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Iterative Method for Solving a Nonlinear Fourth Order Integro-Differential Equation

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Abstract

This study presents the execution of an iterative technique suggested by Temimi and Ansari (TA) method to approximate solutions to a boundary value problem of a 4th-order nonlinear integro-differential equation (4th-ONIDE) of the type Kirchhoff which appears in the study of transverse vibration of hinged shafts. This problem is difficult to solve because there is a non-linear term under the integral sign, however, a number of authors have suggested iterative methods for solving this type of equation. The solution is obtained as a series that merges with the exact solution. Two examples are solved by TA method, the results showed that the proposed technique was effective, accurate, and reliable. Also, for greater reliability, the approximate solutions were compared with the classic Runge-Kutta method (RK4M) where good agreements were observed. For more accuracy the maximum error remainder was found, and the absolute error was computed between the semi-analytical method and the numerical method RK4M. *Mathematica*® 11 was used as a program for calculations.

Keywords an iterative technique, approximate solutions, fourth order integro-differential equation, maximum error remainder.

1.Introduction

Many real-life phenomena engineering and physics are mathematically modeled into functional equations. Many mathematical formulas for physical phenomena and fluid dynamics contain differential and integral equations [1]. [2].

In this paper, the integro-differential equation of the fourth order of Kirchhoff type is considered as

$$\left.\begin{array}{c} \upsilon^{(4)}(t)-\delta\upsilon''(t)-\frac{2}{B}\int_0^B\left[\upsilon'(t)\right]^2dt\,\upsilon''(t)=k(t), \quad 0<t<B, \\ \upsilon(0)=\upsilon(B)=\upsilon''(0)=\upsilon''(B)=0, \end{array}\right] \tag{1}$$

where $\upsilon(t)$ represents the constant deviation of the shafts, $k(t)$ is a continuous function defined over $[0, B]$and δ and B are positive constants. This type of problem represents a bending equilibrium model for expandable shafts that are simply supported on nonlinear bases. The force that the foundation applies to the shafts is represented by the function, $k(t)$. Small changes in shafts length and their effects are represented by : $\frac{2}{B}\int_0^B[\upsilon'(t)]^2dt$ [3]. It is obvious that it is difficult to find the exact solution to Equation (1), so many numerical and semi-analytical methods have been developed, such as [4] reduced the order of the equation to find the root of a nonlinear equation and solve it by Newton's method. [5] used the Optimal Homotopy Asymptotic strategy to solve equation (1) and proved the efficacy of the procedure by comparing it with a finite element technique. [6] proposed a numerical scheme for solving Equation (1) by transforming the problem into a nonlinear algebraic system and then solving it in an iterative method with the collocation technique.

[7] suggested an iterative method namely Tamimi and Ansari (TA) method for solving nonlinear equations, and this technique was applied to solve numerous differential equations, for example, Korteweg-de Vries Equations [8], thin-film flow problems [9], Falkner-Skan problem [10]. In this work, the TA method is used to solve the 4th-ONIDE of the type Kirchhoff which appears in the study of transverse vibration of hinged shafts. The results acquired from the method showed that TA method is accurate, fast, and convenient. Two examples are solved and the results obtained are compared with the RK4M method, and the maximum error reminder is calculated.

This article is arranged as follows: Section 2 provides a clear formulation of the iterative method in general and specifically on Equation (1); Section 3 illustrates the effectiveness of the method with various examples before giving a brief conclusion in Section 4.

1. The proposed method

To clarify the fundamental thought of TA method. Let us consider the general equation

$$\mathcal{L}(\upsilon(t)) + \aleph(\upsilon(t)) + g(t) = 0, \tag{2}$$

with boundary conditions $\mathcal{B}\left(\upsilon, \frac{d\upsilon}{dt}\right) = 0,$

where υ is unknown function, $\mathcal{L}$ is the linear operator, $\aleph$ is the nonlinear operator and g is a known function.

Assuming that $\upsilon_0(t)$ is a solution of Equation (3) of the initial condition

$$\mathcal{L}(\upsilon_0(t)) + g(t) = 0, \text{ with } \mathcal{B}\left(\upsilon_0, \frac{d\upsilon_0}{dt}\right) = 0\ . \tag{3}$$

To find the next iteration, we resolve the following equation:

$$\mathcal{L}(\upsilon_1(t)) + \aleph(\upsilon_0(t)) + g(t) = 0, \mathcal{B}\left(\upsilon_1, \frac{d\upsilon_1}{dt}\right) = 0.$$

Thus, an iterative procedure can be an effective solution to the following problem,

$$\mathcal{L}(\upsilon_{n+1}(t)) + \aleph(\upsilon_n(t)) + \mathrm{g(t)} = 0, \mathcal{B}\left(\upsilon_{n+1}, \frac{\mathrm{d}\upsilon_{n+1}}{dt}\right) = 0. \quad (4)$$

Each of υ_i are solutions to Equation (1) [7].

We will apply the steps of the method in solving equation (1), so it can be formulated as:

$$\left.\begin{array}{c}\mathcal{L}(\upsilon(\mathrm{t}))-\delta\mathrm{R}(\upsilon(\mathrm{t}))-\frac{2}{\mathrm{B}}(\int_0^{\mathrm{B}}\aleph(\upsilon(\mathrm{t}))\mathrm{dt})\ \mathrm{R}(\upsilon(\mathrm{t}))=\mathrm{k(t)}, \quad 0<\mathrm{t}<\mathrm{B}, \\ \upsilon(0)=\upsilon(\mathrm{B})=\upsilon''(0)=\upsilon''(\mathrm{B})=0,\end{array}\right] \quad (5)$$

Where $\mathcal{L}$ is the fourth derivative, R is the second derivative and $\aleph$ the nonlinear term.

$\upsilon_0(t)$ can be considered as a solution to the initial problem:

$$\mathcal{L}(\upsilon_0(t)) = \mathrm{k(t)}, \text{with } \upsilon_0(0) = \upsilon_0(B) = {\upsilon_0}''(0) = {\upsilon_0}''(B) = 0, \quad (6)$$

$\upsilon_1(t)$ can be found by solving Equation (8):

$$\left.\begin{array}{c}\mathcal{L}(\upsilon_1(t))=\delta\mathrm{R}(\upsilon_0(t))+\frac{2}{\mathrm{B}}(\int_0^{\mathrm{B}}\aleph(\upsilon_0(t))\mathrm{dt})\ \mathrm{R}(\upsilon_0(t))+\mathrm{k(t)}, \\ \upsilon_1(0)=\upsilon_1(\mathrm{B})={\upsilon_1}''(0)={\upsilon_1}''(\mathrm{B})=0,\end{array}\right] \quad (7)$$

Likewise, more solutions can be gained for the problems created by:

$$\left.\begin{array}{c}\mathcal{L}(\upsilon_{n+1}(t))=\delta\mathrm{R}(\upsilon_n(t))+\frac{2}{\mathrm{B}}(\int_0^{\mathrm{B}}\aleph(\upsilon_n(t))\mathrm{dt})\ \mathrm{R}(\upsilon_n(t))+\mathrm{k(t)}, \\ \upsilon_{n+1}(0)=\upsilon_{n+1}(\mathrm{B})={\upsilon_{n+1}}''(0)={\upsilon_{n+1}}''(\mathrm{B})=0,\end{array}\right] \quad (8)$$

2. Solution of the problem

In this section, we will solve two problems using the method defined above

Example 1. Consider the 4th-ONIDE [6]

$$\upsilon^{(4)}(t) - \upsilon''(t) - \int_0^1 (\upsilon'(t))^2\, dt\, \upsilon''(t) = 1 \quad, 0 < t < 1,$$

$$\upsilon(0) = \upsilon(1) = \upsilon''(0) = \upsilon''(1) = 0, \quad (9)$$

we start with the assumption that

$$\mathcal{L}(\upsilon(\mathrm{t})) = \upsilon^{(4)}(t)\ , \aleph(\upsilon) = -\upsilon''(t) - \int_0^1 (\upsilon'(t))^2\, dt\, \upsilon''(t), g(t) = -1 \quad (10)$$

With $u(0) = u(1) = u''(0) = u''(1) = 0,$ (11)

Let us start with the first iteration

$${\upsilon_0}^{(4)}(t) = 1, \quad (12)$$

with $\upsilon_0(0) = \upsilon_0(1) = {\upsilon_0}''(0) = {\upsilon_0}''(1) = 0,$ (13)

Then, we get

$$v_0(t) = 0.04166(t - 2t^3 + t^4), \tag{14}$$

The second repetition can be performed as

$$v_1^{(4)}(t) = 1 + v_0''(t) + \int_0^1 \left(v_0'(t)\right)^2 dt\, v_0''(t), \tag{15}$$

with $v_1(0) = v_1(1) = v_1''(0) = v_1''(1) = 0,$ (16)

Thus, we get

$$v_1(t) = 6.889329 \times 10^{-8}(544269t - 1108715t^3 + 604800t^4 - 60531t^5 + 20177t^6) \tag{17}$$

In the third repetition, the following equation must be solved:

$$v_2^{(4)}(t) = 1 + v_1''(t) + \int_0^1 \left(v_1'(t)\right)^2 dt\, v_1''(t), \tag{18}$$

Then, we get

$$v_2(t) = 3.7916 \times 10^{-26}(1.00009 \times 10^{24}t - 2.032907 \times 10^{24}t^3 + 1.0989 \times 10^{24}t^4 - 1.00795 \times 10^{23}t^5 + 3.6655 \times 10^{22}t^6 - 2.6204 \times 10^{21}t^7 + 6.5511 \times 10^{20}t^8) \tag{19}$$

We showed the first three solutions but we did not show the later ones because they get longer.

By applying the (RK4M) using *Mathematica*® as a standard for evaluating TA method performance. In **Figure 1**, good agreement was observed between RK4M and TA method.

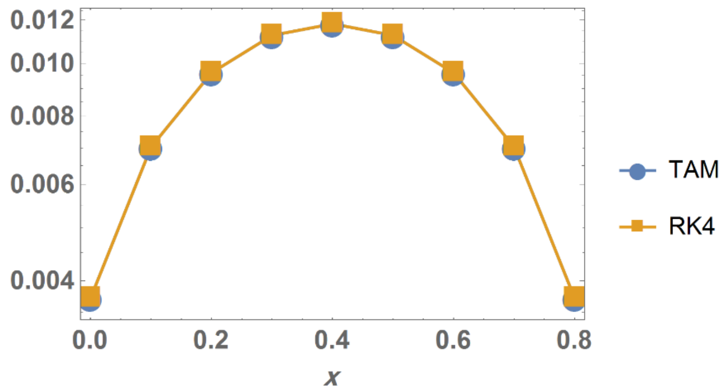

Figure 1. The TA method and RK4M solutions for the problem

The logarithmic plots for MER will be presented in **Figure 2** for the approximate solution obtained by TA method for solving 4th-ONIDE for different values of n=1 to 5.

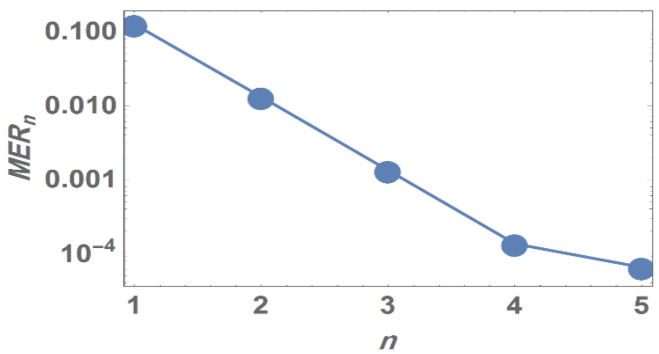

Figure 2. Maximum Error Reminder for Solution by TA method

Example 2: Consider the 4th-ONIDE [6]

$$\upsilon^{(4)}(t) - \upsilon''(t) - 2\int_0^1 \left(\upsilon'(t)\right)^2 dt\, \upsilon''(t) = -t \quad, 0 < t < 1, \tag{20}$$

$$\upsilon(0) = \upsilon(1) = \upsilon''(0) = \upsilon''(1) = 0, \tag{21}$$

we start with the assumption that

$$\mathcal{L}(\upsilon(\mathrm{t})) = \upsilon^{(4)}(t)\,, \aleph(\upsilon) = -\upsilon''(t) - 2\int_0^1 \left(\upsilon'(t)\right)^2 dt\, \upsilon''(t), g(t) = t \tag{22}$$

With $\upsilon(0) = \upsilon(1) = \upsilon''(0) = \upsilon''(1) = 0,$ (23)

Let us start with the first iteration

$$\upsilon_0{}^{(4)}(t) = -t, \tag{24}$$

with $\upsilon_0(0) = \upsilon_0(1) = \upsilon_0{}''(0) = \upsilon_0{}''(1) = 0,$ (25)

Then, we get

$$\upsilon_0(t) = 0.0027(-7t + 10t^3 - 3t^5) \tag{26}$$

The second repetition can be performed as

$$\upsilon_1{}^{(4)}(t) = -t + \upsilon_0{}''(t) + 2\int_0^1 \left(\upsilon_0{}'(t)\right)^2 dt\, \upsilon_0{}''(t), \tag{27}$$

with $\upsilon_1(0) = \upsilon_1(1) = \upsilon_1{}''(0) = \upsilon_1{}''(1) = 0,$ (28)

Thus, we get

$$\upsilon_1(t) = 1.3997 \times 10^{-8}(-1242613t + 1752877t^3 - 496083t^5 - 14181t^7) \tag{29}$$

In the third repetition, the following equation must be solved:

$$\upsilon_2{}^{(4)}(t) = -t + \upsilon_1{}''(t) + 2\int_0^1 \left(\upsilon_1{}'(t)\right)^2 dt\, \upsilon_1{}''(t), \tag{30}$$

Then, we get

$$\upsilon_2(t) = 8.1826 \times 10^{-27}(-2.1513 \times 10^{24}t + 3.0403 \times 10^{24}t^3 - 8.6844 \times 10^{23}t^5 - 2.0212 \times 10^{22}t^7 - 3.3703 \times 10^{20}t^9)) \tag{31}$$

We showed the first three solutions but we did not show the later ones because they get longer.

By applying the (RK4M) using MATHEMATICA as a standard for evaluating TA method performance. In **Figure 3,** good agreement was observed between RK4M and TA method.

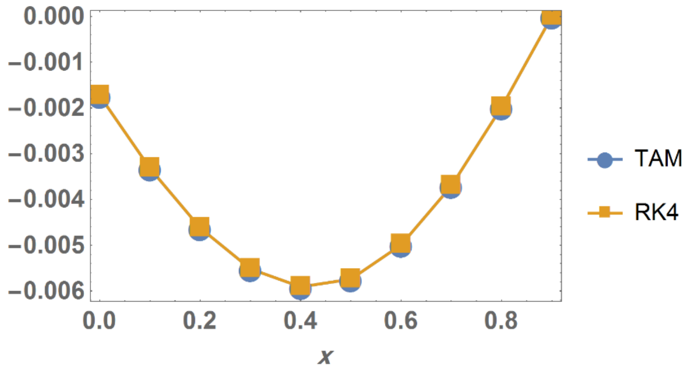

Figure 3. The TA method and RK4M solutions for the problem

The logarithmic plots for MER will be presented in **Figure 2** for the approximate solution obtained by TA method for solving 4th-ONIDE for different values of n=1 to 5.

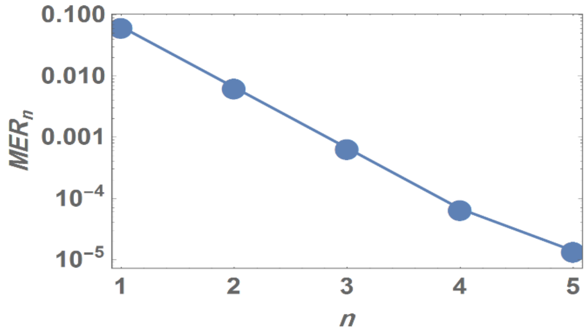

Figure 4. Maximum Error Reminder for Solution by TA method

The absolute error was calculated by RK4M and TA method, we note a good approximation as shown in Table 1 for both examples.

Table 1. The absolute error for RK4M and TA method

x	Absolute error for example 1	Absolute error for example 2
	$\lvert v_{TAM}(t) - v_{RK4}(t)\rvert$	$\lvert v_{TAM}(t) - v_{RK4}(t)\rvert$
0	$1.796565911 \times 10^{-7}$	2.506032872×10^{-8}
0.1	$3.351539886 \times 10^{-7}$	$4.987247103 \times 10^{-8}$
0.2	$4.575623109 \times 10^{-7}$	$7.253721456 \times 10^{-8}$
0.3	$5.365609015 \times 10^{-7}$	$9.058700125" \times 10^{-8}$
0.4	$5.636405136 \times 10^{-7}$	$1.013928555 \times 10^{-7}$
0.5	$5.35561508 \times 10^{-7}$	$1.028496388 \times 10^{-7}$
0.6	$4.558146015 \times 10^{-7}$	$9.384872716 \times 10^{-8}$
0.7	$3.331583074 \times 10^{-7}$	$7.409719157 \times 10^{-8}$
0.8	$1.781605395 \times 10^{-7}$	$4.326019289 \times 10^{-8}$
0.9	0	$1.734723476 \times 10^{-18}$

3. Conclusion

In this work, a proposed iterative technique is presented for solving a class of 4th-ONIDE Kirchhoff type that emerges in the investigation of transversal vibrations of hinge shafts. The TA method is an effective method for solving various kinds of linear and non-linear problems accurately. The fundamental benefit of this technique is that it does not require the estimation of variables. It very well may be seen that when the number of repetition increases, the maximum error reminder have decreased. In comparison with the RK4M, we find a great agreement in the results through the above figures and table.

References

1. Sabaa, M. A.; Mohammed, M. A.; Abd Almjeed, S. H. Approximate Solutions for Alcohol Consumption Model in Spain. *Ibn Al Haitham Journal for Pure and Applied Science*, **2019**, *32, 3*, 153-164.

2. Tawfiq, L. N. M.; Hasan, M. A. Evaluate theRate of Pollution in Soil using SimulinkEnvironment. *Ibn Al Haitham Journal for Pure and Applied Science*, **2019**, *32, 1*, 132-138.

3. To Fu Ma, Positive Solutions for a Nonlocal Fourth Order Equation of Kirchhoff Type, discrete and continuous dynamical systems. *Discrete and Continuous Dynamical Systems Supplement*, **2007**, (Special) 694–703 , doi: 10.3934/proc.2007.2007.694 .

4. Dang,O. A.; Luan,V. T. Iterative method for solving a nonlinear fourth order boundary value problem, *Computers and Mathematics with Applications*, **2010**, *60, 1*, 112_121.

5. Temimia, H; Ansari, A.R.; Siddiqui, A. M. An approximate solution for the static beam problem and nonlinear integro-differential equations. *Computers and Mathematics with Applications***, 2011**, *62, 8*, 3132–3139.

6. Ren, Q.; Tian, H. Numerical solution of the static beam problem by Bernoulli collocation method. *Applied Mathematical Modelling*, **2016**, *40, 21-22*, 8886–8897.

7. Temimia, H; Ansari, A.R A semi-analytical iterative technique for solving nonlinear problems. *Computers and Mathematics with Applications*, **2011,** *61, 2*, 203–210.

8. Ehsani, F.; Hadi, A.; Ehsani,F.; Mahdavi, R. An Iterative Method for Solving Partial Differential Equations and Solution of Korteweg-de Vries Equations for Showing the Capability of the Iterative Method. *World Applied Programming*, **2013**, *3, 8*, 320-327.

9. AL-Jawary, M.A. A semi-analytical iterative method for solving nonlinear thin film flow problems, *Chaos, Solitons and Fractals*,**2017**, *99*, 52–56.

10. AL-JAWARY, M. A.; ADWAN, M. I. Reliable Iterative Methods for Solving the Falkner-Skan Equation, *Gazi University Journal of Science*, **2020**, *l33, 1*, 168-186.