Ibn Al Haitham Journal for Pure and Applied Sciences

Journal homepage: http://jih.uobaghdad.edu.iq/index.php/j/index

(θ_1,θ_2)-Derivation Pair on Rings

Mohammed Khalid Shahoodh

Ministry of Education, Ramadi Directorate of Education, Anbar-Iraq

moha861122@yahoo.com

Article history: Received 21 November, 2021, Accepted,25, January, 2022, Published in April 2022.

Doi: 10.30526/35.2.2723

Abstract

Ring theory is one of the influential branches of abstract algebra. In this field, many algebraic problems have been considered by mathematical researchers who are working in this field. However, some new concepts have been created and developed to present some algebraic structures with their properties. Rings with derivations have been studied fifty years ago, especially the relationships between the derivations and the structure of a ring. By using the notatin of derivation, many results have been obtained in the literature with different types of derivations. In this paper, the concept of the derivation theory of a ring has been considered. This study presented the definition of (θ_1,θ_2)-derivation pair and Jordan (θ_1,θ_2)-derivation pair on an associative ring Γ, and the relation between them. Furthermore, we study the concept of prime rings under this notion by introducing some of its properties where θ_1 and θ_2 are two mappings of Γ into itself.

Keywords: Ring Theory, Derivation theory, Prime ring, Derivation pair, Semiprime ring.

1. Introduction

The study of derivation has been initiated from the development of Galois theory and the theory of invariants. This theory has been studied very widely by many researchers on various algebraic structures. The author in [1] studied this topic on H^*-algebra by introducing Jordan $*$-derivation pair. While the authors in [2] considered the topic of BCI-algebras, and the same topic has been investigated on BCC-algebras by the authors in [3]. Moreover, some other works with different algebraic structures can be found in [4-5]. On the other hand, some other studies have studied this topic with some types of rings such as prime and semiprime rings, see [6-8]. [12] presented a new definition of derivation pair instead of Jordan $*$-derivation pair which was provided by [1]. In this paper, we extended the results of [12] by introducing the notion of (θ_1,θ_2)-derivation pair and studied some of its properties.

2. Basic Concepts

This section contains some of the previous results that are needed in this study which are as follows:

Definition 2.1[9]

A non-empty set Γ is said to be an associative ring, if for all $c_1, c_2, c_3 \in \Gamma$ there exist two binary operations defined on Γ and denoted by + and $\cdot$ respectively, such that

i. $c_1 + c_2 = c_2 + c_1$
ii. $(c_1 + c_2) + c_3 = c_1 + (c_2 + c_3)$
iii. $\forall c_1 \in \Gamma \, \exists \, 0 \in \Gamma$ such that $c_1 + 0 = 0 + c_1 = c_1$
iv. $\forall c_1 \in \Gamma \, \exists - c_1 \in \Gamma$ such that $-c_1 + c_1 = c_1 + (-c_1) = 0$
v. $c_1 \cdot c_2 \in \Gamma$
vi. $(c_1 \cdot c_2) \cdot c_3 = c_1 \cdot (c_2 \cdot c_3)$
vii. $c_1 \cdot (c_2 + c_3) = c_1 \cdot c_2 + c_1 \cdot c_3$
viii. $(c_1 + c_2) \cdot c_3 = c_1 \cdot c_3 + c_2 \cdot c_3$.

Definition 2.2 [9]

A ring Γ is said to be a prime ring if for each $c_1, c_2 \in \Gamma$, $c_1 \Gamma c_2 = 0$ implies that $c_1 = 0$ or $c_2 = 0$.

Definition 2.3 [9]

A ring Γ is said to be k-torsion-free if whenever $kc = 0$ implies that $c = 0$, where $c \in \Gamma$ and $k \neq 0$.

Definition 2.4 [10]

Let Γ be a ring, then $[c_1, c_2]$ is said to be Lie product and given as $[c_1, c_2] = c_1c_2 - c_2c_1$ and $c_1 \circ c_2$ is said to be Jordan product and given as $c_1 \circ c_2 = c_1c_2 + c_2c_1$.

Definition 2.5 [11]

The characteristic of a ring Γ (for short char(Γ)) is the smallest positive integer z such that $zr = 0$ with $r \in \Gamma$. Otherwise, char(Γ) = 0.

Definition 2.6 [12]

Let Γ be a ring and let $\mu, \sigma : \Gamma \longrightarrow \Gamma$ be two additive mappings, then μ, σ are said to be derivation pair (μ, σ) if the following equations are holds:

$\mu(uvu) = \mu(u)vu + u\sigma(v)u + uv\mu(u)$, for each $u, v \in \Gamma$

$\sigma(uvu) = \sigma(u)vu + u\mu(v)u + uv\sigma(u)$, for each $u, v \in \Gamma$

and are called Jordan derivation pair if:

$\mu(u^3) = \mu(u)u^2 + u\sigma(u)u + u^2\mu(u)$, for each $u \in \Gamma$

$\sigma(u^3) = \sigma(u)u^2 + u\mu(u)u + u^2\sigma(u)$, for each $u \in \Gamma$.

3. Main Results

In this section, we presented the notion of (θ_1, θ_2)-derivation pair on the ring Γ where θ_1 and θ_2 are two mappings from the ring Γ into itself. Moreover, some properties of this concept have been proved.

Definition 3.1 Let Γ be a ring. Let $\delta_1, \delta_2 : \Gamma \longrightarrow \Gamma$ be additive mappings, then (δ_1, δ_2) is said to be (θ_1, θ_2)-derivation pair, if the following are holds:

$\delta_1(uvu) = \delta_1(u)\theta_1(vu) + \theta_2(u)\delta_2(v)\theta_1(u) + \theta_2(uv)\delta_1(u)$, for each $u, v \in \Gamma$

$\delta_2(uvu) = \delta_2(u)\theta_1(vu) + \theta_2(u)\delta_1(v)\theta_1(u) + \theta_2(uv)\delta_2(u)$, for each $u, v \in \Gamma$.

and are said to be Jordan (θ_1, θ_2)-derivation pair, if the following are holds:

$\delta_1(u^3) = \delta_1(u)\theta_1(u^2) + \theta_2(u)\delta_2(u)\theta_1(u) + \theta_2(u^2)\delta_1(u)$ for all $u \in \Gamma$

$\delta_2(u^3) = \delta_2(u)\theta_1(u^2) + \theta_2(u)\delta_1(u)\theta_1(u) + \theta_2(u^2)\delta_2(u)$ for all $u \in \Gamma$.

Example 3.1 Let Γ be a non-commutative ring, let $c_1, c_2 \in \Gamma$ such that $\theta_2(u)c_1 = \theta_2(u)c_2 = 0$ (resp. $\theta_2(v)c_1 = \theta_2(v)c_2 = 0$) for all $u, v \in \Gamma$ .Define $\delta_1, \delta_2: \Gamma \longrightarrow \Gamma$ as follows: $\delta_1(u) = c_1\theta_1(u)$ and $\delta_2(u) = c_2\theta_1(u), \forall u \in \Gamma$, where $\theta_1, \theta_2: \Gamma \longrightarrow \Gamma$ are two endomorphism mappings. Then (δ_1, δ_2) is a (θ_1, θ_2)-derivation pair of Γ.

Let $u, v \in \Gamma$, then
$$\begin{aligned}\delta_1(uvu) &= c_1\theta_1(uvu)\\ &= c_1\theta_1(u(vu))\\ &= c_1\theta_1(u)\theta_1(vu)\\ &= \delta_1(u)\theta_1(vu) + \theta_2(u)c_2\theta_1(vu)\\ &= \delta_1(u)\theta_1(vu) + \theta_2(u)c_2\theta_1(v)\theta_1(u)\\ &= \delta_1(u)\theta_1(vu) + \theta_2(u)\delta_2(v)\theta_1(u) + \theta_2(u)\theta_2(v)c_1\theta_1(u)\\ &= \delta_1(u)\theta_1(vu) + \theta_2(u)\delta_2(v)\theta_1(u) + \theta_2(u)\theta_2(v)\delta_1(u)\\ &= \delta_1(u)\theta_1(vu) + \theta_2(u)\delta_2(v)\theta_1(u) + \theta_2(uv)\delta_1(u)\end{aligned}$$

Also,
$$\begin{aligned}\delta_2(uvu) &= c_2\theta_1(uvu)\\ &= c_2\theta_1(u(vu))\\ &= c_2\theta_1(u)\theta_1(vu)\\ &= \delta_2(u)\theta_1(vu) + \theta_2(u)c_1\theta_1(vu)\\ &= \delta_2(u)\theta_1(vu) + \theta_2(u)c_1\theta_1(v)\theta_1(u)\\ &= \delta_2(u)\theta_1(vu) + \theta_2(u)\delta_1(v)\theta_1(u) + \theta_2(u)\theta_2(v)c_2\theta_1(u)\\ &= \delta_2(u)\theta_1(vu) + \theta_2(u)\delta_1(v)\theta_1(u) + \theta_2(u)\theta_2(v)\delta_2(u)\\ &= \delta_2(u)\theta_1(vu) + \theta_2(u)\delta_1(v)\theta_1(u) + \theta_2(uv)\delta_2(u)\end{aligned}$$

Thus, (δ_1, δ_2) is a (θ_1, θ_2)-derivation pair of Γ.

Remark 3.1: Every (θ_1, θ_2)-derivation pair is a Jordan (θ_1, θ_2)-derivation pair, but the converse is not true in general.

Example 3.2: Let Γ be a 2-torsion free non-commutative ring, let $c \in \Gamma$ such that $\theta_2(u)c\theta_1(u) = 0, \forall u \in \Gamma$, but $\theta_2(u)c\theta_1(v) \neq 0$ for some $u \neq v \in \Gamma$. Define $\delta_1, \delta_2: \Gamma \longrightarrow \Gamma$ as follows: $\delta_1(u) = \theta_2(u)c + c\theta_1(u)$ and $\delta_2(u) = \theta_2(u)c - c\theta_1(u), \forall u \in \Gamma$, where $\theta_1, \theta_2: \Gamma \longrightarrow \Gamma$ are two endomorphisms. Then (δ_1, δ_2) is a Jordan (θ_1, θ_2)-derivation but not (θ_1, θ_2)-derivation.

Let $u, v \in \Gamma$, then
$\delta_1(u^3) = \theta_2(u^3)c + c\theta_1(u^3)$ and $\delta_1(u^3) = \delta_1(u)\theta_1(u^2) + \theta_2(u)\delta_2(u)\theta_1(u) + \theta_2(u^2)\delta_1(u)$. Thus, $\delta_1(u)\theta_1(u^2) + \theta_2(u)\delta_2(u)\theta_1(u) + \theta_2(u^2)\delta_1(u) =$
$\big(\theta_2(u)c + c\theta_1(u)\big)\theta_1(u^2) + \theta_2(u)\big(\theta_2(u)c - c\theta_1(u)\big)\theta_1(u) + \theta_2(u^2)\big(\theta_2(u)c + c\theta_1(u)\big) =$
$\big(\theta_2(u)c\theta_1(u)\theta_1(u) + c\theta_1(u)\theta_1(u)\theta_1(u)\big) + \big(\theta_2(u)\theta_2(u)c\theta_1(u) - \theta_2(u)c\theta_1(u)\theta_1(u)\big) + \big(\theta_2(u)\theta_2(u)\theta_2(u)c + \theta_2(u)\theta_2(u)c\theta_1(u)\big) =$
$c\theta_1(u)\theta_1(u)\theta_1(u) + \theta_2(u)\theta_2(u)\theta_2(u)c = \theta_2(u^3)c + c\theta_1(u^3)$.
Also,
$\delta_2(u) = \theta_2(u)c - c\theta_1(u)$ and $\delta_2(u^3) = \delta_2(u)\theta_1(u^2) + \theta_2(u)\delta_1(u)\theta_1(u) + \theta_2(u^2)\delta_2(u)$.
Thus, $\delta_2(u)\theta_1(u^2) + \theta_2(u)\delta_1(u)\theta_1(u) + \theta_2(u^2)\delta_2(u) =$
$\big(\theta_2(u)c - c\theta_1(u)\big)\theta_1(u^2) + \theta_2(u)\big(\theta_2(u)c + c\theta_1(u)\big)\theta_1(u) + \theta_2(u^2)\big(\theta_2(u)c - c\theta_1(u)\big) =$
$\big(\theta_2(u)c\theta_1(u)\theta_1(u) - c\theta_1(u)\theta_1(u)\theta_1(u)\big) + \big(\theta_2(u)\theta_2(u)c\theta_1(u) + \theta_2(u)c\theta_1(u)\theta_1(u)\big) + \big(\theta_2(u)\theta_2(u)\theta_2(u)c - \theta_2(u)\theta_2(u)c\theta_1(u)\big) =$
$-c\theta_1(u)\theta_1(u)\theta_1(u) + \theta_2(u)\theta_2(u)\theta_2(u)c = \theta_2(u^3)c - c\theta_1(u^3)$.
Therefore, (δ_1, δ_2) is a Jordan(θ_1, θ_2)-derivation pair.

Now, $\delta_1(uvu) = \theta_2(uvu)c + c\theta_1(uvu)$ and
$\delta_1(u)\theta_1(vu) + \theta_2(u)\delta_2(v)\theta_1(u) + \theta_2(uv)\delta_1(u) =$
$\big(\theta_2(u)c + c\theta_1(u)\big)\theta_1(vu) + \theta_2(u)\big(\theta_2(v)c - c\theta_1(v)\big)\theta_1(u) + \theta_2(uv)\big(\theta_2(u)c + c\theta_1(u)\big)$
$=$
$c\theta_1(uvu) + \theta_2(uvu)c + 2\theta_2(uv)c\theta_1(u) = \theta_2(uvu)c + c\theta_1(uvu)$.
also,
$\delta_2(uvu) = \theta_2(uvu)c - c\theta_1(uvu)$ and $\delta_2(u)\theta_1(vu) + \theta_2(u)\delta_1(v)\theta_1(u) + \theta_2(uv)\delta_2(u) =$
$\big(\theta_2(u)c - c\theta_1(u)\big)\theta_1(vu) + \theta_2(u)\big(\theta_2(v)c + c\theta_1(v)\big)\theta_1(u) + \theta_2(uv)\big(\theta_2(u)c - c\theta_1(u)\big)$
$=$
$-c\theta_1(uvu) + \theta_2(uvu)c + 2\theta_2(u)c\theta_1(vu)$. Since $\theta_2(u)c\theta_1(v) \neq 0$ for some $u \neq v \in \Gamma$, this means that (δ_1, δ_2) is not (θ_1, θ_2)-derivation pair.

Theorem 3.1 Let Γ be a prime ring. Let θ_1 and θ_2 be two automorphisms of Γ. If Γ is a (δ_1, δ_2)- Derivation pair such that $\delta_1(u) = \mp\, \theta_1(u)$ for each $u \in \Gamma$, then $\delta_2(u) = 0$.

Proof: Let $u \in \Gamma$.
If $\delta_1(u) = \theta_1(u)$ for each $u \in \Gamma$ (1)
Replacing u by uvu in (1), we get:
$\delta_1(uvu) = \theta_1(uvu)$ for each $u, v \in \Gamma$ (2)
That is:
$\delta_1(u)\theta_1(vu) + \theta_2(u)\delta_2(v)\theta_1(u) + \theta_2(uv)\delta_1(u) = \theta_1(uvu)$ for each $u, v \in \Gamma$ (3)
By using (1) we have:
$\theta_1(uvu) + \theta_2(u)\delta_2(v)\theta_1(u) + \theta_2(uv)\theta_1(u) - \theta_1(uvu) = 0$ for each $u, v \in \Gamma$ (4)
That is:
$\theta_2(u)\delta_2(v)\theta_1(u) + \theta_2(uv)\theta_1(u) = 0$ for each $u, v \in \Gamma$ (5)

That is:
$\theta_2(u)\big(\delta_2(v) + \theta_2(v)\big)\theta_1(u) = 0$ for each $u, v \in \Gamma$ (6)
Replacing $\delta_2(v) + \theta_2(v)$ by $\theta_2(v)$ in (6), and using (1) we get:
$\theta_2(uv)\delta_1(u) = 0$ for each $u, v \in \Gamma$ (7)
Left multiplying of (7) by $\delta_2(u)$ we have:
$\delta_2(u)\, \theta_2(uv)\delta_1(u) = 0$ for each $u, v \in \Gamma$ (8)
Since Γ is a prime ring, (8) gives:
$\delta_2(u) = 0$ for each $u \in \Gamma$.
Now,
If $\delta_1(u) = -\, \theta_1(u)$ for each $u \in \Gamma$ (9)
Replacing u by uvu in (9), we get:
$\delta_1(uvu) = -\, \theta_1(uvu)$ for each $u, v \in \Gamma$ (10)
That is:
$\delta_1(u)\theta_1(vu) + \theta_2(u)\delta_2(v)\theta_1(u) + \theta_2(uv)\delta_1(u) = -\, \theta_1(uvu)$ for each $u, v \in \Gamma$ (11)
By using (9) we have:
$-\theta_1(uvu) + \theta_2(u)\delta_2(v)\theta_1(u) - \theta_2(uv)\theta_1(u) + \theta_1(uvu) = 0$ for each $u, v \in \Gamma$ (12)
That is:
$\theta_2(u)\delta_2(v)\theta_1(u) - \theta_2(uv)\theta_1(u) = 0$ for each $u, v \in \Gamma$ (13)
That is:
$\theta_2(u)\big(\theta_2(v) - \delta_2(v)\big)(-\theta_1(u)) = 0$ for each $u, v \in \Gamma$ (14)
By using (9) we have:
$\theta_2(u)\big(\theta_2(v) - \delta_2(v)\big)\delta_1(u) = 0$ for each $u, v \in \Gamma$ (15)

Replacing $\theta_2(v)-\delta_2(v)$ by $\theta_2(v)$ in (15), we get:
$\theta_2(uv)\delta_1(u)=0$ for each $u,v\in\Gamma$ (16)
Left multiplying of (16) by $\delta_2(u)$ we have:
$\delta_2(u)\ \theta_2(uv)\delta_1(u)=0$ for each $u,v\in\Gamma$ (17)
Since Γ is a prime ring, (17) gives:
$\delta_2(u)=0$ for each $u\in\Gamma$. ■

Theorem 3.2 Let Γ be a prime ring. Let θ_1 and θ_2 be two automorphisms of Γ. If Γ is a (δ_1,δ_2)- Derivation pair such that $\delta_2(u)=\mp\,\theta_1(u)$ for each $u\in\Gamma$, then $\delta_1(u)=0$.

Proof: Let $u\in\Gamma$.
If $\delta_2(u)=\ \theta_1(u)$ for each $u\in\Gamma$ (18)
Replacing u by uvu in (18), we get:
$\delta_2(uvu)=\ \theta_1(uvu)$ for each $u,v\in\Gamma$ (19)
That is:
$\delta_2(u)\theta_1(vu)+\theta_2(u)\delta_1(v)\theta_1(u)+\theta_2(uv)\delta_2(u)=\ \theta_1(uvu)$ for each $u,v\in\Gamma$ (20)
By using (18) we have:
$\theta_1(uvu)+\theta_2(u)\delta_1(v)\theta_1(u)+\theta_2(uv)\theta_1(u)-\ \theta_1(uvu)=0$ for each $u,v\in\Gamma$ (21)
That is:
$\theta_2(u)\delta_1(v)\theta_1(u)+\theta_2(uv)\theta_1(u)=0$ for each $u,v\in\Gamma$
(22)

That is:
$\theta_2(u)\big(\delta_1(v)+\theta_2(v)\big)\theta_1(u)=0$ for each $u,v\in$
Γ (23)
Replacing $\delta_1(v)+\theta_2(v)$ by $\theta_2(v)$ in (23), and using (18) we get:
$\theta_2(uv)\delta_2(u)=0$ for each $u,v\in$
Γ (24)
Left multiplying of (24) by $\delta_1(u)$ we have:
$\delta_1(u)\ \theta_2(uv)\delta_2(u)=0$ for each $u,v\in$
Γ (25)
Since Γ is a prime ring, (25) gives:
$\delta_1(u)=0$ for each $u\in\Gamma$.
Now,
If $\delta_2(u)=-\ \theta_1(u)$ for each $u\in\Gamma$ (26)
Replacing u by uvu in (26), we get:
$\delta_2(uvu)=-\ \theta_1(uvu)$ for each $u,v\in\Gamma$ (27)
That is:
$\delta_2(u)\theta_1(vu)+\theta_2(u)\delta_1(v)\theta_1(u)+\theta_2(uv)\delta_2(u)=-\ \theta_1(uvu)$ for each $u,v\in$
Γ (28)
By using (26) we have:
$-\theta_1(uvu)+\theta_2(u)\delta_1(v)\theta_1(u)-\theta_2(uv)\theta_1(u)+\ \theta_1(uvu)=0$ for each $u,v\in\Gamma$ (29)
That is:
$\theta_2(u)\delta_1(v)\theta_1(u)-\theta_2(uv)\theta_1(u)=0$ for each $u,v\in\Gamma$ (30)

That is:
$\theta_2(u)(\theta_2(v)\ -\delta_1(v))(-\theta_1(u))=0$ for each $u,v\in\Gamma$ (31)
By using (26) we have:
$\theta_2(u)\big(\theta_2(v)-\delta_1(v)\big)\delta_2(u)=0$ for each $u,v\in\Gamma$ (32)
Replacing $\theta_2(v)-\delta_1(v)$ by $\theta_2(v)$ in (32), we get:
$\theta_2(uv)\delta_2(u)=0$ for each $u,v\in\Gamma$ (33)
Left multiplying of (33) by $\delta_1(u)$ we have:

$\delta_1(u)\ \theta_2(uv)\delta_2(u) = 0$ for each $u, v \in \Gamma$ (34)
Since Γ is a prime ring, (34) gives:
$\delta_1(u) = 0$ for each $u \in \Gamma$. ■

Theorem 3.3 Let Γ be a prime ring. Let θ_1 and θ_2 be two automorphisms of Γ. If Γ is a (δ_1, δ_2)-Derivation pair such that $\delta_2(u)\delta_1(v) = 0$ (resp. $\delta_1(u)\delta_2(v) = 0$) for each $u, v \in \Gamma$, then $\delta_2(u) = 0$ (resp. $\delta_1(u) = 0$).

Proof: Let $u, v \in \Gamma$.
If $\delta_2(u)\delta_1(v) = 0$ for each $u, v \in \Gamma$ (35)
Replacing u by uvu in (35), we get:
$\delta_2(uvu)\delta_1(v) = 0$ for each $u, v \in \Gamma$ (36)
That is:
$\big(\delta_2(u)\theta_1(vu) + \theta_2(u)\delta_1(v)\theta_1(u) + \theta_2(uv)\delta_2(u)\big)\delta_1(v) = 0$ for each $u, v \in \Gamma$ (37)
That is:
$\delta_2(u)\theta_1(vu)\delta_1(v) + \theta_2(u)\delta_1(v)\theta_1(u)\delta_1(v) + \theta_2(uv)\delta_2(u)\delta_1(v) = 0$ for each $u, v \in \Gamma$ (38)
By using (35), we have:
$\delta_2(u)\theta_1(vu)\delta_1(v) + \theta_2(u)\delta_1(v)\theta_1(u)\delta_1(v) = 0$ for each $u, v \in \Gamma$ (39)
That is:
$\big(\delta_2(u)\theta_1(v) + \theta_2(u)\delta_1(v)\big)\theta_1(u)\delta_1(v) = 0$ for each $u, v \in \Gamma$ (40)
Replacing $\delta_2(u)\theta_1(v) + \theta_2(u)\delta_1(v)$ by $\theta_1(v)$ in (40), we get:
$\theta_1(vu)\delta_1(v) = 0$ for each $u, v \in \Gamma$ (41)
Left multiplying (41) by $\delta_2(u)$ we have:
$\delta_2(u)\ \theta_1(vu)\delta_1(v) = 0$ for each $u, v \in \Gamma$ (42)
Since Γ is a prime ring, (42) gives:
$\delta_2(u) = 0$ for each $u \in \Gamma$.
Now,
If $\delta_1(u)\delta_2(v) = 0$ for each $u, v \in \Gamma$ (43)
Replacing u by uvu in (43), we get:
$\delta_1(uvu)\delta_2(v) = 0$ for each $u, v \in \Gamma$ (44)
That is:
$\big(\delta_1(u)\theta_1(vu) + \theta_2(u)\delta_2(v)\theta_1(u) + \theta_2(uv)\delta_1(u)\big)\delta_2(v) = 0$ for each $u, v \in \Gamma$ (45)
That is:
$\delta_1(u)\theta_1(vu)\delta_2(v) + \theta_2(u)\delta_2(v)\theta_1(u)\delta_2(v) + \theta_2(uv)\delta_1(u)\delta_2(v) = 0$ for each $u, v \in \Gamma$ (46)
By using (43), we have:
$\delta_1(u)\theta_1(vu)\delta_2(v) + \theta_2(u)\delta_2(v)\theta_1(u)\delta_2(v) = 0$ for each $u, v \in \Gamma$ (47)
That is:
$\big(\delta_1(u)\theta_1(v) + \theta_2(u)\delta_2(v)\big)\theta_1(u)\delta_2(v) = 0$ for each $u, v \in \Gamma$ (48)
Replacing $\delta_1(u)\theta_1(v) + \theta_2(u)\delta_2(v)$ by $\theta_1(v)$ in (48), we get
$\theta_1(vu)\delta_2(v) = 0$ for each $u, v \in \Gamma$ (49)
Left multiplying of (49) by $\delta_1(u)$, we have:
$\delta_1(u)\ \theta_1(vu)\delta_2(v) = 0$ for each $u, v \in \Gamma$ (50)
Since Γ is a prime ring, (50) gives:
$\delta_1(u) = 0$ for each $u \in \Gamma$. ■

Theorem 3.4 Let Γ be a prime ring. Let θ_1 and θ_2 be two automorphisms of Γ. If Γ is a (δ_1, δ_2)-Derivation pair such that $c\delta_1(u) = 0$ or $\delta_1(u)c = 0$ (resp. $c\delta_2(u) = 0$ or $\delta_2(u)c = 0$) for each $u, c \in \Gamma$, then either $c = 0$ or $\delta_1(u) = 0$ (resp. $c = 0$ or $\delta_2(u) = 0$).

Proof: Let $u, c \neq 0 \in \Gamma$.
If $c\delta_1(u) = 0$ for each $c, u \in \Gamma$ (51)
Replacing u by uvu in (51), we get:
$c\delta_1(uvu) = 0$ for each $u, v, c \in \Gamma$ (52)
That is
$c(\delta_1(u)\theta_1(vu) + \theta_2(u)\delta_2(v)\theta_1(u) + \theta_2(uv)\delta_1(u)) = 0$ for each $u, v, c \in \Gamma$ (53)
That is
$c\delta_1(u)\theta_1(vu) + c\theta_2(u)\delta_2(v)\theta_1(u) + c\theta_2(uv)\delta_1(u) = 0$ for each $u, v, c \in \Gamma$ (54)
By using (51), we have:
$c\theta_2(u)\delta_2(v)\theta_1(u) + c\theta_2(uv)\delta_1(u) = 0$ for each $u, v, c \in \Gamma$ (55)
That is
$c\theta_2(u)(\delta_2(v)\theta_1(u) + \theta_2(v)\delta_1(u)) = 0$ for each $u, v, c \in \Gamma$ (56)
Replacing $\delta_2(v)\theta_1(u) + \theta_2(v)\delta_1(u)$ by $\delta_1(u)$ in (56), we get:
$c\theta_2(u)\delta_1(u) = 0$ for each $u, c \in \Gamma$ (57)
Since $c \neq 0$ and Γ is a prime rings, then (57) gives $\delta_1(u) = 0$.
Now, let $u, c \neq 0 \in \Gamma$.
If $c\delta_2(u) = 0$ for each $c, u \in \Gamma$ (58)
Replacing u by uvu in (58), we get:
$c\delta_2(uvu) = 0$ for each $u, v, c \in \Gamma$ (59)
That is
$c(\delta_2(u)\theta_1(vu) + \theta_2(u)\delta_1(v)\theta_1(u) + \theta_2(uv)\delta_2(u)) = 0$ for each $u, v, c \in \Gamma$ (60)
That is
$c\delta_2(u)\theta_1(vu) + c\theta_2(u)\delta_1(v)\theta_1(u) + c\theta_2(uv)\delta_2(u) = 0$ for each $u, v, c \in \Gamma$ (61)
By using (58), we have:
$c\theta_2(u)\delta_1(v)\theta_1(u) + c\theta_2(uv)\delta_2(u) = 0$ for each $u, v, c \in \Gamma$ (62)
That is
$c\theta_2(u)(\delta_1(v)\theta_1(u) + \theta_2(v)\delta_2(u)) = 0$ for each $u, v, c \in \Gamma$ (63)
Replacing $\delta_1(v)\theta_1(u) + \theta_2(v)\delta_2(u)$ by $\delta_2(u)$ in (63), we get:
$c\theta_2(u)\delta_2(u) = 0$ for each $u, c \in \Gamma$ (64)
Since $c \neq 0$ and Γ is a prime rings, then (64) gives $\delta_2(u) = 0$. ■

Theorem 3.5 Let Γ be a prime ring with char$(\Gamma) \neq 2$. Let $\theta_1$and θ_2 be two endomorphisms of Γ. If Γ is a (δ_1, δ_2)-Derivation pair such that $c_1qc_2\delta_1(u) + \delta_1(u)c_2qc_1 = 0$ (resp. $c_1qc_2\delta_2(u) + \delta_2(u)c_2qc_1 = 0$) for each $u, c_1, c_2, q \in \Gamma$, then $c_1 = 0$ or $c_2 = 0$.
Proof: From the assumption we have:
$c_1qc_2\delta_1(u) + \delta_1(u)c_2qc_1 = 0$ for each $u, c_1, c_2, q \in \Gamma$ (65)
Replacing u by uvu in (65), we have:
$c_1qc_2\delta_1(uvu) + \delta_1(uvu)c_2qc_1 = 0$ for each $u, v, c_1, c_2, q \in \Gamma$ (66)
That is
$c_1qc_2\big(\delta_1(u)\theta_1(vu) + \theta_2(u)\delta_2(v)\theta_1(u) + \theta_2(uv)\delta_1(u)\big) +$
$(\delta_1(u)\theta_1(vu) + \theta_2(u)\delta_2(v)\theta_1(u) + \theta_2(uv)\delta_1(u))c_2qc_1 = 0$ for each $u, v, c_1, c_2, q \in \Gamma$ (67)
That is
$\big(c_1qc_2\delta_1(u)\theta_1(vu) + c_1qc_2\theta_2(u)\delta_2(v)\theta_1(u) + c_1qc_2\theta_2(uv)\delta_1(u)\big) +$
$\big(\delta_1(u)\theta_1(vu)c_2qc_1 + \theta_2(u)\delta_2(v)\theta_1(u)c_2qc_1 + \theta_2(uv)\delta_1(u)c_2qc_1\big) = 0$ for each
$u, v, c_1, c_2, q \in \Gamma$ (68)
By setting $\theta_1(vu) = \theta_2(uv) = 1$ in (68), we have:
$\big(c_1qc_2\delta_1(u) + c_1qc_2\theta_2(u)\delta_2(v)\theta_1(u) + c_1qc_2\delta_1(u)\big) +$
$\big(\delta_1(u)c_2qc_1 + \theta_2(u)\delta_2(v)\theta_1(u)c_2qc_1 + \delta_1(u)c_2qc_1\big) = 0$ for each $u, v, c_1, c_2, q \in \Gamma$ (69)
By using (65), we get:
$c_1qc_2\theta_2(u)\delta_2(v)\theta_1(u) + \theta_2(u)\delta_2(v)\theta_1(u)c_2qc_1 = 0$ for each $u, v, c_1, c_2, q \in \Gamma$ (70)

By setting $\theta_2(u)\delta_2(v)\theta_1(u) = 1$ in (70), we have:
$c_1qc_2 + c_2qc_1 = 0$ for each $u, v, c_1, c_2, q \in \Gamma$ (71)
Replacing q by xc_1y in (71), we get:
$c_1xc_1yc_2 + c_2xc_1yc_1 = 0$ for each $x, y, c_1, c_2 \in \Gamma$ (72)
That is
$c_1yc_2 = -c_2yc_1$ and $c_2xc_1 = -c_1xc_2$ (73)
Substituting (73) in (72) we have:
$-c_1xc_2yc_1 - c_1xc_2yc_1 = 0$ (74)
That is $2c_1\Gamma c_2\Gamma c_1 = (0)$ (75)
Since $\mathrm{char}(\Gamma) \neq 2$ and Γ is a prime, then (75) gives $c_1 = 0$ or $c_2 = 0$. ∎

Theorem 3.6 Let Γ be a 2-torsion free ring with an identity element. Furthermore, let (δ_1, δ_2) be a Jordan (θ_1, θ_2)-derivation pair such that $\delta_1(1) = \delta_2(1)$. Then $\delta_1(u) = \delta_2(u), \forall u \in \Gamma$ where θ_1 and θ_2 are two mappings of Γ.

Proof: Let $\varphi: \Gamma \to \Gamma$ be a mapping given by $\varphi(u) = \delta_1(u) - \delta_2(u), \forall u \in \Gamma$. By Definition 3.1, we have:
$\delta_1(u^3) = \delta_1(u)\theta_1(u^2) + \theta_2(u)\delta_2(u)\theta_1(u) + \theta_2(u^2)\delta_1(u)$ for all $u \in \Gamma$ (76)
$\delta_2(u^3) = \delta_2(u)\theta_1(u^2) + \theta_2(u)\delta_1(u)\theta_1(u) + \theta_2(u^2)\delta_2(u)$ for all $u \in \Gamma$ (77)
Subtracting (77) from (76), we get:
$\varphi(u^3) = \varphi(u)\theta_1(u^2) - \theta_2(u)\varphi(u)\theta_1(u) + \theta_2(u^2)\varphi(u)$ for all $u \in \Gamma$ (78)
Linearizing (78), we have:
$\varphi(u^2v + vu^2 + uv^2 + v^2u + uvu + vuv) = \varphi(u)\theta_1(uv) + \varphi(u)\theta_1(vu) + \varphi(u)\theta_1(v^2) + \varphi(v)\theta_1(uv) + \varphi(v)\theta_1(vu) + \varphi(v)\theta_1(u^2) - \theta_2(u)\varphi(u)\theta_1(v) - \theta_2(u)\varphi(v)\theta_1(u) - \theta_2(v)\varphi(u)\theta_1(u) - \theta_2(v)\varphi(v)\theta_1(u) - \theta_2(u)\varphi(v)\theta_1(v) - \theta_2(v)\varphi(u)\theta_1(v) + \theta_2(uv)\varphi(u) + \theta_2(vu)\varphi(u) + \theta_2(v^2)\varphi(u) + \theta_2(uv)\varphi(v) + \theta_2(vu)\varphi(v) + \theta_2(u^2)\varphi(v)$ for all $u, v \in \Gamma$ (79)
Replacing u by $-u$ in (79), we get:
$\varphi(u^2v + vu^2 - uv^2 - v^2u + uvu - vuv) = \varphi(u)\theta_1(uv) + \varphi(u)\theta_1(vu) - \varphi(u)\theta_1(v^2) - \varphi(v)\theta_1(uv) - \varphi(v)\theta_1(vu) + \varphi(v)\theta_1(u^2) + \theta_2(u)\varphi(u)\theta_1(v) + \theta_2(u)\varphi(v)\theta_1(u) + \theta_2(v)\varphi(u)\theta_1(u) - \theta_2(v)\varphi(v)\theta_1(u) - \theta_2(u)\varphi(v)\theta_1(v) - \theta_2(v)\varphi(u)\theta_1(v) + \theta_2(uv)\varphi(u) + \theta_2(vu)\varphi(u) - \theta_2(v^2)\varphi(u) - \theta_2(uv)\varphi(v) - \theta_2(vu)\varphi(v) + \theta_2(u^2)\varphi(v)$ for all $u, v \in \Gamma$ (80)
According to (79) and (80), we have:
$\varphi(u^2v + vu^2 + uvu) = \varphi(u)\theta_1(uv) + \varphi(u)\theta_1(vu) + \varphi(v)\theta_1(u^2) - \theta_2(v)\varphi(v)\theta_1(u) - \theta_2(u)\varphi(v)\theta_1(v) - \theta_2(v)\varphi(u)\theta_1(v) + \theta_2(uv)\varphi(u) + \theta_2(vu)\varphi(u) + \theta_2(u^2)\varphi(v)$ for all $u, v \in \Gamma$ (81)
Replacing u by 1 in (81), we get:
$2\varphi(v) = \varphi(v)\theta_1(1) - \theta_2(v)\varphi(v)\theta_1(1) - \theta_2(1)\varphi(v)\theta_1(v) + \theta_2(1)\varphi(v)$ for all $v \in \Gamma$ (82)
By setting $\theta_1(1) = \theta_2(1) = 0$ in (82), then we have:
$2\varphi(v) = 0$ for all $v \in \Gamma$ (83)
Since Γ is a 2-torsion free ring, then
$\varphi(v) = 0$ for all $v \in \Gamma$ (84)
Therefore, (84) gives $\delta_1(u) = \delta_2(u)$ for all $u \in \Gamma$. ∎

Proposition 3.1 Let Γ be a ring, and θ_1, θ_2 be two mappings of Γ. Then

1- If (δ_1, δ_2) is a (θ_1, θ_2)-derivation pair on Γ, then $\delta_1 + \delta_2$ is a (θ_1, θ_2)-derivation.

2- If (δ_1, δ_2) is a Jordan (θ_1, θ_2)-derivation pair on Γ, then $\delta_1 + \delta_2$ is a Jordan (θ_1, θ_2)-derivation.

Proof: (1) Since δ_1 and δ_2 is a (θ_1, θ_2)-derivation pair, then by Definition 3.1, we have:

$\delta_1(uvu) = \delta_1(u)\theta_1(vu) + \theta_2(u)\delta_2(v)\theta_1(u) + \theta_2(uv)\delta_1(u)$ for all $u, v \in \Gamma$
(85)

$\delta_2(uvu) = \delta_2(u)\theta_1(vu) + \theta_2(u)\delta_1(v)\theta_1(u) + \theta_2(uv)\delta_2(u)$ for all $u, v \in \Gamma$
(86)

By adding (85) and (86), we have:

$(\delta_1 + \delta_2)(uvu) = (\delta_1 + \delta_2)(u)\theta_1(vu) + \theta_2(u)(\delta_1 + \delta_2)(v)\theta_1(u) + \theta_2(uv)(\delta_1 + \delta_2)(u)$

Thus, $\delta_1 + \delta_2$ is a (θ_1, θ_2)-derivation. By a similar way to prove (2). ■

4. Conclusion

As a conclusion, this article presented the notion of (θ_1, θ_2)-derivation pair with some of its properties. This study displayed that the sum of two (θ_1, θ_2)-derivation pair is a (θ_1, θ_2)-derivation and the sum of two Jordan (θ_1, θ_2)-derivation pair is a Jordan (θ_1, θ_2)-derivation.

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