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Strongly Maximal Submodules with A Study of Their Influence on Types of Modules

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Abstract

Let S be a commutative ring with identity, and A is an S-module. This paper introduced an important concept, namely strongly maximal submodule. Some properties and many results were proved as well as the behavior of that concept with its localization was studied and shown.

Keywords: Maximal submodule, regular module, regular ring, semi-simple module, prime module.

1.Introduction

Along with this paper, S is a commutative ring with identity, and A is an S-module. A proper submodule N of an S-module A is named maximal if there exists a submodule D of A such that $N \subsetneq D \subseteq A$, then D=A[1][11]. Equivalently, N is the maximal submodule in A if and only if A/N is a simple S-module [1][12]. Maximal submodules may not exist; for instance, the Z-module $\mathbb{C}$ has no maximal submodules. The main goal of this paper is to introduce a new concept called strongly maximal submodule (for short, SM-submodule) where a proper non-zero submodule B of an S-module A is said to be strongly maximal submodule if and only if, for every non-zero ideal E of S implies A/E^2B is a regular module. Field house is defined in [9], a pure submodule of the form: A submodule D of an S-module A is pure if $IA \cap D = ID$ for every ideal I of S and Sahera introduced in [2], the definition of F-regular module, where module A is said to be F-regular if and only if every submodule of A is pure Every strongly maximal submodule is a maximal submodule, but the opposite does not true. This paper is divided into three sections. We reviewed some basic definitions and properties needed in our next work. Section three introduced the definition of strongly maximal submodule. Lots of properties and examples of this concept were shown. In section four, the behavior of strongly maximal submodules under localization was some characterized and results were proved.

2.Basic concepts and Results

This part includes some well-known definitions, concepts, and results that are useful for us in our study of the next section.

Proposition (2.1)

Every submodule of the regular module is regular [2].

Proposition (2.2)

An S-module A is cyclic if and only if it is isomorphic to a factor module of S [4].

Proposition (2.3)

An S-module A is simple if and only if $A \cong S/E$ for some maximal ideal E of S [4] [1].

Definition (2.4)

A submodule B of an S-module A is called prime if $B \neq A$, and whenever $tx \in B$ for $t \in S$ and $x \in A$ we have either $t \in [B: A]$ or $x \in B$ [5].

Definition (2.5)

A submodule B of an S-module A is called a semimaximal submodule if and only if A/B is a semi-simple S-module [3, definition (2.1.1),p32].

Definition (2.6)

Let B be a submodule of an S-module A , the closure of B is denoted by $CL(B) = \{ x \in A : [B:(x)]$ essential in S $\}$. It is clear that CL(B) is a submodule of A containing B. That is $B \subseteq CL(B)$ [7] .

S3: Strongly Maximal Submodules with its advantages

In this section, the concept of strongly maximal submodule (for short, SM-submodule) was introduced, which was a generalization of the concept the strongly maximal ideal in a ring S. Several examples and properties were proved also a lot of characterizations, and different results were presented.

Let us start with our basic definition.

Definition (3.1)

Let A be an S-module, and B be a non-zero proper submodule of A. Then B is named strongly maximal submodule (for short SM-submodule) if and only if, for every non-zero ideal E of S implies A/E^2B is a regular module.

Examples and Remarks (3.2)

1. All the following modules have no SM-submodules.

(i) Z as a Z-module.

(ii) Z_p as a Z-module, p is a prime number.

(iii) Z_p as a Z_p-module, p is a prime number.

2. Every simple S-module has no SM-submodule. But the opposite is not true and the following example shows that: The module $A=Z_4 \oplus Z$ as a Z-module. Since A has no SM-submodules , A is not a simple module. Also notice examples (ii) and (iii) in no.(1) .

3. It is important to note that it is not necessary that all modules contain SM-submodules; for example Zp^{∞} as- Z-module.

Since, all the submodules of Zp^{∞} are of the form $<1/p^i + Z>$, where p is a prime number and i= 0, 1, 2…

Now, we write $N = <1/p^i +Z>$ and let E be an ideal of Z. If we take E= <1>, then, $Zp^{\infty}/E^2B = Zp^{\infty}/<1>^2B = Zp^{\infty}/B \cong Zp^{\infty}$ is not a regular module and hence $B = <1/p^i + Z>$ is not SM-submodule of Zp^{∞}. That is, Zp^{∞} has no SM-submodules.

Also, we can give another example Z_8 as a Z_8-module that has no SM-submodules .Since $<\overline{2}>$ and $<\overline{4}>$ are not SM- submodules in Z_8 .

4. The submodule $<\overline{3}>$ of a Z_6-module Z_6 is an SM-submodule. To clarify, let E

be an ideal of a ring Z_6 .

if $E = <\overline{1}>$, then $Z_6/<\overline{1}>^2 <\overline{3}> = Z_6/ <\overline{3}> \cong Z_3$ is a regular module .

if $E = <\overline{2}>$, then $Z_6/<\overline{2}>^2 <\overline{3}> = Z_6/<\overline{0}> \cong Z_6$ is a regular module .

if $E = <\overline{3}>$, then $Z_6/ <\overline{3}>^2 <\overline{3}> = Z_6/ <\overline{3}> \cong Z_3$ is a regular module .

Therefore $<\overline{3}>$ is an SM-submodule of Z_6 .

In general, all submodules of Z_6 as a Z_6-module are SM-submodules.

5. In Z_{10} as a Z_{20}-module, the submodule $B = <\overline{5}>$ is an SM-submodule. Since if we take E = $<\overline{1}>$, $<\overline{2}>$, $<\overline{10}>$, $<\overline{4}>$, $<\overline{5}>$,where E is ideal of Z_{20} , then $Z_{20}/E^2<\overline{5}>$ is a regular module for all ideal E of Z_{20} . This ends the proof of example.

6. Consider Z_4 as a Z-module. The submodule $<\overline{2}>$ is not SM-submodule of Z_4. Since $Z_4/E^2<\overline{2}>$ is not regular, E is an ideal of Z. To prove that, take $E = <\overline{4}>$. Then, $Z_4/<\overline{4}>^2 <\overline{2}> \cong Z_4$ is not regular module.

7. Let Z_4 as a Z_4-module. Then the submodule $B = <\overline{2}>$ is not an SM-submodule. Notice if E $= <\overline{2}>$ then $Z_4/E^2 <\overline{2}>$ is not regular module.

8. Let B = $<\overline{2}>$ be a submodule of a Z_{20}-module Z_{10} . Since $Z_{10}/ E^2B = Z_{10}/<\overline{2}>^2<\overline{2}> \cong Z_8$ is not a regular module, where E = $<\overline{2}>$ is an ideal of Z_{20}.

9. Let A = $Z_6 \oplus Z_3$ be an Z_{12}-module and B = $<\overline{2}>\oplus<\overline{0}>$ be a submodule of A. Then A/E^2B = $Z_6\oplus Z_3/E^2(<\overline{2}>\oplus<\overline{0}>)$ is regular module where E = $<\overline{1}>$, $<\overline{2}>$, $<\overline{3}>$, $<\overline{4}>$ and $<\overline{6}>$ be ideals of Z_{12} . Therefor B = $<\overline{2}>\oplus<\overline{0}>$ is an SM-Submodule of A = $Z_6\oplus Z_3$.

10. Consider A = $Z_6\oplus Z$ as a Z-module. Then the submodule B = $<\overline{3}>\oplus<\overline{2}>$ of A is not SM-submodule. Since $A/E^2B = Z_6\oplus Z/<\overline{2}>^2 (<\overline{3}>\oplus<\overline{2}>) = Z_6\oplus Z/<\overline{0}>\oplus<\overline{8}>$ is not a regular module, where E = $<\overline{2}>$ is an ideal of Z.

11. Every SM-submodule is maximal but the opposite is not true and the following example shows that: The submodule $<\overline{2}>$ of a Z-module Z_4 is a maximal submodule in Z_4 but it is not a SM-submodule, see no.(6) .

12. A submodule of an SM-submodule is not necessary to be an SM-submodule, for example: A submodule $<\overline{2}>$ in a Z_6-module Z_6 is SM-submodule. See no.(4) , While $<\overline{0}>$ is a submodule of $<\overline{2}>$ and it is not SM-submodule .

13. The intersection of two SM-submodules is not condition to be SM-submodule, for example : The two submodules $<\overline{2}>$, $<\overline{3}>$ in Z_6-module Z_6 are SM- Submodules but $<\overline{2}> \cap <\overline{3}> = <\overline{0}>$ is not an SM-submodule

14. More generally, let $\{B_i\}_{i=1}^{n}$ be a finite collection of SM-submodules of an S-module A. Then $\cap_{i=1}^{n} B_i$ is not always SM-submodule.

15. The direct sum of two SM-submodules of an S-module A is not necessary to be an SM-submodule, for example: Let $<\overline{2}>$, $<\overline{3}>$ be two SM-submodules of a Z_6-module Z_6, but $<\overline{2}>\oplus<\overline{3}>=Z_6$ is not an SM-submodule.

16. From the fact that every maximal submodule is a semimaximal by [3, Remarks and Examples (2.1.2), p_{32}], and the fact no.(11), we obtain that every SM-submodule of an S-module A is a semimaximal while the converse is not true in general and the following shows that: Let 6Z be a submodule of a Z-module Z . Then, 6Z is a semimaximal submodule of Z. Since 6Z =2Z∩5Z where 2Z, 5Z are maximal submodules of Z. But 6Z is not an SM-submodule of Z. Since $Z/(2Z)^2(6Z) = Z/24Z \cong Z_{24}$ is not a regular module .

Proposition (3.3)

Let B, D be two submodules of an S-module A with B⊆D. Then B is SM-submodule in D when B is SM-submodule in A.

Proof:

Suppose B is SM-submodule in A, then A/E^2B is regular module for every non-zero ideal E of S. Since D/E^2B is a submodule of A/E^2B (Notice, $E^2B \subseteq B \subseteq D$), and hence by proposition (2.1), D/E^2B is a regular submodule of A/E^2B. Thus B is SM-submodule in D.

Next, we will give an application to a proposition (3.3)

Corollary (3.4)

Let A be an S-module and B be a proper submodule of A. If B is an SM-submodule of $[B \dot{_A} A]$ and $[B \dot{_A} A]$ is an SM-submodule in A, then, B is an SM-submodule in A.

Proof:

It clear that $B \subseteq [B \dot{_A} A] \subseteq A$. Therefore, by using proposition (3.3), we conclude that B is SM-submodule in A.

Now, we will give the sufficient condition for a submodule to not be SM-submodule.

Proposition (3.5)

Let A be an S-module. If A is cyclic module (if A=Sx for some x∈A) and $ann_s(x)$ is maximal ideal of S, then A has no SM-submodule.

Proof:

Since A= Sx for some x∈A, then A is isomorphic to a factor module of S by proposition (2.2). We can define $f : S \to A$ such that $f(r) = rx$. It is easily to show that f is well- define and epimorphisim, by the first fundamental theorem of isomorphism S/Ker f ≅ A. Next, Ker f $=\{r \in S : f(r) = 0_A\} = \{r \in S : rx = 0_A\} = ann_s(x)$ That is $S/ann_s(x) \cong A$. Also, we have $ann_s(x)$ is maximal ideal of S which implies $S/ann_s(x)$ is simple and hence A is simple. Therefore, by examples and remarks ((3.2) No. (2)), we get the result.

Next is the application of proposition (3.5)

Corollary (3.6)

If a non-zero prime and semi-simple S-module A, then A has no SM-submodule.

Proof:

Suppose that A is a prime and semi-simple module, then, we obtain A is simple module. To prove this, assume that A is not simple which implies A is a direct sum of simple S-modules. Then, there exists a simple module M_1 and M_2 which are a direct summand of A with $M_1 \neq M_2$. $M_1 \cong S/E$, $M_2 \cong S/D$ where E and D are maximal ideals of S, by proposition (2.3). But A is prime module, then $ann_s(M_1) = E = ann_s(A)$ and $ann_s(M_2) = D = ann_s(A)$.Thus E = D which implies that $M_1 = M_2$, and this is a contradiction. Hence, A is a simple module and by using proposition (3.5), we have A has no SM-submodule.

As a direct of corollary (3.6), we have the following.

Corollary (3.7)

Let P be a prime and semimaximal submodule of an S-module A. Then the quotient module by P has no SM-submodule.

Proof:

Since P is a semimaximal submodule of A, then by definition (2.5), we have A/P is a semi-simple S-module. On the other hand, P is a prime submodule of A, then, by [3, proposition (1.1.51)] we obtain that A/P is a prime S-module, and hence by corollary (3.6), then A/P is a simple S-module and hence A/P has no SM-submodule.

S4: The behavior of SM-submodules under localization.

Let K be a subset of a ring S, W is multiplication closed if the two condition hold:

1. $I \in W$.
2. $xy \in W$ for every x , $y \in W$.

We know that every proper ideal E in S is prime if and only if S-E is multiplicatively closed, see [8].If A is an S-module and W be a multiplicatively closed on S such that $W \neq \langle 0 \rangle$, then S_w be the set for all fractional r/w where $r \in S$ and $w \in W$ and A_w be the set of all fractional m/w where $m \in A$ and $w \in W$. For $m_1, m_2 \in A$ and $w_1, w_2 \in W$, $m_1/w_1 = m_2/w_2$ if and only if $\exists\ t \in W$ such that $t(w_1m_1 - w_2m_2) = 0$. Also, we can make A_w in to S_w-module by setting $m_1/w_1 + m_2/w_2 = (w_2m_1 + w_1m_2)/w_1w_2$ and $(r/w_1)\ (m_1/w_2) = rm_1/w_1w_2$ for every $m_1, m_2 \in A$ and every $r \in S$, $w_1, w_2 \in W$. If W =S-E where E is a prime ideal, we used A_E instead of A_w and S_E instead of S_w. If a ring has only one maximal ideal, then it is called a local ring. Hence S_E is often called the localization of S at E, similar A_E is the $r/1, \forall\ r \in S$ and $\boldsymbol{\Phi} : A \rightarrow A_w$ such that $\boldsymbol{\Phi}(m) = m/1$, $\forall m \in A$. Furthermore, if B is a submodule of an S-module A and W be a multiplicatively closed in S, then $B_w = \{n/w: n \in B, w \in W\}$ be a submodule on S_w-module , see [8].

In this section we study the behavior of an SM-submodule under localization and several of results have been proved.

The following lemma is needed in our next result.

Lemma (4.1) [10]

Let A be an S-module and B, L are two submodules of A. Then, B=L if and only if $B_P = L_P$ for every maximal ideal P of S.

The following proposition study the relationship between a module A and its locally and prove that they are equivalent.

Proposition (4.2):

Let A be an S-module and B is nonzero proper submodule of A. Then, B_P is SM-submodule of an S_P-submodule A_P if and only if B is SM-submodule of an S-module A.

Proof:

Suppose that B is nonzero proper submodule of A. We must prove that A/E^2B is a regular S-module for every nonzero ideal E of S; that is, every submodule of A/E^2B is pure. Let L/E^2B be a submodule of A/E^2B. It is clear that $I(L/E^2B) \supseteq I(A/E^2B)\cap(L/E^2B)$ where I is an ideal of S. Now, to prove $I(L/E^2B) \subseteq I(A/E^2B)\cap(L/E^2B)$. Let $x \in I(L/E^2B)$. Then $x = \sum_{i=1}^{n} a_i(l_i+E^2B)$. Therefore $xs/s = (\sum_{i=1}^{n} a_i(l_i+E^2B))s/s \in I_P(L_P/E_P^2B_P)$ but L_P is SM-submodule in A_P, then $I_P(L_P/E_P^2B_P) = (I_P(A_P/E_P^2B_P))\cap(L_P/E_P^2B_P)$ which implies $xs/s \in I_P(A_P/E_P^2B_P)\cap(L_P/E_P^2B_P) = ((I_PA_P+E_P^2B_P)/ E_P^2B_P)\cap(L_P/E_P^2B_P)=(((IA)_P+(E^2B)_P)/(E^2B)_P)\cap(L_P/(E^2B)_P)$ by [10]. And hence $As/s \in((IA+E^2B)_P)/(E^2B)_P)\cap(L_P/(E^2B)_P)= ((IA+E^2B)/(E^2B))_P\cap(L/(E^2B))_P = (((IA+E^2B)/(E^2B))\cap(L/(E^2B)))_P$. Therefore $x \in ((IA+E^2B)/(E^2B))\cap(L/(E^2B))$ which implies $x \in (I(A/(E^2B))\cap(L/(E^2B))$ and hence $I(L/E^2B) \subseteq (I(A/(E^2B)) \cap (L/(E^2B))$. Therefore $I(L/E^2B) = (I(A/(E^2B))\cap(L/(E^2B))$. Thus L/E^2B is pure submodule of A/E^2B. This proves that A/E^2B is regular and finally B is an SM-submodule of A.

Conversely: -

Suppose that B is an SM-submodule of A. To prove B_P is SM-submodule of an S_p-module A_P we must show that $A_P/E_P^2B_P$ is regular S_P-module. It is clear that $(I_P(A_P/E_P^2B_P))\cap(L_P/E_P^2B_P) \subseteq I_P(L_P/E_P^2B_P)$. To prove $I_P(L_P/E_P^2B_P) \subseteq (I_P(A_P/E_P^2B_P))\cap(L_P/E_P^2B_P)$. Let $x/1 \in I$ and $a/s \in I_P(L_P/E_P^2B_P)$ $(xa/s + E_P^2B_P) = \sum_{i=0}^{n}(b_i/s_i)(l_i/t_i+E_P^2B_P^2)$ where s_i , $t_i \notin P$ and $b_i \in I$,$l_i \in L$. Put $c_i=s_it_i$. Therefore $(xa/s + E_P^2B_P) = ((b_1l_1v_1+b_2l_2v_2+\ldots+b_nl_nv_n)/u)+ E_P^2B_P^2$ where $u = c_1c_2c_3\ldots.c_n$ and $v_1 = c_2c_3\ldots.c_n$, $v_2 = c_1c_3c_4\ldots.c_n$, $v_n = c_1c_2c_3\ldots.c_{n-1}$. Thus there exist $k\notin P$ such that $kxau+E^2B = k(b_1l_1v_1+b_2l_2v_2+\ldots+b_nl_nv_n)\in I(L/E^2B)$ but L is pure submodule in A, that is $I(L/E^2B) = I(A/ E^2B)\cap (L/ E^2B)$ (Since A/ E^2B is regular S-module) and hence by (13,theorem (2.6)] we have $(xa+ E^2B) \in I(A/ E^2B \cap L/ E^2B)$. This leads us to write $(xa/s+E_P^2B_P)\in(I_P(A_P/E_P^2B_P)\cap(L_P/E_P^2B_P)$. This gives $I_P(L_P/E_P^2B_P) \subseteq (I_P(A_P/E_P^2B_P))\cap(L_P/E_P^2B_P)$ and $A_P/E_P^2B_P$ is a regular S_P-module and finally , we obtain that B_P is an SM-submodule of A_P .

Proposition (4.3):

Let L, B be two finitely generated submodules of an S-module A. If L_P, B_P are SM-submodules of A_P, then $L\cap B$ is an SM-submodule of A.

Proof:

Since L, B are two finitely generated submodules of A, then by {10,p24}, $[L_P:B_P] + [B_P:L_P] = S_P$ for every maximal ideals P of S. Thus, $L_P\cap B_P = L_P$ or $L_P\cap B_P = B_P$, but L_P and B_P are SM-submodules, then $L_P\cap B_P$ is an SM-submodule, and we have $L_P\cap B_P = (L\cap B)_P$. Therefore $(L\cap B)_P$ is an SM-submodule and by proposition(4.2), $L\cap B$ is an SM-submodule of A.

Proposition (4.4):

Let L, B be two finitely generated submodules of an S-module A. Then, L+B is an SM-submodules of A, if L_P ,B_P are SM-submodules of an S_P-module A_P .

Proof:

Let L, B be two finitely generated submodules of A. Then by {10, p24}, we have $[L_P:B_P]+[B_P:L_P] = S_P$ for every maximal ideal P of S. Let $y_1 \in [L_P:B_P]$ and $y_2 \in [B_P:L_P]$ such that $y_1+y_2 = 1$ = unity of S_P. Then, either y_1 is a unit element or y_2 is a unit element (Since S_P is local ring). Therefore $[L_P:B_P] = S_P$ or $[B_P:L_P] = S_P$ and hence either $L_P \subseteq B_P$ or $B_P \subseteq L_P$ which implies $L_P + B_P = L_P$ or $L_P + B_P = B_P$,but L_P , B_P are SM-submodules of A_P . Thus $L_P + B_P$ is an SM-submodule and $(L+B)_P$ is an SM-submodule and by proposition(4.2) , L+B is an SM-submodule of A.

5.Conclusion

The conclusion of this work is to study an important concept, namely strongly maximal submodule. Some properties and many results were proved and the behavior of that concept with its localization were studied and shown.

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