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The Implementations Special Third-Order Ordinary Differential Equations (ODE) for 5th-order 3rd-stage Diagonally Implicit Type Runge-Kutta Method (DITRKM)

Firas A. Fawzi
firasadil01@gmail.com

Mustafa H. Jumaa
mustafa.hassan551991@gmail.com

Department of Mathematics, Facultyof Computer Sciences and Mathematics,Tikrit University, Iraq.

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Abstract

The derivation of 5th order diagonal implicit type Runge Kutta methods (DITRKM5) for solving 3rd special order ordinary differential equations (ODEs) is introduced in the present study. The DITRKM5 techniques are the name of the approach. This approach has three equivalent non-zero diagonal elements. To investigate the current study, a variety of tests for five various initial value problems (IVPs) with different step sizes h were implemented. Then, a comparison was made with the methods indicated in the other literature of the implicit RK techniques. The numerical techniques are elucidated as the qualification regarding the efficiency and number of function evaluations compared with another literature of the implicit RK approaches from the result of the computations. In addition, the stability polynomial for DITRK method is derived and analyzed.

Keywords: Numerical Methods, Ordinary Differential Equations, Diagonal Implicit Type Runge Kutta Methods, Initial Value Problems, Stability Polynomial Analysis.

1. Introduction

Third-order ODEs are used in neural network engineering and applied sciences, the dynamics of fluid flow, the ship's motion, and electric circuits, among other fields [1-6]. Consider the numerical method for solving the special "initial value problems" (IVPs) for order three as the following form

$$y'''(x) = f(x, y(x)) \qquad (2.1)$$

$$y(x_0) = \alpha,\ y'(x_0) = \beta \quad \text{and} \quad y''(x_0) = \gamma$$

The implicit methods are important because they can reach high orders of accuracy at the equivalent number of stages, which can be represented as an advantage that leads to the more accurate than the explicit approaches. This manufactures it easier to exist the solution to the difficulties of the problems.

So, the implicit RK techniques play an important role for denomination the physical and mathematical problems, like a differential algebraic equation.

In addition, diagonal implicit RK (DIRK) techniques are also pointed to as semi-implicit approaches or semi explicit RK techniques since they obtained at minimum one value does not zero for the lower of the triangular diagonal matrices. Therefore, to solve Eq. (2.1), two general strategies can be employed. The elementary way is to transfer the Eq. (2.1) into a problem with first-order then apply any pattern of the RK approach to it.

As a result, numerous implicit RK approaches, such as Ismail et al. [7] and others, have been developed. The second option is to use the RK Type method to directly solve Eq. (2.1). For second-order systems, several scholars provided an efficient implicit RK technique (see [8-13]). Ghawadri et al. [14], constructed a solution to the ill-posed issue for a beam with an elastically base using special fourth-order ODEs. Moreover, [15-17] developed a solution of the special 3rd order for the ODEs directly by RK technique. Finally, Senu [18] and Fawzi et al. [19] constructed the embedded the RK technique to solve 3rd order for the ODEs.

A significant objective for current research is to show how particular third-order ODEs are solving via the DIRECT method. Additionally, while solving eq. (2.1) numerically, the algebraic order of the technique used must be taken into account, as this is the most important factor in achieving high accuracy.

Section 2.2 demonstrates the basic idea of construction and derivation of the DITRK system for addressing Initial Value Problems (IVPs). The DITRK technique's order criteria are outlined in Section 2.3. Section 2.4 describes the 3rd stage 5th order (DITRKM5) methods. In Section 2.5, the analyses of the stability polynomial for the DITRK method are presented. In Section 2.6, mentions the DITRK approach with five IVPs. In Section 2.7, the validation of the current approach compared with those in the other literatures of the implicit RK techniques.

2. The Methodology of DITRK Techniques

For solving IVPs in eq. (2.1), the prevalent formula of the implicit RK approach for the m stage can be expressed as follows:

$$y_{n+1} = y_n + h\, y_n' + \frac{h^2}{2} y_n'' + h^3 \sum_{i=1}^{m} d_i\, k_i \tag{2.2}$$

$$y_{n+1}' = y_n' + h\, y_n'' + h^2 \sum_{i=1}^{m} b_i\, k_i \tag{2.3}$$

$$y_{n+1}'' = y_n'' + h \sum_{i=1}^{m} g_i\, k_i \tag{2.4}$$

and

$$k_1 = f(x_n,\ y_n) \tag{2.5}$$

$$k_i = f(x_n + c_i h, y_n + h\, c_i y_n' + \frac{h^2}{2} c_i^2\, y_n'' + h^3 \sum_{j=1}^{i-1} a_{ij}\, k_j) \tag{2.6}$$

where $i = 2,3, \dots, m$.

The parameters of diagonal implicit RK type (DITRK) methods are presumed as $c_i, a_{ij}, d_i,\ b_i,\ g_i$ where $i, j = 1, 2, 3 \dots, s$ are real numbers and m is referred to stage digit for the approach. This scheme is known as diagonal implicit when $a_{ij} \neq 0$ for $j > i$. The last denomination includes the single DITRK techniques that A indicate that the lower the triangular diagonal matric of A have same values with $a_{ij} \neq 0$ where $i = j$ at the diagonal. The DITRK approach proposed from the work of Butcher, as illustrated in Table 2.1 [20].

Table 2.1: Butcher form DITRK method.

c_1	$a_{1,1}$		
c_2	$a_{2,1}$	$a_{2,2}$	
c_2	a_{31}	$a_{3,2}$	$a_{3,3}$
d_i	d_1	d_2	d_3
b_i	b_1	b_2	b_3
g_i	g_1	g_2	g_3

3. Order Conditions of the DITRK Technique

According to Mechee et al. [17], the orders of algebraic criteria for RKD approached over order 6 are as follow:

Order conditions of y:

order 3 $\quad \sum d_i = \frac{1}{6}$ (2.7)

order 4 $\quad \sum d_i\, c_i = \frac{1}{24}$ (2.8)

order 5 $\quad \sum d_i\, c_i^2 = \frac{1}{60}$ (2.9)

order 6 $\quad \sum d_i\, c_i^3 = \frac{1}{120}$ and $\sum d_i\, a_{i,j} = 1/720.$ (2.10)

Order conditions of y':

order 2 $\quad \sum b_i = \frac{1}{2}$ (2.11)

order 3 $\quad \sum b_i\, c_i = \frac{1}{6}$ (2.12)

order 4 $\quad \sum b_i\, c_i^2 = \frac{1}{12}$ (2.13)

order 5 $\quad \sum b_i\, c_i^3 = \frac{1}{20}$ and $\sum b_i\, a_{i,j} = \frac{1}{120}$ (2.14)

order 6 $\quad \sum b_i\, c_i^4 = \frac{1}{30}, \sum b_i\, a_{i,j} c_j = \frac{1}{720}$ and $\sum b_i\, c_i\, a_{i,j} = \frac{1}{180}$. (2.15)

Order conditions of y'':

order 1 $\quad \sum g_i = 1$ (2.16)

order 2 $\sum g_i\, c_i = \frac{1}{2}$ (2.17)

order 3 $\sum g_i\, c_i^2 = \frac{1}{3}$ (2.18)

order 4 $\sum g_i\, c_i^3 = \frac{1}{4}$ and $\sum g_i\, a_{i,j} = \frac{1}{24}$ (2.19)

order 5 $\sum g_i\, c_i^4 = \frac{1}{5}$, $\sum g_i\, a_{i,j} c_j = \frac{1}{120}$ and $\sum g_i\, c_i\, a_{i,j} = \frac{1}{30}$ (2.20)

order 6 $\sum g_i\, c_i^2 a_{i,j} = \frac{1}{36}, \sum g_i\, a_{i,j}\, c_j^2 + \sum g_i\, c_i a_{i,j} c_j = \frac{7}{720},$

$\sum g_i\, c_i^5 = \frac{1}{6},\ \sum g_i\, a_{i,j}\, c_j^2 = \frac{1}{360}\ , \sum g_i\, c_i a_{i,j} c_j = \frac{1}{144}$

and

$$\frac{1}{2}\sum g_i\, a_{i,j}\, c_j^2 + \sum g_i\, c_i a_{i,j} c_j = \frac{1}{120} \quad (2.21)$$

4. Formation of the 3rd stage 5th order (DITRKM5) Method

We implement a diagonal implicit type Runge–Kutta approach using order conditions derivations as demonstrated in section 2.3, which is developed according to Mechee work [17]. For the p order DITRK approach, the local truncation error is defined as follows:

for order five:

$$LTE_{\text{order 5}} = \Big[(\sum d_i\, c_i^3 - \tfrac{1}{120})^2 + (\sum d_i\, a_{i,j} - \tfrac{1}{720})^2 + (\sum b_i\, c_i^4 - \tfrac{1}{30})^2 + (\sum b_i\, a_{i,j} c_j - \tfrac{1}{720})^2 + (\sum b_i\, a_{i,j} c_i - \tfrac{1}{180})^2 + (\sum g_i\, a_{i,j} c_i^2 - \tfrac{1}{36})^2 + (\sum g_i\, c_i\, a_{i,j} c_j - \tfrac{1}{144})^2 + (\sum g_i\, c_i^5 - \tfrac{1}{6})^2 + (\sum g_i\, a_{i,j} c_j^2 - \tfrac{1}{360})^2 + (\sum g_i\ \ a_{i,j} c_j^2 + \sum g_i\, c_i\, a_{i,j} c_j - \tfrac{1}{720})^2 + (\tfrac{1}{2}\sum g_i\ \ a_{i,j} c_j^2 + \sum g_i\, c_i\, a_{i,j} c_j - \tfrac{1}{120})^2\Big]^{\frac{1}{2}} \quad (2.22)$$

the error of local truncation terms for y, y' and y''. The fifth-order three-stage of the DITRKM method in the present study can be computed by employing algebraic order conditions over 5. The System of the result includes 16 nonlinear equations with 16 unknown variables, assuming

$$a_{1,1} = a_{2,2} \quad \text{and} \quad a_{2,2} = a_{3,3} \quad (2.23)$$

Thus, the calculations of the system products the set of solutions in terms of the parameters $a_{1,1}$, $a_{2,2}$ and c_1 as follows:

$$a_{2,1} = 0, a_{3,1} = \frac{3}{20} - \frac{3}{10}\mathrm{RootOf}(10\mathrm{z}^2 - 10\mathrm{z} + 1), a_{3,2} = 0, a_{3,3} = \frac{1}{120}\mathrm{RootOf}(10\mathrm{z}^2 - 10\mathrm{z} + 1), b_1 = \frac{-5}{18}\mathrm{RootOf}(10\mathrm{z}^2 - 10\mathrm{z} + 1) + \frac{5}{18}, b_2 = \frac{2}{9}, b_3 = \frac{5}{18}\mathrm{RootOf}(10\mathrm{z}^2 - 10\mathrm{z} + 1), c_1 = \mathrm{RootOf}(10\mathrm{z}^2 - 10\mathrm{z} + 1)\,, c_2 = \frac{1}{2}\,, c_3 = -\mathrm{RootOf}(10\mathrm{z}^2 - 10\mathrm{z} + 1) + 1, \mathrm{d}_1 = \frac{1}{8} - \frac{5}{36}\mathrm{RootOf}(10\mathrm{z}^2 - 10\mathrm{z} + 1), \mathrm{d}_2 = \frac{1}{18}, \mathrm{d}_3 = \frac{5}{36}\ \mathrm{RootOf}(10\mathrm{z}^2 - 10\mathrm{z} + 1) - \frac{1}{72}, g_1 = \frac{5}{18},\ g_2 = \frac{4}{9},\ \ g_3 = \frac{5}{18} \quad (2.24)$$

Finally, coefficients of the DITRKM method for 5-order 3 stages indicated by DITRKM5 can be read as shown in **Table 2.2**.

Table 2.2: third-stage fifth-order DITRK Method (DITRKM5).

$\frac{1}{2}-\frac{\sqrt{15}}{10}$	$\frac{1}{24}-\frac{\sqrt{15}}{120}$		
$\frac{1}{2}$	0	$\frac{1}{24}-\frac{\sqrt{15}}{120}$	
$\frac{1}{2}+\frac{\sqrt{15}}{10}$	$\frac{3\sqrt{15}}{100}$	0	$\frac{1}{24}-\frac{\sqrt{15}}{120}$
	$\frac{1}{18}+\frac{\sqrt{15}}{72}$	$\frac{1}{18}$	$\frac{1}{18}-\frac{\sqrt{15}}{72}$
	$\frac{5}{36}+\frac{\sqrt{15}}{36}$	$\frac{2}{9}$	$\frac{5}{36}-\frac{\sqrt{15}}{36}$
	$\frac{5}{18}$	$\frac{4}{9}$	$\frac{5}{18}$

5. The Stability Polynomial of DITRK Method

In order to study the stability polynomial of the DITRK method, the following equation is suggested :

$$y''' = -\gamma^3 y \qquad (2.25)$$

By substituting the DITRK method from eqs. (2.2) - (2.4) to exam eq. (2.25) with stage $m = 3$ are yield

$$y_{n+1} = y_n + h\, y_n' + \frac{h^2}{2} y_n'' + h^3 \sum_{i=2}^{3} d_i\, k_i \qquad (2.26)$$

$$y_{n+1}' = y_n' + h\, y_n'' + h^2 \sum_{i=2}^{3} b_i\, k_i \qquad (2.27)$$

$$y_{n+1}'' = y_n'' + h \sum_{i=2}^{3} g_i\, k_i \qquad (2.28)$$

and

$$k_1 = f(x_n,\ y_n)$$

$$k_i = f(x_n + c_i h, y_n + h\, c_i y_n' + \frac{h^2}{2} c_i^2\, y_n'' + h^3 \sum_{j=1}^{i-1} a_{ij}\, k_j) \qquad (2.29)$$

where $i = 2,3, \dots, m$ and

$$Y_i = y_n + h\, c_i y_n' + \frac{h^2}{2} c_i^2\, y_n'' + h^3 \sum_{j=1}^{i-1} a_{ij}\, (-\gamma^3 Y_j) \qquad (2.30)$$

where

$$Y_1 = y_n \qquad (2.31)$$

$$Y_i = y_n + h\, c_i y_n' + \frac{h^2}{2} c_i^2\, y_n'' + h^3 \sum_{j=1}^{i-1} a_{ij}\, f(x_n + c_i h, Y_i) \qquad (2.32)$$

more simplification

$$y_{n+1} = y_n + h\, y_n' + \frac{h^2}{2} y_n'' + h^3 \sum_{i=2}^{m} d_i\, (-\gamma^3) Y_j \qquad (2.33)$$

$$y_{n+1}' = y_n' + h\, y_n'' + h^2 \sum_{i=2}^{m} b_i\, (-\gamma^3) Y_j \qquad (2.34)$$

$$y_{n+1}'' = y_n'' + h \sum_{i=2}^{m} g_i\, (-\gamma^3) Y_j \qquad (2.35)$$

From eq. (2.25), the above equations can be written as follow

$$y_{n+1} = y_n + h\, y_n' + \frac{h^2}{2} y_n'' + h^3 \sum_{i=2}^{m} d_i\, f(x_n + c_i h, Y_i) \qquad (2.36)$$

$$y'_{n+1} = y'_n + h\,y''_n + h^2 \sum_{i=2}^{m} b_i\, f(x_n + c_i h, Y_i) \qquad (2.37)$$

$$y''_{n+1} = y''_n + h \sum_{i=2}^{m} g_i\, f(x_n + c_i h, Y_i) \qquad (2.38)$$

From eq. (2.30), multiply eq. (2.34) by h and eq. (2.35) by h^2,yield

$$hy'_{n+1} = hy'_n + h^2\, y''_n + h^3 \sum_{i=2}^{m} b_i\,(-\gamma^3)Y_j \qquad (2.39)$$

$$h^2 y''_{n+1} = h^2 y''_n + h^3 \sum_{i=2}^{m} g_i\,(-\gamma^3)Y_j \qquad (2.40)$$

Then

$$\begin{bmatrix} y_{n+1} \\ hy'_{n+1} \\ h^2 y''_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_n \\ hy'_n \\ h^2 y''_n \end{bmatrix} + (-\gamma^3 h^3) \begin{bmatrix} d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 \\ g_1 & g_2 & g_3 \end{bmatrix} \begin{bmatrix} y_n \\ hy'_n \\ h^2 y''_n \end{bmatrix} \qquad (2.41)$$

Also, the matrix format of eq. (2.30) can be defined as

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & c_2 & c_2^2 \\ \vdots & \vdots & \vdots \\ 1 & c_m & c_m^2 \end{bmatrix} \begin{bmatrix} y_n \\ hy'_n \\ \vdots \\ h^2 y''_n \end{bmatrix} + (-\gamma^3 h^3) \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{m-1} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{bmatrix} \qquad (2.42)$$

where $m = 3$. Therefore,

$$\begin{bmatrix} y_{n+1} \\ hy'_{n+1} \\ h^2 y''_{n+1} \end{bmatrix} = f(H) \begin{bmatrix} y_n \\ hy'_n \\ h^2 y''_n \end{bmatrix}, \quad H = (-\gamma^3 h^3) \qquad (2.43)$$

and

$$f(H) = \begin{bmatrix} 1 + Hd^T P^{-1} E_1 & 1 + Hd^T P^{-1} E_2 & 1 + Hd^T P^{-1} E_3 \\ Hb^T P^{-1} E_1 & 1 + Hb^T P^{-1} E_2 & 1 + Hb^T P^{-1} E_3 \\ Hg^T P^{-1} E_1 & Hg^T P^{-1} E_2 & 1 + Hb^T P^{-1} E_3 \end{bmatrix} \qquad (2.44)$$

where

$$E_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 \\ c_2 \\ c_3 \end{bmatrix}, E_3 = \begin{bmatrix} 0 \\ c_2^2 \\ c_m^2 \end{bmatrix} \quad \text{and } P^{-1} = (1 - HA)^{-1} \qquad (2.45)$$

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{m-1} \end{bmatrix}, \quad B = \begin{bmatrix} d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 \\ g_1 & g_2 & g_3 \end{bmatrix} \qquad (2.46)$$

Thus, the stability polynomial of the DITRK method can be written as

$$\emptyset(\varphi, H) = |\varphi I - f(H)| \qquad (2.47)$$

Where $f(H)$ is given value, the characteristic equation is defined as follow,

$$\emptyset(\varphi, H) = P_0(H)\varphi^3 + P_1(H)\varphi^2 + P_2(H)\varphi + P_3(H) \qquad (2.48)$$

6. Test of Problems

The approaches that demonstrated in section 2.3 tested with 5 various problems in this part. The numerical results of the suggested approaches compared with those of other RK techniques at equivalent order which are already available. The numerical experiments were conducted using the following methods:

(1) DITRKM5: 3rd stage 5th order DITRK approach computed in the present work.

(2) Radau I: 3rd stage 5th order RK technique presented in [20].

(3) Radau IA: 3rd stage 5th order RK method studied in [21].

(4) Radau II: 3rd stage 5th order RK scheme tested in [20].

(5) Radau IIA: 3rd stage 5th order RK approach noted in [21].

Problem (1): Consider a nonhomogeneous linear ODE given in [23]

$y'''(x) = y(x) + \cos(x)$, with $y(0) = 0,\ y'(0) = 0,\ y''(0) = 1$ where $x \in [0,1]$,

and analytic solution $y(x) = \frac{(e^x - \cos(x) - \sin(x))}{2}$.

Problem (2): Consider the nonhomogeneous nonlinear ODE

$y'''(x) = (y(x))^2 + \cos^2(x) - \cos(x) - 1$, with $y(0) = 0,\ y'(0) = 1,\ y''(0) = 1$ where $0 \le x \le 2$, the exact solution $y(x) = \sin(x)$.

Problem (3): The nonhomogeneous nonlinear ODEs is considered as

$y_1'''(x) = y_2(x)$, with $y_1(0) = 1,\ y_1'(0) = 0, y_1''(0) = 1$,

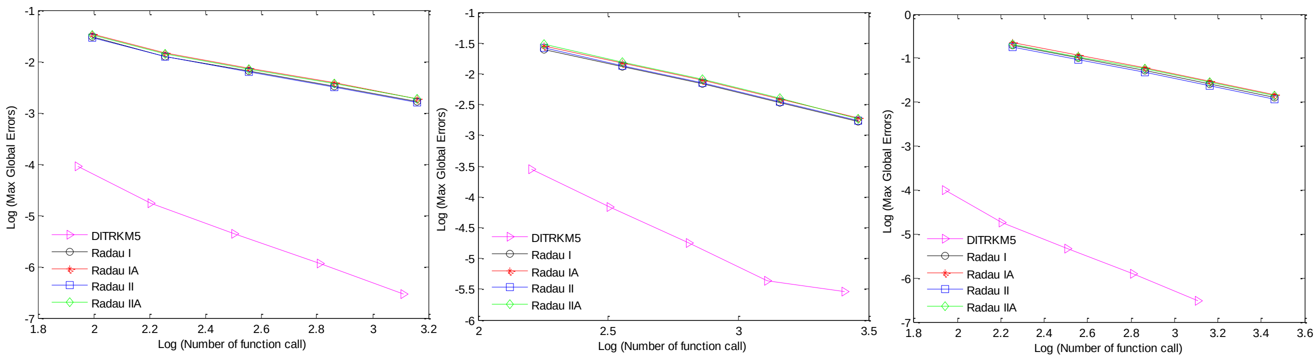
$y_2'''(x) = -y_1(x) - 2\,y_2(x) + 2\,y_3(x)$ with $y_2(0) = 0,\ y_2'(0) = 1, y_2''(0) = 0$,

$y_3'''(x) = y_1(x) + y_2(x)$ with $y_3(0) = 1,\ y_3'(0) = 1, y_3''(0) = 1$,With analytic

solution $y_1(x) = \cosh(x)$, $y_2(x) = \sinh(x)$ and $y_3(x) = e^x$ where $0 \le x \le 1$.

7. Numerical Results

Figure (2.1) shows the efficiency of the DITRKM methods created by charting of decimal logarithm for the highest "global error" versus logarithm of function estimate.



problem 1 problem 2 problem 3

Figure 2.1: Accuracy curve for DITRKM5, Radau I, Radau IA, Radau II and Radau IIA with h = 0.1, 0.05 0.025, 0.00125, 0.00625 for the problem 1, problem 2 and problem 3.

When compared the current study with another implicit RK approach for equivalent order, the DITRKM5 method requires fewer "function evaluations". The digit of equations increased three times with the problems turned to a system of 1st order ODEs. In the comparison, the existing implicit the RK approach with the equivalent order, the "global error" and digit of "function estimate" contain the smallest maximum for the DITRKM5 method at each iteration, as shown in **Figure (2.1)** that obtained from **Table (2.1).** As shown in **Figure (2.1**), the fifth-order three stage results DITRK method (DITRKM5) produces more accurate findings than the other results in the literature (Radau I, Radau IA, Radau II, and Radau IIA). In this work, the logarithm of "maximum global error" is known as a logarithm function for "function evaluation" with different step size $h = 0.1, 0.05, 0.025, 0.0125, 0.00625$ for five test problems.

Table 2.1: Comparisons of number of function call and maximum global error for DITRKM5, Radau I, Radau IA, Radau II and Radau IIA Methods with h = 0.1, 0.05 0.025, 0.00125, 0.00625 for the problem 1, problem 2 and problem 3.

Step size (h)	Method	Problem 1		Problem 2		Problem 3	
		No. of Function Call	Maximum Global Error	No. of Function Call	Maximum Global Error	Maximum Global Error	Maximum Global Error
0.1	DITRKM5	88	9.16352E-05	160	0.000277124	88	9.74073E-05
	Radau I	99	0.02986368	180	0.02489423	99	0.05488292
	Radau IA	99	0.03449261	180	0.02867864	99	0.06335586
	Radau II	99	0.02878467	180	0.02647426	99	0.05488292
	Radau IIA	99	0.03336877	180	0.0298746	99	0.06335586
0.05	DITRKM5	160	1.74282E-05	320	6.8347E-05	160	1.8304E-05
	Radau I	180	0.01273117	360	0.01292113	180	0.02293305
	Radau IA	180	0.01475188	360	0.01494992	180	0.02656216
	Radau II	180	0.01229666	360	0.01360768	180	0.02293305
	Radau IIA	180	0.0143079	360	0.015539	180	0.02656216
0.025	DITRKM5	320	4.45635E-06	648	1.79521E-05	320	4.67234E-06
	Radau I	360	0.006421128	729	0.006847777	360	0.0115538
	Radau IA	360	0.007451075	729	0.007940234	360	0.01340439
	Radau II	360	0.006195875	729	0.007152807	360	0.0115538
	Radau IIA	360	0.007223443	729	0.008219884	360	0.01340439
0.0125	DITRKM5	648	1.1728E-06	1288	4.35276E-06	648	1.22981E-06
	Radau I	729	0.003298104	1449	0.003387183	729	0.005945227
	Radau IA	729	0.003829934	1449	0.003931975	729	0.006903284
	Radau II	729	0.003179202	1449	0.003535766	729	0.005945227
	Radau IIA	729	0.003710422	1449	0.004074293	729	0.006903284
0.00625	DITRKM5	1288	2.88938E-07	2560	2.88938E-06	1288	3.02667E-07
	Radau I	1449	0.00163419	2880	0.001667861	1449	0.002941504
	Radau IA	1449	0.001898433	2880	0.001937206	1449	0.003416977
	Radau II	1449	0.001575303	2880	0.001741931	1449	0.002941504
	Radau IIA	1449	0.001839395	2880	0.001741931	1449	0.003416977

8. Conclusion

The test of the fifth-order three-stage DITRKM5 methods for the integration of ODEs obtained by testing the minimized error norm and studied the digit of the evaluations of function that described in this paper. The digit of "function evaluations" and the maximum error of the supposed technique is lower than those in implicit of the RK approaches, as indicated by numerical results in all **Figure (2.1)** and **Table (2.1)**, and the introduced techniques are most accurate by solving the directly specific ODEs of the order of three. The stability polynomial for DITRK method is analyzed and presented.

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