Ibn Al-Haitham Jour. for Pure & Appl. Sci. 35(1)2022

Ibn Al Haitham Journal for Pure and Applied Science

Journal homepage: http://jih.uobaghdad.edu.iq/index.php/j/index

The Homomorphism of Cubic bipolar ideals of a KU-semigroup

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Article history: Received 22, June 2021, Accepted 4,October 2021, Published in January 2022.

Doi: 10.30526/35.1.2801

Abstract

The idea of a homomorphism of a cubic set of a KU-semigroup is studied and the concept of the product between two cubic sets is defined. And then, a new cubic bipolar fuzzy set in this structure is discussed, and some important results are achieved. Also, the product of cubic subsets is discussed and some theorems are proved.

2010 AMS Classification: 06F35, 03G25, 08A72 .

Key words: A KU-semigroup, a cubic sub KU-semigroup, a cubic bipolar *k*-ideal, homomorphism.

1.Introduction

In 2009, Prabpayak and Leerawat [1,2] studied a new algebra called a KU-algebra. They introduced homomorphism in a KU-algebra and discussed some recent results. After that, Mostafa et al. [3, 4] studied the concepts of fuzzy KU-ideals and an interval value fuzzy KU-ideals. In [5] Kareem and Hasan presented the structure of a KU-semigroup and introduced some ideals of this structure. After that, they introduced the fuzzy ideals of this structure in [6]. In [7] Kareem and Talib gave the concept of an interval value fuzzy some ideal in KU-semigroup.

Jun et al. [8, 9] introduced the concept of cubic subalgebras/ideals in BCK/BCI-algebras. Yaqoob et al [10] presented a cubic KU-algebra and discussed a few interesting theorems. This work studied the idea of a homomorphism of a cubic set of a KU-semigroup, and a new cubic bipolar fuzzy set in this structure is defined, and some important results are achieved. Also, the product of cubic subsets was discussed, and a few theorems were proved.

2. Basic concepts

We introduce some definitions, propositions and theorems of KU-algebra and KU-semigroup in this part.

Definition2.1[1]. A KU-algebra $(\aleph,*,0)$ is satisfied the following conditions, for all $\alpha, \beta, \delta \in \aleph$,

(ku_1) $(\alpha * \beta) * [(\beta * \delta) * (\alpha * \delta)] = 0$

(ku_2) $\alpha * 0 = 0$

(ku_3) $0 * \alpha = \alpha$

(ku_4) $\alpha * \beta = 0$ and $\beta * \alpha$ implies $\alpha = \beta$ and

(ku_5) $\alpha * \alpha = 0$

The relation $\leq$ on a KU-algebra $\aleph$ is define $\alpha \leq \beta \Leftrightarrow \beta * \alpha = 0$.

Example 2.2 [1]. The following table define the binary operation $*$ on the set $\aleph = \{0, a, b, c\}$

$*$	0	a	b	c
0	0	a	b	c
a	0	0	0	b
b	0	b	0	a
c	0	0	0	0

Then $(\aleph,*,0)$ is a KU-algebra.

Theorem2.3[2]. The following axioms are satisfying, in a KU-algebra $\aleph$.

For all $\alpha, \beta, \delta \in \aleph$,

(1) if $\alpha \leq \beta$ imply $\beta * \delta \leq \alpha * \delta$

(2) $\alpha * (\beta * \delta) = \beta * (\alpha * \delta)$

(3) $((\beta * \alpha) * \alpha) \leq \beta$

Definition2.4[5]. The set $\aleph \neq \varphi$ and two binary operations $*, \circ$ with a constant 0 is named a KU-semigroup if

(I) The triple $(\aleph,*,0)$ is a KU-algebra

(II) the ordered pair $(\aleph,\circ)$ is a semigroup

(III) $\alpha, \beta, \delta \in \aleph$,, $\alpha \circ (\beta * \delta) = (\alpha \circ \beta) * (\alpha \circ \delta)$ and $(\alpha * \beta) \circ \delta = (\alpha \circ \delta) * (\beta \circ \delta)$.

Example 2.5[5]. If $\aleph = \{0,1,2,3\}$ is a set and two binary operations $*$ and $\circ$ are defined by the following.

$*$	0	1	2	3
0	0	1	2	3
1	0	0	0	2
2	0	2	0	1
3	0	0	0	0

$\circ$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Then $(\aleph,*,\circ,0)$ is a KU-semigroup.

Definition 2.6[5]. A non-empty subset A of $\aleph$ is named A *sub*KU-*semigroup* if it is satisfied $\alpha * \beta, \alpha \circ \beta \in A$, for all $\alpha, \beta \in A$.

Definition2.7[5]. A non-empty subset $I \subseteq \aleph$ is called an *S-ideal* of $\aleph$, if

(i) $0 \in I$

(ii) $\alpha * \beta \in I$ and $\alpha \in I$ imply $\beta \in I$.

(iii) $\forall \alpha \in \aleph$, $\boldsymbol{e} \in I$, we have $\alpha \circ \boldsymbol{e} \in I$ and $\boldsymbol{e} \circ \alpha \in I$.

Definition2.8[5]. The set $\varphi \neq A \subseteq \aleph$ is named a *k-ideal* of $\aleph$, if

i) $0 \in I$

ii) $\forall \alpha, \beta, \delta \in \aleph$, $(\alpha * (\beta * \delta)) \in I$, $\beta \in I$ imply $\alpha * \delta \in I$.

iii) $\forall \alpha \in \aleph$, $\boldsymbol{e} \in A$, we have $\alpha \circ \boldsymbol{e} \in A$ and $\boldsymbol{e} \circ \alpha \in A$.

Definition2.9[5]. Let $\aleph$ and $\aleph'$ be two KU-semigroups. A mapping $f: \aleph \to \aleph'$ is called a KU-semigroup homomorphism if $f(\alpha * \beta) = f\alpha * f(\beta)$ and $f(\alpha \circ \beta) = f(\alpha) \circ f(\beta)$ for all $\alpha, \beta \in \aleph$. The set $\{\alpha \in \aleph: f(\alpha) = 0\}$ is called the kernel of f and denoted by $ker f$ Moreover, the set $\{ f(\alpha) \in \aleph' : \alpha \in \aleph\}$ is called the image of f and denoted by $im f$.

We recall that a cubic bipolar valued fuzzy subset in [11] as follows:

Definition 2.10[11]. Let $\aleph$ be a non-empty set. A cubic bipolar set in a set $\aleph$ is the structure $\Omega = \{\langle \alpha, \tilde{\mu}_\Omega^+(\alpha), \tilde{\mu}_\Omega^-(\alpha), \lambda_\Omega^+(\alpha), \lambda_\Omega^-(\alpha) : \alpha \in \aleph \rangle\}$ where $N(\alpha) = \{\tilde{\mu}_\Omega^+(\alpha), \tilde{\mu}_\Omega^-(\alpha)\}$ is called interval valued bipolar fuzzy set and $K(\alpha) = \{\lambda_\Omega^+(\alpha), \lambda_\Omega^-(\alpha)\}$ is a bipolar fuzzy set. Consider $\tilde{\mu}_\Omega^+: \aleph \to D[0,1]$ such that $\tilde{\mu}_\Omega^+(\alpha) = [\delta_{\Omega_L}^+(\alpha), \delta_{\Omega_U}^+(\alpha)]$ and $\tilde{\mu}_\Omega^-: \aleph \to D[-1,0]$ such that $\tilde{\mu}_\Omega^-(\alpha) = [\delta_{\Omega_L}^-(\alpha), \delta_{\Omega_U}^-(\alpha)]$, also $\lambda_\Omega^+: \aleph \to [0,1]$, and $\lambda_\Omega^-: \aleph \to [-1,0]$ it follows that

$\Omega = \{< \alpha, \{[\delta_{\Omega_L}^+(\alpha), \delta_{\Omega_U}^+(\alpha)], [\delta_{\Omega_L}^-(\alpha), \delta_{\Omega_U}^-(\alpha)]\}, \lambda_\Omega^+(\alpha), \lambda_\Omega^-(\alpha)\} >: \alpha \in \aleph\}$

It is a cubic bipolar set and can be denoted by $\Omega = \langle N, K \rangle$.

Definition2.11[11]. A Cubic bipolar set $\Omega = \langle N, K \rangle$ is named a cubic bipolar sub KU-semigroup if: $\forall \alpha, \beta \in \aleph$,

(1) $\tilde{\mu}_\Omega^+(\alpha * \beta) \geq rmin\{\tilde{\mu}_\Omega^+(\alpha), \tilde{\mu}_\Omega^+(\beta)\}$, $\tilde{\mu}_\Omega^-(\alpha * \beta) \leq rmax\{\tilde{\mu}_\Omega^-(\alpha), \tilde{\mu}_\Omega^-(\beta)\}$
$\lambda_\Omega^+(\alpha * \beta) \geq min\{\lambda_\Omega^+(\alpha), \lambda_\Omega^+(\beta)\}$, $\lambda_\Omega^-(\alpha * \beta) \leq max\{\lambda_\Omega^-(\alpha), \lambda_\Omega^-(\beta)\}$,

(2) $\tilde{\mu}_\Omega^+(\alpha \circ \beta) \geq rmin\{\tilde{\mu}_\Omega^+(\alpha), \tilde{\mu}_\Omega^+(\beta)\}$, $\tilde{\mu}_\Omega^-(\alpha \circ \beta) \leq rmax\{\tilde{\mu}_\Omega^-(\alpha), \tilde{\mu}_\Omega^-(\beta)\}$
$\lambda_\Omega^+(\alpha \circ \beta) \geq min\{\lambda_\Omega^+(\alpha), \lambda_\Omega^+(\beta)\}$, $\lambda_\Omega^-(\alpha \circ \beta) \leq max\{\lambda_\Omega^-(\alpha), \lambda_\Omega^-(\beta)\}$,

Example2.12[11]. If $\aleph = \{0,1,2,3\}$ is a set and two binary operations $*$ and $\circ$ are define by the following.

*	0	1	2	3
0	0	1	2	3
1	0	0	0	2
2	0	2	0	1
3	0	0	0	0

∘	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Then $(\aleph, *, \circ, 0)$ is a KU-semigroup. Define $\Omega = \langle N, K \rangle$ as follows

$N(\alpha) = \begin{cases} \{[-0.2, -0.5], [0.1, 0.9]\} & if \quad \alpha = \{0,1\} \\ \{[-0.6, -0.2], [0.2, 0.5]\} & if \quad otherwise \end{cases}$, $\lambda_\Omega^+(\alpha) = \begin{cases} 0.5 & if \quad \alpha = \{0,1\} \\ 0.3 & if \quad otherwise \end{cases}$ $\lambda_\Omega^-(\alpha) = \begin{cases} -0.6 & if \quad \alpha = \{0,1\} \\ -0.3 & if \quad otherwise \end{cases}$

then $\Omega = \langle N, K \rangle$ is a cubic bipolar sub KU-semigroup of $\aleph$.

Definition 2.13[11]. A cubic bipolar set $\Omega = \langle N, K \rangle$ in $\aleph$ is called a cubic bipolar ideal of $\aleph$ if, $\forall \alpha, \beta \in \aleph$

(BC1) $\tilde{\mu}_\Omega^+(0) \geq \tilde{\mu}_\Omega^+(\alpha)$, $\lambda_\Omega^+(0) \geq \lambda_\Omega^+(\alpha)$ and $\tilde{\mu}_\Omega^-(0) \leq \tilde{\mu}_\Omega^-(\alpha)$, $\lambda_\Omega^-(0) \leq \lambda_\Omega^-(\alpha)$

(BC2) $\tilde{\mu}_\Omega^+(\beta) \geq rmin\{\tilde{\mu}_\Omega^+(\alpha*\beta), \tilde{\mu}_\Omega^+(\alpha)\}, \tilde{\mu}_\Omega^-(\beta) \leq rmax\{\tilde{\mu}_\Omega^-(\alpha*\beta), \tilde{\mu}_\Omega^-(\alpha)\}$and
$\lambda_\Omega^+(\beta) \geq min\{\lambda_\Omega^+(\alpha*\beta), \lambda_\Omega^+(\alpha)\}, \lambda_\Omega^-(\beta) \leq max\{\lambda_\Omega^-(\alpha*\beta), \lambda_\Omega^-(\alpha)\}$,
(BC3)$\tilde{\mu}_\Omega^+(\alpha\circ\beta) \geq rmin\{\tilde{\mu}_\Omega^+(\alpha), \tilde{\mu}_\Omega^+(\beta)\}, \tilde{\mu}_\Omega^-(\alpha\circ\beta) \leq rmax\{\tilde{\mu}_\Omega^-(\alpha), \tilde{\mu}_\Omega^-(\beta)\}$ and
$\lambda_\Omega^+(\alpha\circ\beta) \geq min\{\lambda_\Omega^+(\alpha), \lambda_\Omega^+(\beta)\}, \lambda_\Omega^-(\alpha\circ\beta) \leq max\{\lambda_\Omega^-(\alpha), \lambda_\Omega^-(\beta)\}$.

Example 2.14[11]. If $\aleph = \{0,1,2\}$ is a set and two binary operations $*$ and $\circ$ are defined by the following.

$*$	0	1	2
0	0	1	2
1	0	0	1
2	0	1	0

$\circ$	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

Then$(\aleph,*,\circ,0)$ is a KU-semigroup. Define$\Omega = \langle N, K\rangle$ as follows

$N(\alpha) = \begin{cases}\{[-0.3,-0.1],[0.1,0.8]\} & if \quad \alpha = 0\\ \{[-0.7,-0.3],[0.4,0.6]\} & otherwise\end{cases}$, $\lambda_\Omega^+(\alpha) = \begin{cases}0.9 & if \quad \alpha = 0\\ 0.4 & if \quad otherwise\end{cases}$ $\lambda_\Omega^-(\alpha) = \begin{cases}-0.8 & if \quad \alpha = 0\\ -0.3 & if \quad otherwise\end{cases}$

We can easily prove that$\Omega = \langle N, K\rangle$is a cubic bipolarideal of$\aleph$.

Definition 2.15[11]. A cubicbipolar set $\Omega = \langle N, K\rangle$in $\aleph$is called a cubic bipolar$k$-ideal of $\aleph$ if, $\forall \alpha, \beta, \delta \in \aleph$

(a) $\tilde{\mu}_\Omega^+(0) \geq \tilde{\mu}_\Omega^+(\alpha), \lambda_\Omega^+(0) \geq \lambda_\Omega^+(\alpha)$ and $\tilde{\mu}_\Omega^-(0) \leq \tilde{\mu}_\Omega^-(\alpha), \lambda_\Omega^-(0) \leq \lambda_\Omega^-(\alpha)$.

(b) $\tilde{\mu}_\Omega^+(\alpha*\delta) \geq rmin\{\tilde{\mu}_\Omega^+((\alpha*(\beta*\delta)), \tilde{\mu}_\Omega^+(\beta)\}$,
$\tilde{\mu}_\Omega^-(\alpha*\delta) \leq rmax\{\tilde{\mu}_\Omega^-((\alpha*(\beta*\delta)), \tilde{\mu}_\Omega^-(\beta)\}$and
$\lambda_\Omega^+(\alpha*\delta) \geq min\{\lambda_\Omega^+((\alpha*(\beta*\delta)), \lambda_\Omega^+(\beta)\}$,
$\lambda_\Omega^-(\alpha*\delta) \leq max\{\lambda_\Omega^-((\alpha*(\beta*\delta)), \lambda_\Omega^-(\beta)\}$,

(c)$\tilde{\mu}_\Omega^+(\alpha\circ\beta) \geq rmin\{\tilde{\mu}_\Omega^+(\alpha), \tilde{\mu}_\Omega^+(\beta)\}, \tilde{\mu}_\Omega^-(\alpha\circ\beta) \leq rmax\{\tilde{\mu}_\Omega^-(\alpha), \tilde{\mu}_\Omega^-(\beta)\}$
$\lambda_\Omega^+(\alpha\circ\beta) \geq min\{\lambda_\Omega^+(\alpha), \lambda_\Omega^+(\beta)\}, \lambda_\Omega^-(\alpha\circ\beta) \leq max\{\lambda_\Omega^-(\alpha), \lambda_\Omega^-(\beta)\}$.

3. A Cubic Bipolar$k$-ideal underHomomorphism

We study some definitions of homomorphism; the product of cubic bipolar k-ideals and a cubic bipolar ideal. Also some theorems are discussed.

Definition 3.1. For any $\alpha \in \aleph$. We define a new cubicbipolar fuzzy set $\Omega_f = (\alpha, \tilde{\mu}_f^+, \tilde{\mu}_f^-, \lambda_f^-, \lambda_f^+)$in $\aleph$ by $\tilde{\mu}_f^-(\alpha) = \tilde{\mu}^-(f(\alpha))$ and$\tilde{\mu}_f^+(\alpha) = \tilde{\mu}^+(f(\alpha))$,$\lambda_f^-(\alpha) = \lambda^-(f(\alpha))$ and$\lambda_f^+(\alpha) = \lambda^+(f(\alpha))$,where$f: \aleph \to \aleph'$ is a KU-semigroup homomorphism. For short $\Omega_f = (\alpha, \tilde{\mu}_f^+, \tilde{\mu}_f^-, \lambda_f^-, \lambda_f^+)$ is written Ω_f and a cubic bipolar is ACB.

Example3.2. In Example2.14, we have $\aleph' = \{0', a, b\}$ is a set and $f: \aleph \to \aleph'$ is mapping such that $f(\chi) = \chi'$ with two tables

$*$	$0'$	a	b
$0'$	$0'$	a	b
a	$0'$	$0'$	b
b	$0'$	b	$0'$

$\circ$	$0'$	a	b
$0'$	$0'$	$0'$	$0'$
a	$0'$	a	$0'$
b	$0'$	$0'$	a

Then $f: \aleph \rightarrow \aleph'$ is a KU-semigroup homomorphism and $N(\chi') =$
$\begin{cases}\{[-0.2,-0.1],[0.2,0.9]\} & if \quad \chi = 0 \\ \{[-0.8,-0.2],[0.3,0.5]\} & otherwise\end{cases}$, $\lambda_{\Omega}^{+}(x) = \begin{cases} 0.6 & if \quad \chi = 0 \\ 0.2 & if \quad otherwise\end{cases}$

$\lambda_{\Omega}^{-}(x) = \begin{cases} -0.9 & if \quad \chi = 0 \\ -0.4 & if \quad otherwise\end{cases}$

We have: $\tilde{\mu}_f^-(0) = \tilde{\mu}^-(f(0)) = \tilde{\mu}^-(0') = [-0.2,-0.1]$ and
$\tilde{\mu}_f^-(1) = \tilde{\mu}^-(f(1)) = \tilde{\mu}^-(a) = [-0.8,-0.2]$ and so on.

Theorem 3.3. Let $f: \aleph \rightarrow \aleph'$ be a KU-semigroup homomorphism and onto mapping. Then Ω_f is ACB *k*-ideal of $\aleph'$ if and only if Ω_f is ACB *k*-ideal of $\aleph$.

Proof. For any $\alpha' \in \aleph'$ there exists $\alpha \in \aleph$ such that $f(\alpha) = \alpha'$, we have

$$\tilde{\mu}_f^+(0) = \tilde{\mu}^+(f(0)) = \tilde{\mu}^+(0') \geq \tilde{\mu}^+(\alpha') = \tilde{\mu}^+(f(\alpha)) = \tilde{\mu}_f^+(\alpha)$$

And $\tilde{\mu}_f^-(0) = \tilde{\mu}^-(f(0)) = \tilde{\mu}^-(0') \leq \tilde{\mu}^-(\alpha') = \tilde{\mu}^-(f(\alpha)) = \tilde{\mu}_f^-(\alpha)$.

Also, $\lambda_f^+(0) = \lambda^+(f(0)) = \lambda^+(0') \geq \lambda^+(\alpha) = \lambda^+(f(\alpha)) = \lambda_f^+(\alpha)$

And $\lambda_f^-(0) = \lambda^-(f(0)) = \lambda^-(0') \leq \lambda^-(\alpha') = \lambda^-(f(\alpha)) = \lambda_f^-(\alpha)$.

Let $\alpha, \delta \in \aleph, \gamma' \in \aleph'$ then there exists $\beta \in \aleph$ such that $f(\beta) = \beta'$.

We have

$$\tilde{\mu}_f^+(\alpha * \delta) = \tilde{\mu}^+(f(\alpha * \delta)) = \tilde{\mu}^+(f(\alpha) * f(\delta)) \geq rmin\{\tilde{\mu}^+\left(f(\alpha) * \left(\beta' * f(\delta)\right), \tilde{\mu}^+(\beta)\right\}$$
$$= rmin\{\tilde{\mu}^+(f(\alpha) * (f(\beta) * f(\delta))\}, \tilde{\mu}^+(f(\beta)\}$$
$$= rmin\{\tilde{\mu}_f^+(\alpha * (\beta * \delta)), \tilde{\mu}_f^+(\beta)\}.$$

And

$$\tilde{\mu}_f^-(\alpha * \delta) = \tilde{\mu}^-(f(\alpha * \delta)) = \tilde{\mu}^-(f(\alpha) * f(\delta)) \leq rmax\{\tilde{\mu}^-\left(f(\alpha) * \left(\beta' * f(\delta)\right), \tilde{\mu}^-(\beta')\right\}$$
$$= rmax\{\tilde{\mu}^-(f(\alpha) * (f(\beta) * f(\delta))\}, \tilde{\mu}^-(f(\beta)\}$$
$$= rmax\{\tilde{\mu}_f^-(\alpha * (\beta * \delta)), \tilde{\mu}_f^-(\beta)\}.$$

Also,

$$\lambda_f^+(\alpha * \delta) = \lambda^+(f(\alpha * \delta)) = \lambda^+(f(\alpha) * f(\delta)) \geq min\{\lambda^+\left(f(\alpha) * \left(\beta' * f(\delta)\right), \lambda^+(\beta')\right\}$$
$$= min\{\lambda^+(f(\alpha) * (f(\beta) * f(\delta))\}, \lambda^+(f(\beta)\} = min\{\lambda_f^+(\alpha * (\beta * \delta)), \lambda_f^+(\beta)\}.$$

And

$$\lambda_f^-(\alpha * \delta) = \lambda^-(f(\alpha * \delta)) = \lambda^-(f(\alpha) * f(\delta)) \leq max\{\lambda^-\left(f(\alpha) * \left(\beta' * f(\delta)\right), \lambda^-(\beta')\right\}$$
$$= max\{\lambda^-(f(\alpha) * (f(\beta) * f(\delta))\}, \lambda^-(f(\beta)\}$$
$$= max\{\lambda_f^-(\alpha * (\beta * \delta)), \lambda_f^-(\beta)\}.$$

And the condition (c) is

$$\tilde{\mu}_f^+(\alpha \circ \beta) = \tilde{\mu}^+(f(\alpha \circ \beta)) = \tilde{\mu}^+(f(\alpha) \circ f(\beta)) \geq rmin\{\tilde{\mu}^+(f(\alpha), \tilde{\mu}^+(f(\beta)\}$$
$$= rmin\{\tilde{\mu}_f^+(\alpha), \tilde{\mu}_f^+(\beta)\}$$

And

$$\tilde{\mu}_f^-(\alpha \circ \beta) = \tilde{\mu}^-(f(\alpha \circ \beta)) = \tilde{\mu}^-(f(\alpha) \circ f(\beta)) \leq rmax\{\tilde{\mu}^-(f(\alpha), \tilde{\mu}^-(f(\beta)\}$$
$$= rmax\{\tilde{\mu}_f^-(\alpha), \tilde{\mu}_f^-(\beta)\}$$

Also,

$$\lambda_f^+(\alpha \circ \beta) = \lambda^+(f(\alpha \circ \beta)) = \lambda^+(f(\alpha) \circ f(\beta)) \geq min\{\lambda^+(f(\alpha), \lambda^+(f(\beta)\}$$
$$= min\{\lambda_f^+(\alpha), \lambda_f^+(\beta)\}$$

And

$$\lambda_f^-(\alpha\circ\beta)=\lambda^-\big(f(\alpha\circ\beta)\big)=\lambda^-\big(f(\alpha)\circ f(\beta)\big)\leq max\{\lambda^-(f(\alpha),\lambda^-(f(\beta)\}$$
$$=max\{\lambda_f^-(\alpha),\lambda_f^-(\beta)\}$$

Conversely, since $f:\aleph\to\aleph'$ is an onto mapping, then for any $\alpha,\beta,\delta\in\aleph'$.
It follows that, there exists $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\in\aleph$such that $f(\boldsymbol{a})=\alpha, f(\boldsymbol{b})=\beta$
and $f(\boldsymbol{c})=\delta$. We have

$$\tilde{\mu}_f^+(\alpha*\delta)=\tilde{\mu}^+\big(f(\boldsymbol{a})*f(\boldsymbol{c}))\big)=\tilde{\mu}^+(f(\boldsymbol{a}*\boldsymbol{c}))=\tilde{\mu}_f^+(\boldsymbol{a}*\boldsymbol{c})$$
$$\geq rmin\{\tilde{\mu}_f^+\big(\boldsymbol{a}*(\boldsymbol{b}*\boldsymbol{c})),\tilde{\mu}_f^+(\boldsymbol{b})\big\}$$
$$=rmin\{\tilde{\mu}^+\big(f(\boldsymbol{a})*\big(f(\boldsymbol{b})*f(\boldsymbol{c})\big)\big\},\tilde{\mu}^+(f(\boldsymbol{b})\}$$
$$=rmin\{\tilde{\mu}^+(\alpha*(\beta*\delta)),\tilde{\mu}^+(\beta)\}.$$

And

$$\tilde{\mu}_f^-(\alpha*\delta)=\tilde{\mu}^-\big(f(\boldsymbol{a})*f(\boldsymbol{c}))\big)=\tilde{\mu}^-(f(\boldsymbol{a}*\boldsymbol{c}))=\tilde{\mu}_f^-(\boldsymbol{a}*\boldsymbol{c})$$
$$\leq rmax\{\tilde{\mu}_f^-\big(\boldsymbol{a}*(\boldsymbol{b}*\boldsymbol{c})),\tilde{\mu}_f^-(\boldsymbol{b})\big\}$$
$$=rmax\{\tilde{\mu}^-\big(f(\boldsymbol{a})*\big(f(\boldsymbol{b})*f(\boldsymbol{c})\big)\big\},\tilde{\mu}^-(f(\boldsymbol{b})\}$$
$$=rmax\{\tilde{\mu}^-(\alpha*(\beta*\delta)),\tilde{\mu}^-(\beta)\}.$$

Also,

$$\lambda_f^+(\alpha*\delta)=\lambda^+\big(f(\boldsymbol{a})*f(\boldsymbol{c}))\big)=\lambda^+(f(\boldsymbol{a}*\boldsymbol{c}))=\lambda_f^+(\boldsymbol{a}*\boldsymbol{c})\geq min\{\lambda_f^+\big(\boldsymbol{a}*(\boldsymbol{b}*\boldsymbol{c})),\lambda_f^+(\boldsymbol{b})\big\}$$
$$=min\{\lambda^+\big(f(\boldsymbol{a})*\big(f(\boldsymbol{b})*f(\boldsymbol{c})\big)\big\},\lambda^+(f(\boldsymbol{b})\}=min\{\lambda^+(\alpha*(\beta*\delta)),\lambda^+(\beta)\}.$$

And

$$\lambda_f^-(\alpha*\delta)=\lambda^-\big(f(\boldsymbol{a})*f(\boldsymbol{c}))\big)=\lambda^-(f(\boldsymbol{a}*\boldsymbol{c}))=\lambda_f^-(\boldsymbol{a}*\boldsymbol{c})$$
$$\leq max\{\lambda_f^-\big(\boldsymbol{a}*(\boldsymbol{b}*\boldsymbol{c})),\lambda_f^-(\boldsymbol{b})\big\}$$
$$=max\{\lambda^-\big(f(\boldsymbol{a})*\big(f(\boldsymbol{b})*f(\boldsymbol{c})\big)\big\},\lambda^-(f(\boldsymbol{b})\}$$
$$=max\{\lambda^-(\alpha*(\beta*\delta)),\lambda^-(\beta)\}.$$

And the condition (c) is

$$\tilde{\mu}_f^+(\alpha\circ\beta)=\tilde{\mu}^+\big(f(\boldsymbol{a})\circ f(\boldsymbol{b}))\big)\geq rmin\{\tilde{\mu}^+(f(\boldsymbol{a})),\tilde{\mu}^+(f(\boldsymbol{b})\}=rmin\{\tilde{\mu}_f^+(\alpha),\tilde{\mu}_f^+(\beta)\}$$

And

$$\tilde{\mu}_f^-(\alpha\circ\beta)=\tilde{\mu}^-\big(f(\boldsymbol{a})\circ f(\boldsymbol{b}))\big)\leq rmax\{\tilde{\mu}^-(f(\boldsymbol{a}),\tilde{\mu}^-(f(\boldsymbol{b})\}=rmax\{\tilde{\mu}_f^-(\alpha),\tilde{\mu}_f^-(\beta)\}$$

Also,

$$\lambda_f^+(\alpha\circ\beta)=\lambda^+\big(f(\boldsymbol{a})\circ f(\boldsymbol{b}))\big)\geq min\{\lambda^+(f(\boldsymbol{a}),\lambda^+(f(\boldsymbol{b})\}=min\{\lambda_f^+(\alpha),\lambda_f^+(\beta)\}$$

And

$$\lambda_f^-(\alpha\circ\beta)=\lambda^-\big(f(\boldsymbol{a})\circ f(\boldsymbol{b}))\big)\leq max\{\lambda^-(f(\boldsymbol{a}),\lambda^-(f(\boldsymbol{b})\}=max\{\lambda_f^-(\alpha),\lambda_f^-(\beta)\}$$

Therefore $\Omega_f$is ACB *k*-ideal of $\aleph'$.

In the following, we introduce the product of the cubic bipolar *k*-ideals and a cubic bipolar ideal as follows.

Definition 3.4. Let $\Omega_{f_1}$and Ω_{f_2}be two CB fuzzy sets of $\aleph$. The product $\Omega_{f_1}\times\Omega_{f_2}=((\alpha,\beta),\ \tilde{\mu}_1^-\times\tilde{\mu}_2^-,\ \tilde{\mu}_1^+\times\ \tilde{\mu}_2^+,\ \lambda_1^-\times\ \lambda_2^-,\ \lambda_1^+\times\ \lambda_2^+)$ is defined by the following:
$(\tilde{\mu}_1^-\times\tilde{\mu}_2^-)(\alpha,\beta)=rmax\{\ \tilde{\mu}_1^-(\alpha),\ \tilde{\mu}_2^-(\beta)\}$, $(\tilde{\mu}_1^+\times\ \tilde{\mu}_2^+)(\alpha,\beta)=rmin\{\tilde{\mu}_1^+(\alpha),\tilde{\mu}_2^+(\beta)\}$ and $(\ \lambda_1^-\times\ \lambda_2^-)(\alpha,\beta)=max\{\ \lambda_1^-(\alpha),\ \lambda_2^-(\beta)\}$, $(\ \lambda_1^+\times\ \lambda_2^+)(\alpha,\beta)=min\{\ \lambda_1^+(\alpha),\ \lambda_2^+(\beta)\}$where
$\tilde{\mu}_1^-\times\ \tilde{\mu}_2^-:\aleph\times\aleph\to[-1,0]$, $\tilde{\mu}_1^+\times\tilde{\mu}_2^+:\aleph\times\aleph\to[0,1]$ and $\lambda_1^-\times\lambda_2^-:\aleph\times\aleph\to[-1,0]$, $\lambda_1^+\times\lambda_2^+:\aleph\times\aleph\to[0,1]$, for all $\alpha,\beta\in\aleph$.

Theorem 3.5. Let Ω_{f_1} and Ω_{f_2} be two CB *k*-ideals of $\aleph$, then $\Omega_{f_1} \times \Omega_{f_2}$ is ACB *k*-ideal of $\aleph \times \aleph$.

Proof. Let $(\alpha, \beta) \in \aleph \times \aleph$, we have
$(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)(0,0) = rmin\{\tilde{\mu}_1^+(0), \tilde{\mu}_2^+(0)\} \geq rmin\{\tilde{\mu}_1^+(\alpha), \tilde{\mu}_2^+(\beta)\} = (\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)(\alpha, \beta)$ and
$(\tilde{\mu}_1^- \times \tilde{\mu}_2^-)(0,0) = rmax\{\tilde{\mu}_1^-(0), \tilde{\mu}_2^-(0)\} \leq rmax\{\tilde{\mu}_1^-(\alpha), \tilde{\mu}_2^-(\beta)\} = (\tilde{\mu}_1^- \times \tilde{\mu}_2^-)(\alpha, \beta)$.

Let $(\alpha_1, \alpha_2), (\beta_1, \beta_2)$ and $(\delta_1, \delta_2) \in \aleph \times \aleph$, then

$$(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)(\alpha_1 * \delta_1, \alpha_2 * \delta_2) = rmin\{\tilde{\mu}_1^+(\alpha_1 * \tau_1), \tilde{\mu}_2^+(\alpha_2 * \delta_2)\}$$
$$\geq rmin\{rmin\{\tilde{\mu}_1^+(\alpha_1 * (\beta_1 * \delta_1)), \tilde{\mu}_1^+(\beta_1)\}, rmin\{\tilde{\mu}_2^+(\alpha_2 * (\beta_2 * \delta_2)), \tilde{\mu}_2^+(\beta_2)\}\} =$$
$$rmin\{rmin\{\tilde{\mu}_1^+(\alpha_1 * (\beta_1 * \delta_1)), \tilde{\mu}_2^+(\alpha_2 * (\beta_2 * \delta_2)\}, rmin\{\tilde{\mu}_1^+(\beta_1), \tilde{\mu}_2^+(\beta_2)\}$$
$$= rmin\left\{rmin(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)\left\{(\alpha_1 * (\beta_1 * \delta_1)), (\alpha_2 * (\beta_2 * \delta_2))\right\}, \{(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)(\beta_1, \beta_2)\}\right\}$$

And

$$(\tilde{\mu}_1^- \times \tilde{\mu}_2^-)(\alpha_1 * \delta_1, \alpha_2 * \delta_2) = rmax\{\tilde{\mu}_1^-(\alpha_1 * \delta_1), \tilde{\mu}_2^-(\alpha_2 * \delta_2)\}$$
$$\leq rmax\{rmax\{\tilde{\mu}_1^-(\alpha_1 * (\beta_1 * \delta_1)), \tilde{\mu}_1^-(\beta_1)\}, rmax\{\tilde{\mu}_2^-(\alpha_2 * (\beta_2 * \delta_2), \tilde{\mu}_2^-(\beta_2)\} =$$
$$rmax\{rmax\{\tilde{\mu}_1^-(\alpha_1 * (\beta_1 * \delta_1)), \tilde{\mu}_2^-(\alpha_2 * (\beta_2 * \delta_2)\}, rmax\{\tilde{\mu}_1^-(\beta_1), \tilde{\mu}_2^-(\beta_2)\}$$
$$= rmax\left\{(\tilde{\mu}_1^- \times \tilde{\mu}_2^-)\left\{(\alpha_1 * (\beta_1 * \delta_1)), (\alpha_2 * (\beta_2 * \delta_2))\right\}, \{(\tilde{\mu}_1^- \times \tilde{\mu}_2^-)(\beta_1, \beta_2)\}\right\}.$$

Also,

$$(\lambda_1^+ \times \lambda_2^+)(\alpha_1 * \delta_1, \alpha_2 * \delta_2) = min\{\lambda_1^+(\alpha_1 * \delta_1), \lambda_2^+(\alpha_2 * \delta_2)\}$$
$$\geq min\{min\{\lambda_1^+(\alpha_1 * (\beta_1 * \delta_1)), \lambda_1^+(\beta_1)\}, min\{\lambda_2^+(\alpha_2 * (\beta_2 * \delta_2)), \lambda_2^+(\beta_2)\}\} =$$
$$min\{min\{\lambda_1^+(\alpha_1 * (\beta_1 * \delta_1)), \lambda_2^+(\alpha_2 * (\beta_2 * \delta_2)\}, min\{\lambda_1^+(\beta_1), \lambda_2^+(\beta_2)\}$$
$$= min\left\{min(\lambda_1^+ \times \lambda_2^+)\left\{(\alpha_1 * (\beta_1 * \delta_1)), (\alpha_2 * (\beta_2 * \delta_2))\right\}, \{(\lambda_1^+ \times \lambda_2^+)(\beta_1, \beta_2)\}\right\}$$

And

$$(\lambda_1^- \times \lambda_2^-)(\alpha_1 * \delta_1, \alpha_2 * \delta_2) = max\{\lambda_1^-(\alpha_1 * \delta_1), \lambda_2^-(\alpha_2 * \delta_2)\}$$
$$\leq max\{max\{\lambda_1^-(\alpha_1 * (\beta_1 * \delta_1)), \lambda_1^-(\beta_1)\}, max\{\lambda_2^-(\alpha_2 * (\beta_2 * \delta_2), \lambda_2^-(\beta_2)\} =$$
$$max\{max\{\lambda_1^-(\alpha_1 * (\beta_1 * \delta_1)), \lambda_2^-(\alpha_2 * (\beta_2 * \delta_2)\}, max\{\lambda_1^-(\beta_1), \lambda_2^-(\beta_2)\}$$
$$= max\left\{(\lambda_1^- \times \lambda_2^-)\left\{(\alpha_1 * (\beta_1 * \delta_1)), (\alpha_2 * (\beta_2 * \delta_2))\right\}, \{(\lambda_1^- \times \lambda_2^-)(\beta_1, \beta_2)\}\right\}.$$

And $(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)(\alpha_1 \circ \beta_1, \alpha_2 \circ \beta_2) = rmin\{\tilde{\mu}_1^+(\alpha_1 \circ \beta_1), \tilde{\mu}_2^+(\alpha_2 \circ \beta_2)\} \geq rmin\{rmin\{\tilde{\mu}_1^+(\alpha_1), \tilde{\mu}_1^+(\beta_1)\}, rmin\{\tilde{\mu}_2^+(\alpha_2), \tilde{\mu}_2^+(\beta_2)\} = rmin\{rmin\{\tilde{\mu}_1^+(\alpha_1), \tilde{\mu}_2^+(\alpha_2)\}, rmin\{\tilde{\mu}_1^+(\beta_1), \tilde{\mu}_2^+(\beta_2)\} = rmin\{\{(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)(\alpha_1, \alpha_2)\}, \{(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)(\beta_1, \beta_2)\}\}$

And

$$(\tilde{\mu}_1^- \times \tilde{\mu}_2^-)(\alpha_1 \circ \beta_1, \alpha_2 \circ \beta_2) = rmax\{\tilde{\mu}_1^-(\alpha_1 \circ \beta_1), \tilde{\mu}_2^-(\alpha_2 \circ \beta_2)\}$$
$$\leq rmax\{rmax\{\tilde{\mu}_1^-(\alpha_1), \tilde{\mu}_1^-(\beta_1)\}, rmax\{\tilde{\mu}_2^-(\alpha_2), \tilde{\mu}_2^-(\beta_2)\} =$$
$$rmax\{rmax\{\tilde{\mu}_1^-(\alpha_1), \tilde{\mu}_2^-(\alpha_2)\}, rmax\{\tilde{\mu}_1^-(\beta_1), \tilde{\mu}_2^-(\beta_2)\}$$
$$= rmax\{\{(\tilde{\mu}_1^- \times \tilde{\mu}_2^-)(\alpha_1, \alpha_2)\}, \{(\tilde{\mu}_1^- \times \tilde{\mu}_2^-)(\beta_1, \beta_2)\}\}.$$

Also,

$$(\lambda_1^+ \times \lambda_2^+)(\alpha_1 \circ \beta_1, \alpha_2 \circ \beta_2) = min\{\lambda_1^+(\alpha_1 \circ \beta_1), \lambda_2^+(\alpha_2 \circ \beta_2)\}$$
$$\geq min\{min\{\lambda_1^+(\alpha_1), \lambda_1^+(\beta_1)\}, min\{\lambda_2^+(\alpha_2), \lambda_2^+(\beta_2)\} =$$

$$min\{min\{\lambda_1^+(\alpha_1),\lambda_2^+(\alpha_2)\},min\{\lambda_1^+(\beta_1),\lambda_2^+(\beta_2)\}$$
$$= min\{\{(\lambda_1^+\times\lambda_2^+)(\alpha_1,\alpha_2)\},\{(\lambda_1^+\times\lambda_2^+)(\beta_1,\beta_2)\}\}$$

And

$$(\lambda_1^-\times\lambda_2^-)(\alpha_1\circ\beta_1,\alpha_2\circ\beta_2) = max\{\lambda_1^-(\alpha_1\circ\beta_1),\lambda_2^-(\alpha_2\circ\beta_2)\}$$
$$\leq max\{max\{\lambda_1^-(\alpha_1),\lambda_1^-(\beta_1)\},max\{\lambda_2^-(\alpha_2),\lambda_2^-(\beta_2)\} =$$
$$max\{max\{\lambda_1^-(\alpha_1),\lambda_2^-(\alpha_2)\},max\{\lambda_1^-(\beta_1),\lambda_2^-(\beta_2)\}$$
$$= max\{\{(\lambda_1^-\times\lambda_2^-)(\alpha_1,\alpha_2)\},\{(\lambda_1^-\times\lambda_2^-)(\beta_1,\beta_2)\}\}.$$

Then $\Omega_{f_1}\times\Omega_{f_2}$ is ACB k-ideal of $\aleph\times\aleph$.

Theorem 3.6. Let Ω_{f_1} and Ω_{f_2} be two CB ideal of KU-semigroup $\aleph$, such that $\Omega_{f_1}\times\Omega_{f_2}$ is ACB ideal of $\aleph\times\aleph$. We have

(i) Either $\tilde{\mu}_1^+(0)\geq\tilde{\mu}_1^+(\alpha)$, $\tilde{\mu}_1^-(0)\leq\tilde{\mu}_1^-(\alpha)$ or $\tilde{\mu}_2^+(0)\geq\tilde{\mu}_2^+(\beta)$, $\tilde{\mu}_2^-(0)\leq\tilde{\mu}_2^-(\beta)$, also, $\lambda_1^+(0)\geq\lambda_1^+(\alpha)$, $\lambda_1^-(0)\leq\lambda_1^-(\alpha)$ or $\lambda_2^+(0)\geq\lambda_2^+(\beta)$, $\lambda_2^-(0)\leq\lambda_2^-(\beta)$ for all $\alpha,\beta\in\aleph$.

(ii) If $\tilde{\mu}_1^+(0)\geq\tilde{\mu}_1^+(\alpha)$, $\tilde{\mu}_1^-(0)\leq\tilde{\mu}_1^-(\alpha)$ and $\lambda_1^+(0)\geq\lambda_1^+(\alpha)$, $\lambda_1^-(0)\leq\lambda_1^-(\alpha)$ for all $\alpha\in\aleph$. Then either $\tilde{\mu}_2^+(0)\geq\tilde{\mu}_1^+(\alpha)$, $\tilde{\mu}_2^-(0)\leq\tilde{\mu}_1^-(\alpha)$ and $\lambda_2^+(0)\geq\lambda_1^+(\alpha)$, $\lambda_2^-(0)\leq\lambda_1^-(\alpha)$ or $\tilde{\mu}_2^+(0)\geq\tilde{\mu}_2^+(\beta)$, $\tilde{\mu}_2^-(0)\leq\tilde{\mu}_2^-(\beta)$ and $\lambda_2^+(0)\geq\lambda_2^+(\beta)$, $\lambda_2^-(0)\leq\lambda_2^-(\beta)$ for all $\alpha,\beta\in\aleph$.

(iii) If $\tilde{\mu}_2^+(0)\geq\tilde{\mu}_2^+(\alpha)$, $\tilde{\mu}_2^-(0)\leq\tilde{\mu}_2^-(\alpha)$, and $\lambda_2^+(0)\geq\lambda_2^+(\alpha)$, $\lambda_2(0)\leq\lambda_2^-(\alpha)$, for all $\alpha\in\aleph$, then either $\tilde{\mu}_1^+(0)\geq\tilde{\mu}_1^+(\alpha)$, $\tilde{\mu}_1^-(0)\leq\tilde{\mu}_1^-(\alpha)$, and $\lambda_1^+(0)\geq\lambda_1^+(\alpha)$, $\lambda_1^-(0)\leq\lambda_1^-(\alpha)$ or $\tilde{\mu}_1^+(0)\geq\tilde{\mu}_2^+(\alpha)$, $\tilde{\mu}_1^-(0)\leq\tilde{\mu}_2^-(\alpha)$ and $\lambda_1^+(0)\geq\lambda_2^+(\alpha)$, $\lambda_1^-(0)\leq\lambda_2^-(\alpha)$ for all $\alpha\in\aleph$.

Proof. (i) Suppose that $\tilde{\mu}_1^+(0)\geq\tilde{\mu}_1^+(\alpha)$, $\tilde{\mu}_1^-(0)\leq\tilde{\mu}_1^-(\alpha)$ and $\tilde{\mu}_2^+(0)\geq\tilde{\mu}_2^+(y)$, $\tilde{\mu}_2^-(0)\leq\tilde{\mu}_2^-(\beta)$, also, $\lambda_1^+(0)\geq\lambda_1^+(\alpha)$, $\lambda_1^-(0)\leq\lambda_1^-(\alpha)$ and $\lambda_2^+(0)\geq\lambda_2^+(\beta)$, $\lambda_2^-(0)\leq\lambda_2^-(\beta)$, for some $\alpha,\beta\in\aleph$. Then

$$(\tilde{\mu}_1^+\times\tilde{\mu}_2^+)(\alpha,\beta) = rmin\{\tilde{\mu}_1^+(\alpha),\tilde{\mu}_2^+(\beta)\}\geq rmin\{\tilde{\mu}_1^+(0),\tilde{\mu}_2^+(0)\} = (\tilde{\mu}_1^+\times\tilde{\mu}_2^+)(0,0)$$

And

$$(\tilde{\mu}_1^-\times\tilde{\mu}_2^-)(\alpha,\beta) = rmax\{\tilde{\mu}_1^-(\alpha),\tilde{\mu}_2^-(\beta)\}\leq rmax\{\tilde{\mu}_1^-(0),\tilde{\mu}_2^-(0)\} = (\tilde{\mu}_1^-\times\tilde{\mu}_2^-)(0,0)$$

Also,

$$(\lambda_1^+\times\lambda_2^+)(\alpha,\beta) = min\{\lambda_1^+(\alpha),\lambda_2^+(\beta)\}\geq min\{\lambda_1^+(0),\lambda_2^+(0)\} = (\lambda_1^+\times\lambda_2^+)(0,0)$$

And

$$(\lambda_1^-\times\lambda_2^-)(\alpha,\beta) = max\{\lambda_1^-(\alpha),\lambda_2^-(\beta)\}\leq max\{\lambda_1^-(0),\lambda_2^-(0)\} = (\lambda_1^-\times\lambda_2^-)(0,0)$$

, for all $\alpha,\beta\in\aleph$. This is a contradiction. Therefore, either $\tilde{\mu}_1^+(0)\geq\tilde{\mu}_1^+(\alpha)$, $\tilde{\mu}_1^-(0)\leq\tilde{\mu}_1^-(\alpha)$ or $\tilde{\mu}_2^+(0)\geq\tilde{\mu}_2^+(\beta)$, $\tilde{\mu}_2^-(0)\leq\tilde{\mu}_2^-(\beta)$, also, $\lambda_1^+(0)\geq\lambda_1^+(\alpha)$, $\lambda_1^-(0)\leq\lambda_1^-(\alpha)$ or $\lambda_2^+(0)\geq\lambda_2^+(\beta)$, $\lambda_2^-(0)\leq\lambda_2^-(\beta)$ for all $\alpha,\beta\in\aleph$.

(ii) Suppose that $\tilde{\mu}_2^+(0)\leq\tilde{\mu}_1^+(\alpha)$, $\tilde{\mu}_2^-(0)\geq\tilde{\mu}_1^-(\alpha)$ and $\tilde{\mu}_2^+(0)\leq\tilde{\mu}_2^+(\beta)$, $\tilde{\mu}_2^-(0)\geq\tilde{\mu}_2^-(\beta)$ also, $\lambda_2^+(0)\leq\lambda_1^+(\alpha)$, $\lambda_2^-(0)\geq\lambda_1^-(\alpha)$ and $\lambda_2^+(0)\leq\lambda_2^+(\beta)$, $\lambda_2^-(0)\geq\lambda_2^-(\beta)$, for all $\alpha,\beta\in\aleph$.

Then $(\tilde{\mu}_1^+\times\tilde{\mu}_2^+)(0,0) = rmin\{\tilde{\mu}_1^+(0),\tilde{\mu}_2^+(0)\} = \tilde{\mu}_2^+(0)$

And

$$(\tilde{\mu}_1^+\times\tilde{\mu}_2^+)(\alpha,\beta) = rmin\{\tilde{\mu}_1^+(\alpha),\tilde{\mu}_2^+(\beta)\}\geq\{\tilde{\mu}_2^+(0),\tilde{\mu}_2^+(0)\} = \tilde{\mu}_2^+(0) = (\tilde{\mu}_1^+\times\tilde{\mu}_2^+)(0,0)$$

And $(\tilde{\mu}_1^-\times\tilde{\mu}_2^-)(0,0) = rmax\{\tilde{\mu}_1^-(0),\tilde{\mu}_2^-(0)\} = \tilde{\mu}_2^-(0)$.

$(\tilde{\mu}_1^-\times\tilde{\mu}_2^-)(\alpha,\beta) = rmax\{\tilde{\mu}_1^-(\alpha),\tilde{\mu}_2^-(\beta)\}\leq rmax\{\tilde{\mu}_2^-(0),\tilde{\mu}_2^-(0)\} = \tilde{\mu}_2^-(0) =$
$(\tilde{\mu}_1^-\times\tilde{\mu}_2^-)(0,0)$aaa

Also,

$$(\lambda_1^+ \times \lambda_2^+)(0,0) = min\{ \lambda_1^+(0) , \lambda_2^+(0) \} = \lambda_2^+(0)$$

And

$(\lambda_1^+ \times \lambda_2^+)(\alpha,\beta) = min\{ \lambda_1^+(\alpha) , \lambda_2^+(\beta) \} \geq \{ \lambda_2^+(0) , \lambda_2^+(0) \} = \lambda_2^+(0) =$
$(\lambda_1^+ \times \lambda_2^+)(0,0)$.

And $(\lambda_1^- \times \lambda_2^-)(0,0) = max\{ \lambda_1^-(0), \lambda_2^-(0)\} = \lambda_2^-(0)$.

$$(\lambda_1^- \times \lambda_2^-)(\alpha,\beta) = max\{ \lambda_1^-(\alpha), \lambda_2^-(\beta)\} \leq max\{ \lambda_2^-(0), \lambda_2^-(0)\} = \lambda_2^-(0)$$
$$= (\lambda_1^- \times \lambda_2^-) (0,0)$$

. This is a contradiction. Therefore, either $\tilde{\mu}_2^+(0) \geq \tilde{\mu}_1^+(\chi)$,$\tilde{\mu}_2^-(0) \leq \tilde{\mu}_1^-(\alpha)$ and $\lambda_2^+(0) \geq \lambda_1^+(\alpha)$,$\lambda_2^-(0) \leq \lambda_1^-(x)$or $\tilde{\mu}_2^+(0) \geq \tilde{\mu}_2^+(\beta)$,$\tilde{\mu}_2^-(0) \leq \tilde{\mu}_2^-(\beta)$and $\lambda_2^+(0) \geq \lambda_2^+(\beta)$,$\lambda_2^-(0) \leq \lambda_2^-(\beta)$.

(iii)The proof is similar to (ii).

The partial converse of Theorem (3.5) is the following.

Theorem3.7. In a KU-semigroupℵ. If $\Omega_{f_1} \times \Omega_{f_2}$ is ACB ideal of $\aleph \times \aleph$, then Ω_{f_1} or Ω_{f_2} is ACB ideal of ℵ.

Proof. By use the Theorem (3.5) (i), without loss of generality we suppose that $\tilde{\mu}_2^+(0) \geq \tilde{\mu}_2^+(\alpha), \tilde{\mu}_2^-(0) \leq \tilde{\mu}_2^-(\alpha)$, and $\lambda_2^+(0) \geq \lambda_2^+(\alpha)$,$\lambda_2^-(0) \leq \lambda_2^-(\alpha)$,for allα ∈ ℵ. It follows from Theorem (4.6)(iii) that either then either

$\tilde{\mu}_1^+(0) \geq \tilde{\mu}_1^+(\alpha)$,$\tilde{\mu}_1^-(0) \leq \tilde{\mu}_1^-(\alpha)$, and$\lambda_1^+(0) \geq \lambda_1^+\alpha$,$\lambda_1^-(0) \leq \lambda_1^-(\alpha)$ or $\tilde{\mu}_1^+(0) \geq \tilde{\mu}_2^+(\alpha)$, $\tilde{\mu}_1^-(0) \leq \tilde{\mu}_2^-(\alpha)$ and $\lambda_1^+(0) \geq \lambda_2^+(\alpha)$, $\lambda_1^-(0) \leq \lambda_2^-(\alpha)$.

, for all α ∈ ℵ.

Then $\left(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+\right)(0,\alpha) = rmin\left\{ \tilde{\mu}_1^+(0), \tilde{\mu}_2^+(\alpha)\right\} = \tilde{\mu}_2^+(\alpha) \ldots\ldots (1)$

$\left(\tilde{\mu}_1^- \times \tilde{\mu}_2^-\right)(0,\alpha) = rmax\left\{ \tilde{\mu}_1^-(0), \tilde{\mu}_2^-(\alpha)\right\} = \tilde{\mu}_2^-(\alpha) \ldots\ldots (2)$

Also,

$\left(\lambda_1^+ \times \lambda_2^+\right)(0,\alpha) = min\left\{\lambda_1^+(0), \lambda_2^+(\alpha)\right\} = \lambda_2^+(\alpha) \ldots\ldots (3)$

$$(\lambda_1^- \times \lambda_2^-)(0,\alpha) = max\{\lambda_1^-(0), \lambda_2^-(\alpha)\} = \lambda_2^-(\alpha) \ldots\ldots (4)$$

Since $\Omega_{f_1} \times \Omega_{f_2}$ is ACB ideal of ℵ×ℵ, then

$$\left(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+\right)\left(\beta_1,\beta_2\right) \geq rmin\{(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)((\alpha_1,\alpha_2) * (\beta_1,\beta_2)), (\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)(\alpha_1,\alpha_2)\}$$
$$= rmin\{\left(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+\right)\left(\alpha_1 * \beta_1, \alpha_2 * \beta_2\right), \left(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+\right)(\alpha_1,\alpha_2)\}$$

Put$\alpha_1 = \beta_1 = 0$, then we have

$$\left(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+\right)\left(0,\beta_2\right) \geq rmin\{(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)((0,\alpha_2) * (0,\beta_2)), (\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)(0,\alpha_2)\}$$
$$= rmin\{\left(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+\right)\left(0,\alpha_2 * \beta_2\right), (\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)(0,\alpha_2)\}$$

and by equation (1), then

$\tilde{\mu}_2^+(\beta_2) \geq rmin\{\tilde{\mu}_2^+(\alpha_2 * \beta_2), \tilde{\mu}_2^+(\alpha_2)\}$. And

$$\left(\tilde{\mu}_1^- \times \tilde{\mu}_2^-\right)\left(\beta_1,\beta_2\right) \leq rmax\left\{(\tilde{\mu}_1^- \times \tilde{\mu}_2^-)((\alpha_1,\alpha_2) * (\beta_1,\beta_2)), (\tilde{\mu}_1^- \times \tilde{\mu}_2^-)(\alpha_1,\alpha_2)\right\}$$
$$= rmax\{\left(\tilde{\mu}_1^- \times \tilde{\mu}_2^-\right)\left(\alpha_1 * \beta_1, \alpha_2 * \beta_2\right), \left(\tilde{\mu}_1^- \times \tilde{\mu}_2^-\right)(\alpha_1,\alpha_2)\}$$

Put $\alpha_1 = \beta_1 = 0$, then we have

$$\left(\tilde{\mu}_1^- \times \tilde{\mu}_2^-\right)\left(0,\beta_2\right) \leq rmax\left\{(\tilde{\mu}_1^- \times \tilde{\mu}_2^-)((0,\alpha_2) * (0,\beta_2)), (\tilde{\mu}_1^- \times \tilde{\mu}_2^-)(0,\alpha_2)\right\}$$
$$= rmax\{\left(\tilde{\mu}_1^- \times \tilde{\mu}_2^- \right)\left(0,\alpha_2 * \beta_2\right), (\tilde{\mu}_1^- \times \tilde{\mu}_2^-)(0,\alpha_2)\}$$

and by using equation (2), we have
$\tilde{\mu}_2^-(\beta_2) \leq rmax\{\tilde{\mu}_2^-(\alpha_2 * \beta_2), \tilde{\mu}_2^-(\alpha_2)\}$.
Also,

$$(\lambda_1^+ \times \lambda_2^+)(\beta_1, \beta_2) \geq min\{(\lambda_1^+ \times \lambda_2^+)((\alpha_1, \alpha_2) * (\beta_1, \beta_2)), (\lambda_1^+ \times \lambda_2^+)(\alpha_1, \alpha_2)\}$$
$$= min\{(\lambda_1^+ \times \lambda_2^+)(\alpha_1 * \beta_1, \alpha_2 * \beta_2), (\lambda_1^+ \times \lambda_2^+)(\alpha_1, \alpha_2)\}$$

Put $\alpha_1 = \beta_1 = 0$, then we have

$$(\lambda_1^+ \times \lambda_2^+)(0, \beta_2) \geq min\{(\lambda_1^+ \times \lambda_2^+)((0, \alpha_2) * (0, \beta_2)), (\lambda_1^+ \times \lambda_2^+)(0, \alpha_2)\}$$
$$= min\{(\lambda_1^+ \times \lambda_2^+)(0, \alpha_2 * \beta_2), (\lambda_1^+ \times \lambda_2^+)(0, \alpha_2)\}$$

and by using equation (3), we have
$\lambda_2^+(\beta_2) \geq min\{\lambda_2^+(\alpha_2 * \beta_2), \lambda_2^+(\alpha_2)\}$.
And

$$(\lambda_1^- \times \lambda_2^-)(\beta_1, \beta_2) \leq max\{(\lambda_1^- \times \lambda_2^-)((\alpha_1, \alpha_2) * (\beta_1, \beta_2)), (\lambda_1^- \times \lambda_2^-)(\alpha_1, \alpha_2)\}$$
$$= max\{(\lambda_1^- \times \lambda_2^-)(\alpha_1 * \beta_1, \alpha_2 * \beta_2), (\lambda_1^- \times \lambda_2^-)(\alpha_1, \alpha_2)\}$$

Put$\alpha_1 = \beta_1 = 0$, then we have
$(\lambda_1^- \times \lambda_2^-)(0, \beta_2) \leq max\{(\lambda_1^- \times \lambda_2^-)((0, \alpha_2) * (0, \beta_2)), (\lambda_1^- \times \lambda_2^-)(0, \alpha_2)\} = max\{(\lambda_1^- \times \lambda_2^-)(0, \alpha_2 * \beta_2), (\lambda_1^- \times \lambda_2^-)(0, \alpha_2)\}$and by using equation (4), we have
$\lambda_2^-(\beta_2) \leq max\{\lambda_2^-(\alpha_2 * \beta_2), \lambda_2^-(\alpha_2)\}$. And the condition **(BC$_3$)** is
$(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)((\alpha_1, \alpha_2) \circ (\beta_1, \beta_2) \geq rmin\{(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)(\alpha_1, \alpha_2), (\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)(\beta_1, \beta_2)\}(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)(\alpha_1 \circ \beta_1, \alpha_2 \circ \beta_2) \geq rmin\{(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)(\alpha_1, \alpha_2), (\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)(\beta_1, \beta_2)\}$

Put $\alpha_1 = \beta_1 = 0$, then we have

$$(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)(0, \alpha_2 \circ \beta_2) \geq rmin\{(\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)(0, x_2), (\tilde{\mu}_1^+ \times \tilde{\mu}_2^+)(0, \beta_2)\}$$

and by using equation (1), we have $\tilde{\mu}_2^+(\alpha_2 \circ \beta_2) \geq rmin\{\tilde{\mu}_2^+(\alpha_2), \tilde{\mu}_2^+(\beta_2)\}$
And

$$(\tilde{\mu}_1^- \times \tilde{\mu}_2^-)((\alpha_1, \alpha_2) \circ (\beta_1, \beta_2) \leq rmax\{(\tilde{\mu}_1^- \times \tilde{\mu}_2^-)(\alpha_1, \alpha_2), (\tilde{\mu}_1^- \times \tilde{\mu}_2^-)(\beta_1, \beta_2)\}$$
$$(\tilde{\mu}_1^- \times \tilde{\mu}_2^-)(\alpha_1 \circ \beta_1, \alpha_2 \circ \beta_2) \leq rmax\{(\tilde{\mu}_1^- \times \tilde{\mu}_2^-)(\alpha_1, \alpha_2), (\tilde{\mu}_1^- \times \tilde{\mu}_2^-)(\beta_1, \beta_2)\}$$

Put $\alpha_1 = \beta_1 = 0$, then we have

$$(\tilde{\mu}_1^- \times \tilde{\mu}_2^-)(0, \alpha_2 \circ \beta_2) \leq rmax\{(\tilde{\mu}_1^- \times \tilde{\mu}_2^-)(0, \alpha_2), (\tilde{\mu}_1^- \times \tilde{\mu}_2^-)(0, \beta_2)\}$$

And by using equation (2), we have $\tilde{\mu}_2^-(\alpha_2 \circ \beta_2) \leq rmax\{\tilde{\mu}_2^-(\alpha_2), \tilde{\mu}_2^-(\beta_2)\}$
Also, we have

$$(\lambda_1^+ \times \lambda_2^+)((\alpha_1, \alpha_2) \circ (\beta_1, \beta_2) \geq min\{(\lambda_1^+ \times \lambda_2^+)(\alpha_1, \alpha_2), (\lambda_1^+ \times \lambda_2^+)(\beta_1, \beta_2)\}$$
$$(\lambda_1^+ \times \lambda_2^+)(\alpha_1 \circ \beta_1, \alpha_2 \circ \beta_2) \geq min\{(\lambda_1^+ \times \lambda_2^+)(\alpha_1, \alpha_2), (\lambda_1^+ \times \lambda_2^+)(\beta_1, \beta_2)\}$$

Put $\alpha_1 = \beta_1 = 0$, then we have

$$(\lambda_1^+ \times \lambda_2^+)(0, \alpha_2 \circ \beta_2) \geq min\{(\lambda_1^+ \times \lambda_2^+)(0, \alpha_2), (\lambda_1^+ \times \lambda_2^+)(0, \beta_2)\}$$

and by using equation (3), we have $\lambda_2^+(\alpha_2 \circ \beta_2) \geq min\{\lambda_2^+(\alpha_2), \lambda_2^+(\beta_2)\}$
And

$$(\lambda_1^- \times \lambda_2^-)((\alpha_1, \alpha_2) \circ (\beta_1, \beta_2) \leq max\{(\lambda_1^- \times \lambda_2^-)(\alpha_1, \alpha_2), (\lambda_1^- \times \lambda_2^-)(\beta_1, \beta_2)\}$$
$$(\lambda_1^- \times \lambda_2^-)(\alpha_1 \circ \beta_1, \alpha_2 \circ \beta_2) \leq max\{(\lambda_1^- \times \lambda_2^-)(\alpha_1, \alpha_2), (\lambda_1^- \times \lambda_2^-)(\beta_1, \beta_2)\}$$

Put $\alpha_1 = \beta_1 = 0$, then we have

$$(\lambda_1^- \times \lambda_2^-)(0, \alpha_2 \circ \beta_2) \leq max\{(\lambda_1^- \times \lambda_2^-)(0, \alpha_2), (\lambda_1^- \times \lambda_2^-)(0, \beta_2)\}$$

and by using equation (4), we have $\lambda_2^-(\alpha_2 \circ \beta_2) \leq max\{\lambda_2^-(\alpha_2), \lambda_2^-(\beta_2)\}$

Then, it follows that Ω_{f_2} is ACB ideal of $\aleph$.

This completes the proof.

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