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Loc-hollow Fuzzy Modules with Related Modules

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Abstract

The concept of a small f- subm was presented in a previous study. This work introduced a concept of a hollow f- module, where a module is said to be hollow fuzzy when every subm of it is a small f- subm. Some new types of hollow modules are provided namely, Loc- hollow f- modules as a strength of the hollow module, where every Loc- hollow f- module is a hollow module, but the converse is not true. Many properties and characterizations of these concepts are proved, also the relationship between all these types is researched. Many important results that explain this relationship are demonstrated also several characterizations and properties related to these concepts are given.

Keywords:f- Modules, small f- subm, maximal f-subm, hollow f- modules, Loc- hollow f- modules.

1. Introduction

Call M is L-hollow module where a module has a unique maximal submodule that contains all small submodules of M [10]. In this paper, we fuzzify these concepts from L-hollow module to Loc-hollow f- modules. Moreover, we generalize numerous properties of Loc- hollow f- modules. This work contains four Parts.

*I*n part one, we recollect several *d*efinitions and *ρ*roperties that are useful are needed later.

*I*n part two, several fundamental ρroperties of L-hollow f- modules are argued.

part three includes the Relation between hollow f- modules, and Loc- hollow f- modules.

Finally part four, we shall give the relation between Loc-hollow f- modules and different modules like as amply supplemented f- modules, indecomposable f- modules, and lifting f- modules

1. Preliminaries:

1.1 Definition [1] Let $M \neq \emptyset$, let Ɨ be the closed interval [0, 1] on the real line (real number), f-set A in M (a f- subset A of M) is fun from X to Ɨ.''

The following example describes the above definition:

1.2 Example [1]

Let M be the real line R and A be a fuzzy set of numbers that is much greater than I. Then one can accord an accurate characterization of A specifying:

$$A(x) = \begin{cases} 1 - \frac{1}{x^2} & if \quad x > 1 \\ 0 & if \quad x \leq 1 \end{cases}$$ ''

1.3''Definition [2]

Let X_t: M → [0, 1] be a fuzzy set in M, where x ∈M, t ∈ (0,1] defined by

$X_t(y) = \begin{cases} 1 & if \quad t = y \\ 0 & if \quad t \neq y \end{cases}$ ∀y∈ M. X_t is said a fuzzy singleton where x= 0, t = 1, therefor

$0_1(y) = \begin{cases} 1 & if \quad y = 0 \\ 0 & if \quad y \neq 0 \end{cases}$, is fuzzy singleton fuzzy zero singleton ''

1.4 Definition [3]

Let u and p be two f- sets inM, then:

1- (u ∪ p)(x)= max {u(x), p(x)}, ∀x ∈ M.

2- (u ∩ p)(x)= min {u(x), p(x)}, ∀ x ∈ M. (u ∪ p) a, (u ∩ p) are fuzzy sets in M in general if { u_a , a ∈ A }, is u family of fuzzy sets in M , then

$\left(\left(\bigcap_{a \in A}\right) va\right)$ (x) = inf {u_a (x), a ∈ A}, ∀ x ∈ S.

$\left(\left(\bigcup_{a \in A}\right) va\right)$ (x) = sup {u_a (x), a ∈ A}, ∀x ∈ S.''

1.5 Definition [5] [4]

If U is f- set in M ,for all t ∈ (0,1].the set u_t = { x ∈ M., u(x) ≥ t }, is named a level subset of u. Note that, u_t is a subset of M in the ordinary sense.''

1.6 Remark [1]

Let v, p are two f- sets inM, then:

1-(u ∪ p)$_t$ = u_t ∪ p_t }, for any t ∈ (0, 1].

2-(u ∩ p)$_t$ = u_t ∩ p_t for any t ∈ (0, 1].

3- U = p if and only if u_t = p_t for any t ∈ (0, 1].''

1.7 Definition [1]

Let U is f- sets inM, u is named empty fuzzy set, denoted by θ if ⟺ u(x) =0, ∀ x∈ M.''

1.8 D*d*efinition [2] [4]

Let M be an Ř- module, a f- sets X of M is named is f- module where:

1- A (0) = 1.

2-A(x – y) ≥ min {A (x), A(y)}.

3-A (rx) ≥ A (x), ∀x ∈ M, γ ∈ Ř. ''

1.9 Definition [5][4]

Let A, B be two f- modules of an R- module M. B is named f- module of A, if A ⊆ B.''

1.10 Definition [2]

Let A, C are f- subm of an f- module X, then A+C is f- module m.''

1.11 Definition [7]

Let X be f- module, X is named simple f- module if X has only one proper f- subm, which is 0_1.''

1.12 Definition [12]

let A and B be two f- modules of an R-module X_1 and X_2. Let f : A→B be a fuzzy homomorphism. If υ and σ are two f- subms of A, B, therefore

1- $f(\upsilon)$ is fuzzy submodules of B, whenever f is an epimorphism.

2-$f^{-1}(\sigma)$ is fuzzy submodules of A''

1.13 Definition [11]

Let A be a proper f- sub of M. then A is named a maximal fuzzy submodule of M. whether to any other proper fuzzy submodule β of M containing A then $A = \beta$.''

1.14 Definition [8]

A proper fuzzy subm A of an R-module X, is named a small fuzzy if A is a fuzzy subm of χ , then A is named a small fuzzy in χ if every fuzzy submodule β of χ , s.t $A + \beta = \chi$. Implies $\beta = \chi$ ''

1.15 Definition [13]

Let A be f- module of an R- module M. A is named finitely generated f- Module if there exists x_{t1} , x_{t2}, , , , , , x_{tn} ⊆ A such that $A = \{ a_1 (x_t)_{t1} + a_2 (x_t)_{t2} + ... + a_n (x_n)_{tn}\}$, where $a_i \in R$, $a(x)_t = (ax)_t$, ∀ t ∈ (0,1] ,

$(Ax)_t(y) = \begin{cases} 1 & if \quad y = ax \\ 0 & otherwise \end{cases}$ ''

1.16 Definition [5]

Let μ, Υ be two f- modules, f: $X_1 \to X_2$ is a homomorphism between X_1 and X_2 respectively.Then F-kernel of f , F-kernel f

Is the fuzzy subset of M_1, defined by?

F-ker $f(x) = \begin{cases} \mu(0) & if \quad x \in ker\, f \\ 0 & if \quad x \notin ker\, f \end{cases}$ ''

1. Characterization of Loc- hollow fuzzy Module

Throughout the part, we introduce the definition of Loc- hollow f- modules and study the basic properties of these kinds of modules.

2.1 Definition

An R-module X is Loc- hollow fuzzy module if X has a unique maximal fuzzy submodule that contains each a small fuzzy submodule of X.

2.2 Example

Let M= Z_4, R = Z, define

X: M → [0,1] *as follow*

$X(x) = \begin{cases} 1 & if \quad x \in M \\ 0 & otherwise \end{cases}$ and Define

A: M → [0,1] *such that*

$A(x) = \begin{cases} t & if \quad x \in N \\ 0 & otherwise \end{cases}$, t ∈ (0, 1] ,where N= $2Z_4$

Clear that x is a fuzzy module and A_t =N is a submodule in X_t , since A_t is only the maximal submodule of X_t by [9], therefore A is only the maximal fuzzy submodule of X and it is a fuzzy small submodule in X by [8] , which contains all fuzzy small submodule. On the other side

1- Let M=Z_6 as Z-module, R=Z.

Define X: M⟶ [0, 1] By $X_{x)} = \begin{cases} 1 & if \quad x \in M \\ 0 & otherwise \end{cases}$

A: M⟶ [0, 1] by $A(x) = \begin{cases} t & if \quad x \in N \\ 0.75 & otherwise \end{cases}$ ∀ t ∈ (0,1] , N=2Z

B: M⟶ [0, 1] by B (t) = $\begin{cases} t & if \quad x \in L \\ 0{,}25 & otherwise \end{cases}$ ∀ t ∈ (0,1] , L=3Z

Clear that X is fuzzy module and X_t=M, A_t=⟨$\bar{2}$⟩, B_t =⟨$\bar{3}$⟩ Aare submodules in X_t.

But A_t, B_t are two maximal submodules in X_t by [9], therefore A, B are two fuzzy maximal submodules in X . Thus X =Z_6 is not a Loc-hollow fuzzy module.

2.3 ρroposition

Let X be a f- module. Then X is Loc-hollow f- modules ⇔X_t is Loc-hollow f- modules ∀ t ∈ (0,1].

Proof:

⟹ Suppose that X is a Loc-hollow f- modules, then there exists a unique maximal fuzzy submodule A which contains all a small f -subm in X. suppose that ***A*** is a f -subm in χ, therefore A+B=X for some f -subm B in A of X, and $(A+B)_t$=X_t ∀ t ∈ (0,1] By [6] . But A_t is a unique maximal f -subm in X_t, by [10]. Therefore A_t is a unique maximal submodule in X_t. Thus X_t is a Loc-hollow module ∀ t ∈ (0,1].

⟸ Let X_t be a Loc-hollow module, suppose that B_t be a submodule of X_t and $(A+B)_t$=X_t for some f -subm B_t in A_t of X_t ∀ t ∈ (0,1] , hence A+B=X or some f -subm B in A of X. and B_t is a f -subm in X_t by [8] . But A is a unique maximal f- subm of X which contains all a small f-subm by our assumption. Therefore X is Loc- hollow f- modules.

(2.4) Remark and Example

1- Every Loc- hollow f- module is hollow f- module.

Proof

Suppose that X is a Loc-hollow f- module, then there exists a unique maximal f- subm containing all a small f-subm say B in X. And since B is f -subm of X. Therefore B contain in X. By definition of hollow f- module. This implies that X is a hollow f- module.

2- The convers Remark **(2.4)** (1) is not true in general for instant

let M=Z_{p^∞} . R=Z , Define X: M → [0,1] *as follow*

$$X_{(x)} = \begin{cases} 1 & if \quad x \in M \\ 0 & otherwise \end{cases}$$

Clear that χ is f- module and X_t = Z_{p^∞} = M. But Z_{p^∞} has a small submodule by [5], and it is a fuzzy small by [1].Then X_t =Z_p^∞ is a hollow module. But not Loc- hollow module. Then X is not Loc-hollow module by Proposition: (2.3)

The 3-Every local fuzzy module is Loc- hollow f- module, while the converse is not true in general.

Proof:

Let M= Z_2 ⊕ Q, R = Z, define by

X: M → [0,1] *as follow*

X(x, y) = $\begin{cases} 1 & \forall (x,y) \quad x \in M \\ 0 & otherwise \end{cases}$ such that x ∈ $\bar{0}$, y ∈ Q ∀ t ∈ (0,1].

Clear that χ is f- module and M=X_t , Z_2 ⊕ Q is Loc- hollow module. But not local module since X_t = {$\bar{0}_1$} ⊕Q ≅ Q is a unique maximal submodule of Z_2 ⊕ Q and {0_1} ⊕ {0_1} is a small submodule of Z_2 ⊕ Q and contained in {0_1} ⊕ Q ≅ Q. But Z_2 ⊕ {0_1} is a proper submodule of Z_2 ⊕ Q, also Z_2 ⊕ {0_1} is not contained in {0_1} ⊕ Q. Thus X is Loc- hollow fuzzy module but a local fuzzy module.

4-Every simple fuzzy submodule is not Loc- hollow fuzzy module.

Proof:

Let $M=Z_5$, $R=Z$,

$\chi : M \rightarrow [0,1]$ *as follow*

$\chi(x) = \begin{cases} 1 & if \quad x \in M \\ 0 & otherwise \end{cases}$, $t \in (0, 1]$, $A(x) = \begin{cases} t & if \quad x \in N \\ 0 & otherwise \end{cases}$ $t \in (0, 1]$, $N=\bar{0}$.

Clear that X is a simple fuzzy module. But not local fuzzy module since $M= \{\bar{0}_1\} \oplus Z_5 = Z_5$ is a unique maximal fuzzy submodule of Z_5 and $\{0_1\}$ is only one proper fuzzy submodule, which is a small submodule of Z_5 and contained in $\{0_1\} \oplus Z_5 = Z_5$. Also $Z_5 \oplus \{0_1\}$ is not contained in $\{0_1\} \oplus Z_5$.That is X is a simple fuzzy module and not Loc- hollow fuzzy module.

5-Every Loc- hollow f- module is not a simple fuzzy submodule:

Proof:

Let $M= Z_8$, R = Z, Define $X: M \rightarrow [0,1]$ *as follow*

$X(x) = \begin{cases} 1 & if \quad x \in M \\ 0 & otherwise \end{cases}$ and Define $A: M \rightarrow [0,1]$ *such that*

$A(x) = \begin{cases} t & if \quad x \in N \\ 0 & otherwise \end{cases}$, $t \in (0, 1]$. $N= 2Z_8$

$B: M \rightarrow [0,1]$ *such that*

$B(x) = \begin{cases} t & if \quad x \in L \\ 0 & otherwise \end{cases}$, $t \in (0, 1]$. $L= 4Z_8$

Consequently, X fuzzy module. and $A_t = N$, $B_t = L$ are two submodules of X_t and A_t a small submodule of X_t by [5], then A is a small submodule of X by [9], Which is a maximal submodule of χ . But $(\oplus \boldsymbol{B})_t = X_t$ by [10] are two direct sum submodules of X_t. and $A \oplus B = \chi$ .This implies that X is not a simple f- module.

2.5 ρroposition:

Epimorphic image of Loc- hollow f- module is Loc- hollow f- module.

ρroof:

Let X_1 be Loc- hollow f- module and let $f: X_1 \rightarrow X_2$ be an epimorphism with X_2 is a f- module [9]. Suppose that A_t be a unique maximal submodule of X_2 with $A_t + C_t =(X_2)_t$ where C_t proper submodule of $(X_2)_t = X_t$, $\forall$ $t \in (0, 1]$, but A is a unique maximal submodule [10].Then $A + C = X_2$. Hence $A_t + C_t =(X_1)_t$ .Now. $f^{-1}(A)$ is a unique maximal fuzzy submodule of X_1 since otherwise $f^{-1}(A)= X_1= X_t$, $\forall$ $t \in (0, 1]$ and hence $f(\Omega f^{-1}(A)) = f(X_1) = X_2$ implies that $A= X_2$, which is a contradiction, with A is a unique maximal fuzzy submodule of X_2, thus $f^{-1}(A)$ is a unique maximal fuzzy submodule of X_1 and since X_1 is Loc- hollow fuzzy module, therefore $f^{-1}(A)$ contains all a fuzzy small submodule of X_1 and hence $f(f^{-1}(A))$ is a fuzzy small submodule of $f(X_1)$. This A is a fuzzy small submodule of X_2, where A_t is an f small submodule of $(X_2)_t$, $\forall$ $t \in (0, 1]$. Therefore X_2 is Loc- hollow fuzzy module.

The next proposition appears more particulars of Loc- hollow fuzzy modules.

2.6 Proposition:

Let C be a small Ƒuzzy subm▯dule of f- modules X. whether χ/C is Loc-hollow f- modules then X is Loc- hollow f- modules.

Proof

Let X/C is Loc- hollow f- modules, where C is a small Ƒuzzy submodule of X. Then there exists a unique maximal small Ƒuzzy submodule A/C in χ /C, with X=B+ D and $X_t=B_t+D_t$, ∀ t ∈ (0,1] by [6]. Where B is a small Ƒuzzy submodule of χ and D is a proper Ƒuzzy submodule of χX. then (D+***B***)/C= χ /C implies that((D+C/C)) + (B+C)/C= χ /C, since (D+C/C) is a Ƒuzzy submodule of A/C , χ /C is Loc-hollow f- modules ,then (D+C/C) is a small fuzzy of X/C. Thus (B+C)/C =X/C, so B+C=X. Since C_t is a submodule of X_t, $B_t=X_t$ by [9]. Therefore C is a f- subm of X, and B=X. Thus, X is Loc- hollow f- modules.

2.7 Corollary:

Suppose that χ be f- modules. If X be a Loc-hollow f- modules then χ /***A*** is Loc- hollow f-modules for every proper f- subm ***A*** ofχ.

Proof:

Suppose that χ is Loc-hollow f- Modules. Then there exists a unique maximal f- subm A contains all a small f- subm. Let ***A*** be a proper Ƒuzzy submodule of Loc-hollow f- modules χ and let $\pi: \chi \rightarrow \chi$ /***A*** be nature epimorphism then χ /***A*** is Loc-hollow f- modules by proposition **(2.5)**.

2.8 *ρ*roposition:

Let A be a proper f- subm of an R-module X. If X is Loc-hollow f- module and X/A is finitely generated then X is finitely generated f- module.

Proof:

Let A be a proper fuzzy submodule of Loc-hollow fuzzy module X with X/A is finitely generated then X/A=$a_1((x_t)_{t1)}$ +A) + $a_2((x_t)_{t2}$ +A) + . . . + a_n (($(x_n)_{tn}$ +A)}, where a_i ∈R and $a(x)_t$ = $(ax)_t$, ∀ t ∈ (0,1] , xi ∈ X, for all i= 1,2, , ,n , where

$$(ax)_t\,(y) = \begin{cases} 1 & if \quad y = ax \\ 0 & otherwise \end{cases} \qquad \forall\, t \in (0,1]\,.$$

We claim that X =$a_1((x_t)_{t1)}$ + $a_2((x_t)_{t2}$ + . . . + a_n (($(x_n)_{tn}$ ∀ a_i ∈R. Let x ⊆X ,then x +A ∈ X/A , implies that C+ A= $a_1(x_1$+A)+ $a_2(x_2)$+A +. . . + $a_n(x_n$ +A)= $a_1(x_1)$+ $a_2(x_2)$ +. . . + $a_n(x_n$)+A . Thus implies that c= $a_1(x_1)$ + $a_2(x_2)$ + . . . + a_n (x_n) +u for some u ⊆ A. Thus X = $a_1(x_1)$+ $a_2(x_2)$ +. . . + $a_n(x_n$)+u , and since X is Loc-hollow f- modules., therefore X is hollow fuzzy module by remark((2.1.4)(1)). Then A is a small f- subm of X which implies that X= $a_1(x_1)$ + $a_2(x_2)$ + . . . + a_n (x_n) where a_i ∈R. Thus X is finitely generated f- modules.

3. Relation between Hollow fuzzy module and Loc- Hollow fuzzy modules

We introduce the following definition let χ be an f-module of an R-Module X. X is named Hollow f - module If every proper f- submodule of X is a small f- module of X. Hence we can say every Loc-hollow fuzzy module is Hollow fuzzy module, and we introduced an examples display that the converse is not true. In this section, we discuss conditions under which Hollow fuzzy modules could be Loc- hollow fuzzy modules.

3.1 Proposition:

Let X be a f- module of an R-module, X is a Loc –hollow f- module ⟺X is a Hollow and cyclic f- module.

Proof;

⟹ Suppose that X is Loc –hollow f- modules then there exists a unique maximal fuzzy submodule A which contains all a small f- submodule of X. This mean there exists A_t is a small submodule of X_t, ∀ t∈ (0,1]. Let $x_t \subseteq X$ with $x_t \not\subseteq A$, (x_t) is a submodule of X_t. then Rx is a f- submodule of X. We claim that X = (x_t) such that $y_s \subseteq X$ has written $y_s = x_t\, a_\ell$ for some fuzzy singleton a_ℓ of R where t, s , ℓ ∈ (0, 1] , R_x = X , So $R_{xt} = X_t$, ∀ t∈ (0,1].If $R_x \neq X$ then R_x is proper a small f- submodule of X and hence Rx is a small fuzzy submodule of A which implies that x ∈A , which contradiction. Thus Rx =X then X is cyclic modules. Now since X Loc –hollow f- modules then X is Hollow f- modules by remark (2.4) (1).

⟸ suppose that X is hollow cyclic f- modules and then it is finitely generated f- modules and hence X has maximal f- submodule suppose that X is hollow cyclic f- modules and then it is fuzzy finitely generated module and hence X has maximal f- submodule contained all proper a small f- submodule say A, Let B f- submodule of X. If B is not contained in A, then B+A =X, but X is a Loc –hollow f- modules. Thus A =X hence $A_t = X_t$ where t ∈ (0, 1] by [8], which contradiction. This implies that every proper f- submodule of X is contained in A, which implies that X has a unique maximal f- submodule that contained all proper f- submodules of X. Therefore X is a local f- module.

3.2 Corollary:

Let X be a f- Modules an Ř – module. X is a Loc- hollow f- module ⟺ X is a hollow and finitely generated f- module.

Proof:

⟹ Suppose that X is Loc- hollow f- modules, then X is hollow f- modules and cyclic by proposition (3.1), and since X is cyclic f- modules,. Thus X is finitely generated

⟸ let X finitely generated hollow module the X = { $Ra_1 (x_t)_{t1}$ + $Ra_2 (x_t)_{t2}$ + . . . + $Ra_n (x_n)_{tn}$}, where a_i ∈R , $a(x)_t = (ax)_t$, ∀ t ∈ (0,1] .If X ≠ $Ra_1 (x_t)_{t1}$ then Rx_{t1} is proper fuzzy submodule of X which implies that R x_{t1} is fuzzy small submodule of X. hence X = {$Ra_1 (x_t)_{t1}$ + $Ra_2 (x_t)_{t2}$ + . . . + $Ra_n (x_n)_{tn}$}.So, we delete the summand one by until we have X= $Ra_1 (x_t)_{ti}$ for some i .thus X is cyclic f- Modules and by proposition (3.1). Therefore X is Loc- hollow f- modules.

3.3 *ρ*roposition:

Suppose that χ be f- modules of an R – module M, χ is a Loc- hollow f- module⟺ X is hollow and has a unique maximal fuzzy submodule.

Proof

⟹ Suppose that X is Loc- hollow fuzzy module then X is hollow a f- modules by remark (2.4) (1). And by definition (2.1), then X has a unique maximal fuzzy submodule fuzzy module.

⟸ Let X be t hollow f- modules which have unique maximal fuzzy submodule fuzzy module, say A, we only have to show that X is a cyclic fuzzy module, let $x_t \subseteq X$ and $x_t \subseteq A$ clear that x ∈ X_t and x ∉ A_t, ∀ ∈ (0,1], then $Rx_t + A_t = X_t$ ¸ ∀ t ∈ (0,1]. But χ is a hollow f- module s then ***A*** small fuzz submodule of χ and hence X= Rx. Therefore X is cyclic fuzzy module, and by proposition (3.1). Therefore X is Loc- hollow f- modules.

3. 4 ρroposition

Let X be f- module, X is a Loc- hollow f- module ⟺X is cyclic and X/A is indecomposable.

Proof:

⟹ Suppose that X is Loc- hollow f- module then X is hollow and cyclic f- module by proposition (2.1). Then X/A indecomposable.

⟸ Let X be a cyclic f- module and every X/A is indecomposable, then by part one. X is hollow f- module and proposition (2.1). Thus X Loc- hollow f- module.

3.5 Proposition

Let X be a fuzzy module of an R – module M, X is a Loc- hollow f- module⟺ X is a hollow f- module, and F-Rad(X)≠ X.

Proof:

⟹ Suppose that X is Loc- hollow f- module, then X is X hollow and cyclic f- module by ρproposition (2.1). X_t is cyclic module and it's finitely generated, ∀ t ∈ (0,1] by [9]. Hence F-Rad(X)≠ X.

⟸ Let X hollow fuzzy module and F-Rad(X)≠ X is a small f- submodule of X. F-Rad(X) is unique maximal f- submodule of X and this X/ Rad(X) is a simple fuzzy module and hence cyclic. Implies that X/ Rad(X) = <x +Rad(x)> for some x ⊆ X.

We clam that X= Rx. Let u ∈ X then u+ Rad(x) ∈ X/ Rad(X), therefore a ∈ R such that u+ Rad(x) = a (u+ Rad(x)) = au+ Rad(x), implies that u- ax ∈ Rad(x) which implies that u- ax = B for some B ∈ Rad(x). Thus u= ax + B ∈ Rx +Rad(x), hence X= Rx +Rad(x). But F-Rad(x) is fuzzy small submodule of X implies that X= Rx. Thus X is cyclic f- module by ρroposition (2.1).we get X is a Loc- hollow f- module.

4. The Relationships between Loc- Hollow Fuzzy Module and Other Types of Modules

In this section, we shall give the relation between the Loc-hollow fuzzy module and different modules like as amply supplemented modules, indecomposable modules and lifting modules.

We shall fuzzify the following definitions:

4.1 Definition:

let A, B are f- submodules of fuzzy module X Then A is named a fuzzy supplement of B in X , if A is minimal with A + B = X. equivalently, A is named f- supplement of B ⟺ $\boldsymbol{A} + \boldsymbol{B} = \chi$ and $\boldsymbol{A} \cap \boldsymbol{B}$ is a small f- subm of $\boldsymbol{A}$.

An f- subm A of X is named f- supplement, if there is f- subm B of X such that A is f-supplement of B.

4.2 Example

Let M =Z_4 , Define X: M → [0, 1] and define:

A: M → [0, 1], define by

A (t) = $\begin{cases} t & if \quad x \in N \\ 0 & otherwise \end{cases}$ ∀ t ∈ (0, 1] , N =2Z

And let B: M → [0, 1], define by

B (t) = $\begin{cases} t & if \quad x \in L \\ 0 & otherwise \end{cases}$ $\quad \forall \ t \in (0, 1]$, L=$\langle \bar{0} \rangle$

Clear that $A_t = \langle \bar{2} \rangle$, $B_t = \langle \bar{4} \rangle$ are two submodules of χ_t , χ_t = M = Z_4. Hence $(A+B)_t = (\chi)_t$ of χ_t , $\forall$ t ∈ (0, 1] by[3],therefor A+B = χ ∀ A,B αre a f- subm of χ. But B_t is a supplement in A_t of χ_t by [9], hence B= 0_1 of χ is f- supplement in A ofχ. On the other side, A is not f- supplement because 0_1 is a minimal in χ =Z_4.

4.3 Definition:

Let A, B are fuzzy submodules of f- module X. Then X is called amply f- supplemented with A + B = X if there is a supplement u of A such that u ⊆ B in X**.**

4.4 Example

Let M=Z_{12} as Z-module, define

X: M → [0,1] $as\ follow$, X(x) = $\begin{cases} 1 & if \quad x \in M \\ 0 & otherwise \end{cases}$ Define

A: M → [0,1] $such\ that$ A(x) = $\begin{cases} t & if \quad x \in N \\ 0 & otherwise \end{cases}$, t ∈ (0, 1]. N= 3Z Also

B: M → [0,1] $such\ that$ $B_{(x)}$ = $\begin{cases} t & if \quad x \in L \\ 0 & otherwise \end{cases}$, t ∈ (0, 1], L= 2Z

C: M → [0,1] $such\ that$ $B_{(x)}$ = $\begin{cases} t & if \quad x \in K \\ 0 & otherwise \end{cases}$, t ∈ (0, 1], K= 4Z

Clear that X is fuzzy module and X_t =M ,A_t =3Z , B_t=2Z and C_t=4Z are submodules in X_t with $(A + B)_t=(\chi)_t$, ∀ t ∈ (0, 1] ,therefore $\boldsymbol{A}$ + $\boldsymbol{B}$= χ where $\boldsymbol{A}$, $\boldsymbol{B}$ are fuzzy submodules of X . But $\boldsymbol{A}$ is fuzzy supplement of B in X .Also we have A + C = X where C is a f- submodule in X, again A_t is a supplement of C_t in X_t ∀ t ∈ (0, 1] therefore A is f- supplement of C in X. Thus X is amply f-supplemented.

4.5 Remark

Each a direct summand of f- Module is f- supplement submodule of X.

Proof

Let A be f- submodule of X, then there exist B of X such that (A ⊕ B)t= X_t then (A+B)t= X_t by [6] , therefore A ⊕ B = X then A + B = X and B∩ A is a proper f- submodule of A. To prove that A is supplement of B in X .Suppose that there exist C be a f- submodule in A such that C+B= X. Then A= X∩ A= (C+B) ∩ A but $((C+B) \cap A)_t = (Ct+ B_t) \cap A_t$ ∀ t ∈ (0, 1] implies that C+ (B∩ A) by [14] .But by our assumption (B∩ A) is a proper f- submodule in A implies that A=C . Thus A is supplement of B in X .

4.6 Proposition

Every Loc- hollow fuzzy module is an amply supplemented (supplement fuzzy) is a Ƒuzzy submodule.

Proof:

Suppose that χ is a Loc- hollow f- module, is a unique maximal Ƒuzzy submodule of χ. Since χ is Loc- hollow f- module, then we have $A_t+X_t=(X)_t$, $t\in(0,1]$ .This leads A+X=X and A∩X= A is a small fuzzy submodule of X by [9]. Therefore X is an amply supplemented module.

ɕonverse of ρroposition (4.6) is not satisfied by the following

4.7 Example

Let M =Z_6 , Define χ: M → [0, 1] and define by:

A: M → [0, 1], define by

$$A(t)=\begin{cases} t & if \quad x\in N \\ 0 & otherwise \end{cases} \qquad \forall\ t\in(0,1] \ , N=2Z$$

And let B: M → [0, 1], define by

$$B(t)=\begin{cases} t & if \quad x\in L \\ 0 & otherwise \end{cases} \qquad \forall\ t\in(0,1],\ L=3Z$$

Clear that $A_t=\langle\bar{2}\rangle$, $B_t=\langle\bar{3}\rangle$ are two submodule of X and X_t = M = Z_6. Hence $(A+B)_t=(X)_t$ direct summand of X_t is fuzzy submodule of X by[3] and there exists a supplement x_t of A in X_t such that $x_t \subseteq B_t$, ∀ t ∈ (0, 1]. Then X =Z_6 amply supplemented module. But Z_6 is not Loc- hollow y module. Since Z_6 has no unique maximal submodule. Thus X is supplemented fuzzy module but not Loc- hollow fuzzy module

4.8 Proposition

Every Loc- hollow f- module is an indecomposable f- module.

Proof:

Suppose that χ be a Loc- hollow f- module. ∃ a unique maximal fuzzy submodule say A which contains all fuzzy small submodule of X , let X is decomposable , Then There exists C,B are a proper fuzzy submodules of X and A,B≠ 0_1 , such that C , ***B*** are Ƒ- subms of A and χ= C ⊕ ***B*** hence $(X)_t=C_t \oplus B_t$, ∀ t ∈ (0, 1],But X is hollow then either

B is a small fuzzy of X with B is f- subm of A implies that X=C or C is a small fuzzy of X with C is f- submodule of A implies that X=B, which contradiction. Then X is indecomposable module.

4.9 Remark:

The convers of proposition (4.8) it is not always true, as well as, in the next example.

4.10 Example

Let M = Z –module Z, and A = 5Z

$$X: M\to[0,1] \quad s.t\ X(x)=\begin{cases} t & if \quad x\in M \\ 0 & otherwise \end{cases} \quad , \ \forall\ t\in(0,1].$$

clear that X_t =M and M is indecomposable module but Loc- hollow module[5] . Therefore X is indecomposable module but not Loc- hollow fuzzy module.

Recall that if X finitely generated f- Module then X/C is finitely generated f- module for every C a f- subm X. But the converse is not true. The following proposition shows if X is a hollow f- module and X/C a finitely generated f- module. Then X also a finitely generated f- module.

4.11 Proposition

Every Loc- hollow f- module is lifting f- module.

***ρ*roof:**

Let X be a Loc- hollow fuzzy module. ∃ a unique maximal fuzzy submodule says A which contains all fuzzy small submodule of X. So X= X ⊕ $\{0_1\}$ where $\{0_1\}$ is a fuzzy submodule of A, $\boldsymbol{A}$ ∩ χ= $\boldsymbol{A}$. But X Loc- hollow f- modules. Then $\boldsymbol{A}$ χ = $\boldsymbol{A}$ is a Ƒuzzy small submodule by [8] implies that X is lifting a fuzzy module.

4.12 Remark:

The convers of proposition (4.11) it is not satisfy we can show that by the following

4.13 Example:

Let X =Z_{10} , define X: M → [0, 1] define by:

$\boldsymbol{A}$: M → [0, 1], s.t

A (t) = $\begin{cases} t & if \quad x \in N \\ 0.5 & otherwise \end{cases}$ ∀ t ∈ (0, 1] , N= $\langle \bar{0} \rangle$

And let B: M → [0, 1], define by

B (t) = $\begin{cases} t & if \quad x \in L \\ 0.3 & otherwise \end{cases}$ ∀ t ∈ (0, 1] , L= 2Z

C (t) = $\begin{cases} t & if \quad x \in K \\ 0.25 & otherwise \end{cases}$ ∀ t ∈ (0, 1] , K= 5Z

clear that X_t = M = Z_{10} is fuzzy module, A_t = N , B_t = L and C_t = K are submodules in X_t .Hence(A ⊕ X$)_t$=(X$)_t$ is only direct summand of X_t is fuzzy submodule of X_t, and $X_t \cap A_t = 0_1$ is a small submodule in $B_t = \langle \bar{2} \rangle$. But Z_{10} has no unique maximal fuzzy submodule.

The convers of proposition (4.12) is achieved if the following condition is hold

4.13 Proposition:

Let X be a cyclic fuzzy indecomposable module of an R-module M .If X is lifting fuzzy module, then X is Loc- hollow fuzzy module.

Proof:

Let A be a proper f- submodule of X, but χ is lifting f- module. Then χ = $\boldsymbol{A}$+B, where B is f- submodule and A ∩ B is a small f- submodule of A. But X is an indecomposable fuzzy module, implies that B=0_1 and hence A_t = X_t, ∀ t ∈ (0, 1] by [9], since X_t is a hollow fuzzy module and hence A= X. Which implies that A ∩ X = A, clear that we have $A_t \cap X_t = A_t$, ∀ t ∈ (0, 1]. Then A is a small f- submodule of X. So X is hollow fuzzy module and since X is cyclic fuzzy module. Then X is Loc- hollow fuzzy module through Proposition (3.1).

4.15 Proposition

Let C be a maximal f- submodule of a fuzzy module X. If B is a f- supplement of C in X, therefore B is a Loc- hollow fuzzy module.

Proof:

Let B is f- supplement of C, let B_1 be a proper f- subm of B with B_1+B_2= B for some f-submodule B_2 of B and $(B_1+B2)_t= (B)_t$, ∀ t ∈ (0, 1]. Let C+B = X = C+ B_1+B_2= X and B_1 is a f- submodule of C, B_1= X , by [1],also C is a maximal f- subm of X therefore B_1=B, which contradiction with our assumption .Thus C +B=X and since C is a maximal f- submodule of X so we have B_2=B implies that C is a hollow fuzzy module. To prove that C is a cyclic fuzzy module, let $x_t \subseteq X$ and $x_t \subseteq$ C, so there exists (x_t) a small submodule of X_t, ∀ t∈ (0,1]. R_x +C =X and this implies that R_x= C, where $(R_x)_t = C_t$ ∀ t∈ (0,1] through dependent on minimality of C and by Proposition (3.1). Thus B is Loc- hollow fuzzy module.

5 . Conclusion:

In this work, we introduced a concept of a hollow f- module, where a module is said to be a hollow fuzzy when every subm is a small f-subm. Also, fuzzify these concepts L-hollow module to Loc-hollow f- modules. Moreover, we generalized numerous properties of Loc-hollow f- modules and got several results and fundamental properties of L-hollow f- modules which are argued. Includes the relation between hollow f- modules and Loc-hollow f-modules. However, allocated the relation between Loc-hollow f- modules and different modules like amply supplemented f- modules, indecomposable f- modules, and lifting f- modules. Finally, it is significant to remark that the characterizations and properties that are fundamentally related to these concepts are introduced.

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