5	γ =0.5 δ=0.5	γ =1.5 δ =1.5	γ =0.5 δ=0.5	γ =1.5 δ =1.5
$\hat{\vartheta}_{10}$	Mean	1.891493	1.661508	1.649716	1.518772	1.536577	1.463327
	MSE	0.281091	0.094864	0.071919	0.018216	0.020733	0.004713
$\hat{\vartheta}_{11}$	Mean	1.778901	1.396064	1.567025	1.404391	1.487024	1.405830
	MSE	0.200647	0.004210	0.035395	0.000653	0.008776	0.000127
$\hat{\vartheta}_{12}$	Mean	1.528850	1.176655	1.454463	1.299242	1.429922	1.350981
	MSE	0.044197	0.053117	0.005332	0.010630	0.001209	0.002500
$\hat{\vartheta}_{20}$	Mean	1.908679	1.837298	1.697597	1.615853	1.575174	1.517195
	MSE	0.278134	0.222254	0.096621	0.053875	0.033146	0.015314
$\hat{\vartheta}_{21}$	Mean	1.893409	1.666984	1.651239	1.521140	1.537546	1.464519
	MSE	0.282716	0.098366	0.072711	0.018885	0.021015	0.004884
$\hat{\vartheta}_{22}$	Mean	1.783570	1.401668	1.569262	1.406779	1.488186	1.407020
	MSE	0.204673	0.004534	0.036263	0.000725	0.009003	0.000149

Table 8: Expected values and MSE's for $\hat{\vartheta}$ when $\vartheta = 1.4$ and $\lambda = 1.5$

Estimate	Criterion	n = 20		n = 50		n = 100	
		γ =0.5 δ=0.5	γ =1.5 δ =1.5	γ =0.5 δ=0.5	γ =1.5 δ =1.5	γ =0.5 δ=0.5	γ =1.5 δ =1.5
$\hat{\vartheta}_{10}$	Mean	1.512297	1.469568	1.451659	1.433722	1.430547	1.421613
	MSE	0.013801	0.005302	0.002773	0.001172	0.000945	0.000471
$\hat{\vartheta}_{11}$	Mean	1.484740	1.437263	1.439344	1.420686	1.424305	1.415198
	MSE	0.007943	0.001524	0.001603	0.000436	0.000597	0.000232
$\hat{\vartheta}_{12}$	Mean	1.452524	1.403687	1.426311	1.407464	1.417894	1.408746
	MSE	0.003071	0.000048	0.000710	0.000058	0.000322	0.000077
$\hat{\vartheta}_{20}$	Mean	1.534550	1.498874	1.463174	1.446452	1.436663	1.428020
	MSE	0.019556	0.010645	0.004150	0.002237	0.001364	0.000794
$\hat{\vartheta}_{21}$	Mean	1.512832	1.470226	1.451908	1.433990	1.430673	1.421744
	MSE	0.013931	0.005403	0.002800	0.001191	0.000953	0.000477
$\hat{\vartheta}_{22}$	Mean	1.485386	1.437965	1.439612	1.420960	1.424438	1.415334
	MSE	0.008064	0.001581	0.001625	0.000448	0.000603	0.000236

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