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Fuzzy Soc-Semi-Prime Sub-Modules

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Abstract

In this paper, we study a new concept of fuzzy sub-module, called fuzzy socle semi-prime sub-module that is a generalization the concept of semi-prime fuzzy sub-module and fuzzy of approximately semi-prime sub-module in the ordinary sense. This leads us to introduce level property which studies the relation between the ordinary and fuzzy sense of approximately semi-prime sub-module. Also, some of its characteristics and notions such as the intersection, image and external direct sum of fuzzy socle semi-prime sub-modules are introduced. Furthermore, the relation between the fuzzy socle semi-prime sub-module and other types of fuzzy sub-module presented.

Keyword: $\mathcal{F}$-module, $\mathcal{F}$-sub-module, $\mathcal{F}$-prime sub-module, Socle of $\mathcal{F}$-module.

1.Introduction

The concept of fuzzy sets was introduced by Zadeh in1965[1]. Many authors indeed presented fuzzy subrings and fuzzy ideals. The concept of fuzzy module was introduced by Negoita and Relescu in 1975 [2]. Since then several authors have studied fuzzy modules. The concept of semi-prime fuzzy sub-module was introduced by Rabi 2004[3]. The concept of approximately semi-prime sub-module was introduced by Ali 2019[4]. The socle of M is a summation of simple sub-modules of an $\mathcal{R}$-module M and denoted by $Soc(M)$. But, the fuzzy socle of $\mathcal{F}$-module X an $\mathcal{R}$-module M is a summation of simple $\mathcal{F}$-sub-modules of X and denoted by $F - Soc(X)$.

Preliminaries

" There are various definitions and characteristics in this section of $\mathcal{F}$-sets , $\mathcal{F}$-modules , and prime $\mathcal{F}$-sub-modules.

Definition 1.1 [1]

Let D be a non- empty set and I is closed interval $[0,1]$ of real numbers. An $\mathcal{F}$-set B in D (an $\mathcal{F}$-subset of D) is a function from D into I.

Definition 1.2 [1]

AN $\mathcal{F}$-set B of a set D is said to be $\mathcal{F}$-constant if $B(x) = t, \forall\, x \in D\; t \in [0,1]$

Definition 1.3 [1]

Let $x_t: D \to [0,1]$ be an $\mathcal{F}$-set in D, where x ∈D , t ∈ [0, 1] defined by:

$$x_t(y) = \begin{cases} t & if\ x = y \\ 0 & if\ x \neq y \end{cases}$$

for all $y \in D$. x_t is said to be an $\mathcal{F}$-singleton or $\mathcal{F}$-point in D.

Definition 1.4 [5]

Let B be an $\mathcal{F}$-set in D, for all t ∈ [0, 1], the set $B_t = \{x \in D; B(x) \geq t\}$ is said to be a level subset of B.

Remark 1.5 [6]

Let A and B be two $\mathcal{F}$-sets in S, then:

1- $A = B$ if and only if $A(x) = B(x)\ for\ all\ x \in S$.
2- $A \subseteq B$ if and only if $A(x) \leq B(x)\ for\ all\ x \in S$.
3- $A = B$ if and only if $A_t = B_t\ for\ all\ t \in [0,1]$.

If $A < B$ and there exists x ∈S such that $A(x) < B(x)$, then A is a proper $\mathcal{F}$-subset of B and written as $A < B$.

By part (2), we can deduce that $x_t \subseteq A$ if and only if $A(x) \geq t$.

Definition 1.6 [6]

If M is an $\mathcal{R}$-module. An $\mathcal{F}$-set X of M is called $\mathcal{F}$-module of an $\mathcal{R}$-module M if :

1- $X(x-y) \geq \min\{X(x), X(y)\} for\ all\ x, y \in M\}$.

2- $X(rx) \geq X(x)\ for\ all\ x \in M\ and\ r \in \mathcal{R}$.

3- $X(0) = 1$.

Proposition 1.7 [7]

Let C be an $\mathcal{F}$-set of an $\mathcal{R}$-module M. Then the level subset C_t of M , ∀ t ∈ [0, 1] is a sub-module of M if and only if C is an $\mathcal{F}$-sub-module of $\mathcal{F}$-module of an $\mathcal{R}$-module M.

Definition 1.8 [8]

Let X and A be two $\mathcal{F}$-modules of $\mathcal{R}$-module M. A is said to be an $\mathcal{F}$-sub-module of X if $A \subseteq X$.

Proposition 1.9 [5]

Let A be an $\mathcal{F}$-set of an $\mathcal{R}$-module M. Then the level subset A_t , t ∈ [0, 1] is a sub-module of
M if A is an $\mathcal{F}$-sub-module of X where X is an $\mathcal{F}$-module of an $\mathcal{R}$-module M.

Now, we go over various $\mathcal{F}$-sub-module attributes that will be useful in the next section.

Lemma 1.10 [6]

If r_t be an $\mathcal{F}$-singleton of $\mathcal{R}$ and A be an $\mathcal{F}$-module of an $\mathcal{R}$-module M.Then for any w ∈ M

$$(r_t \mathrm{A})(w) = \begin{cases} \sup\{\inf(t, A(x))\}: \; if\ w = rx\} & for\ some\ \ x \in \mathrm{M} \\ 0 & otherwise \end{cases}$$

Where $r_t\colon \mathcal{R} \to [0,1]$, defined by

$$r_t(z) = \begin{cases} t & if\ r = z \\ 0 & if\ r \neq z \end{cases}$$

For all $z \in \mathcal{R}$

Definition 1.11 [6]

Let A and B be two $\mathcal{F}$-sub-modules of an $\mathcal{F}$-module X of $\mathcal{R}$-module M. The residual quotient of A and B denoted by (A : B) is the $\mathcal{F}$-subset of $\mathcal{R}$ defined by:
$(\mathrm{A} : B)(\mathrm{r}) = \sup\{\mathrm{t} \in [0,1] : r_t B \subseteq \mathrm{A}\}$, for all $r \in \mathcal{R}$. That is $(\mathrm{A} : B) = \{r_t \ : r_t \mathrm{B} \subseteq \mathrm{A};\ r_t$ is an $\mathcal{F}-$ singleton of $\mathcal{R}\}$. If $B = \langle x_k \rangle$, then $(\mathrm{A} : \langle x_k \rangle) = \{r_t \ : r_t x_k \subseteq \mathrm{A};\ r_t$ is an $\mathcal{F} -$ singleton of $\mathcal{R}$ }.

Lemma 1.12 [9]

Let A be an $\mathcal{F}$-sub-module of $\mathcal{F}$-module X, $(\mathrm{A}_t : X_t) \geq (\mathrm{A}: X)_t$,For all t ∈ [0, 1].
Also , we can prove that by Lemma 2.3.3.[6].

It follows that if , $X = \mathrm{A} \oplus B$,where $A, B \leq X$ then $X_t = (\mathrm{A} \oplus B)_t = \mathrm{A}_t \oplus B_t$.

Definition 1.13 [10]

Let f be a mapping from a set M into a set N and let A be $\mathcal{F}$-set in M. The image of A is denoted by f(A), where f(A) is defined by:

$$f\,(\mathrm{A})\,(y) = \begin{cases} \sup\{\mathrm{A}(z) : z \in f^{-1}(y) \neq \emptyset\} & for\ \ all\ y \in \mathrm{N} \\ 0 & otherwise \end{cases}$$

Note that, if f is a bijective mapping, then $f\,(\mathrm{A})(y) = \mathrm{A}(f^{-1}(y))$

Proposition 1.14 [11]

Let f be a mapping from a set M into a set N. Assume that X and Y are $\mathcal{F}$-modules of M and N respectively, let A be an $\mathcal{F}$-sub-module of X, then f(A) is an $\mathcal{F}$-sub-module of Y.

Definition 1.15 [12]

An $\mathcal{F}$-subset K of a ring $\mathcal{R}$ is called $\mathcal{F}$-ideal of $\mathcal{R}$, if $\forall\ x, y \in \mathcal{R}$:

1- $K(x - y) \geq \min\{K(x), K(y)\}$.
2- $K(xy) \geq \max\{K(x), K(y)\}$.

Definition 1.16 [13]

Let X be an $\mathcal{F}$-module of an $\mathcal{R}$-module M, let A be an $\mathcal{F}$-sub-module of X and K be an $\mathcal{F}$-ideal of $\mathcal{R}$, the product KA of K and A is defined by:

$$\mathrm{KA}(x) = \begin{cases} \sup\{inf\{K(r_1), \ldots, K(r_n), \mathrm{A}(x_1), \ldots, \mathrm{A}(x_n)\}\} & for\ some\ \ r_i \in \mathcal{R}, x_i \in \mathrm{M}, n \in \mathrm{N} \\ 0 & otherwise \end{cases}$$

Note that K A is an $\mathcal{F}$-sub-module of X, and $(\mathrm{KA})_t = \mathrm{K}_t \mathrm{A}_t$, $\forall$ t ∈ [0, 1].

Definition 1.17 [9]
Let X be an $\mathcal{F}$-module of an $\mathcal{R}$-module M, An $\mathcal{F}$-sub-module U of X is called completely prime if whenever $r_b m_t \subseteq U$,with $r_b \neq 0_1$ is an $\mathcal{F}$-singleton of $\mathcal{R}$ and m_t is an $\mathcal{F}$-singleton of Ximplies that $m_t \subseteq U$ for each $t, b \in [0,1]$.

Definition 1.18 [6]
Let A and B be two $\mathcal{F}$-sub-modules of an R-module M. The addition A + B is defined by:

$$(\mathrm{A}+\mathrm{B})(\mathrm{x}) = \sup\{min\{\mathrm{A}(y), \mathrm{B}(z)\}\ with\ x = y + z, for\ all\ x, y, z \in \mathrm{M}\}.$$

Furthermore, A + B is an $\mathcal{F}$-sub-module of an $\mathcal{R}$-module M.

Corollary 1.19 [8]
If X is an $\mathcal{F}$-module of an $\mathcal{R}$-module M and $x_t \subseteq$ X, then for all $\mathcal{F}$-singleton r_k of $\mathcal{R}$, $r_k\, x_t = (rx)_\lambda$, where $\lambda = \min\{t, k\}$.

Proposition 1.20 [6]
Let A and B be two $\mathcal{F}$-sub-modules of an $\mathcal{F}$-module X of an $\mathcal{R}$-module M. Then the residual quotient of A and B (A : B) is an $\mathcal{F}$-ideal of $\mathcal{R}$.

Proposition 1.21 [14]
Let $f: M \longrightarrow \mathrm{N}$ be an $\mathcal{R}$-homomorphisim, then $f(Soc(M)) \subseteq Soc(\mathrm{N})$.

Definition 1.22 [15]
Let X be an $\mathcal{F}$-module of an R-module M, X is called $\mathcal{F}$-simple if and only if X has no proper $\mathcal{F}$-sub-modules (in fact X is $\mathcal{F}$-simple if and only if X has only itself and 0_1).

Definition 1.23 [16]
A $\mathcal{F}$-module X is called semi-simple if X is a summation of simple $\mathcal{F}$-sub-modules of X . Moreover, X is called semi-simple if $X = F - Soc(X)$.

Definition 1.24 [9]
Let X be an $\mathcal{F}$-module of an $\mathcal{R}$-module M, X is said to be faithful if $F - annX = 0_1$. Where $F - annX = \{r_t\ :\ r_t\, x_l = 0_1$; for all $x_l \subseteq$ X and r_t be an $\mathcal{F} -$ singleton of $\mathcal{R}\}$.

Definition 1.25 [17]
Let X be an $\mathcal{F}$-module of an $\mathcal{R}$-module M, X is said to be cancellative if whenever $r_t\, x_l = r_t\, y_d$ for all $x_l, y_d \subseteq$ X and r_t be an $\mathcal{F} -$ singleton of $\mathcal{R}$ then $x_l = y_d$.

Definition 1.26 [3]
A proper $\mathcal{F}$-sub-module U of an $\mathcal{F}$-module X of an $\mathcal{R}$-module M is called semi-prime $\mathcal{F}$-sub-module of X if whenever ${r_b}^n m_t \subseteq U$,where r_b is an $\mathcal{F}$-singleton of $\mathcal{R}$, m_t is an $\mathcal{F}$-singleton of X and n ∈ Z^+implies that $r_b m_t \subseteq U$ for each $t, b \in [0,1]$.

Definition 1.27 [4]
A proper sub-module E of an $\mathcal{R}$-module M is called pproximately semi prime (for a short app-semi-prime) sub-module of M if whenever $am \in E$, for $a \in \mathcal{R}$, $m \in \mathrm{M}$ $implies$ that $am \in E + Soc(\mathrm{M})$.

Definition 1.28 [9]
An $\mathcal{F}$-sub-module N of an $\mathcal{F}$-module X of an $\mathcal{R}$-module M is called weakly pure $\mathcal{F}$-sub-module of X if for any $\mathcal{F}$-singleton r_b of $\mathcal{R}$ implies that $r_bN = r_bX \cap N$ with b $\in$ [0,1].

Lemma 1.29 [18]
Let X be an $\mathcal{F}$-module of an $\mathcal{R}$-module M and let A , B and C are $\mathcal{F}$-sub-modules of X such that C $\subseteq$ B. Then $C + (\mathrm{B} \cap \mathrm{A}) = (C + \mathrm{A}) \cap \mathrm{B}$.

Proposition 1.30 [14]
If M be a faithful multiplication $\mathcal{R}$-module, then $Soc(\mathcal{R})\mathrm{M} = Soc(\mathrm{M})$

Definition 1.31 [15]
Let X be an $\mathcal{F}$-module of an $\mathcal{R}$- module M. X is called multiplication $\mathcal{F}$-module if and only if for each $\mathcal{F}$sub-module A of X ,there exists an $\mathcal{F}$-ideal K of $\mathcal{R}$ such that A = KX.

Proposition 1.32 [15]
AN $\mathcal{F}$-module X of an $\mathcal{R}$-module M is a multiplication if and only if every non-empty $\mathcal{F}$ -sub-module A of X such that $\mathrm{A} = (\mathrm{A}:_R X)X$.

Definition 1.33 [19]
A sub-module V of $\mathcal{R}$-module M is called essential if $H \cap \mathrm{V} \neq 0$. For non-trivial sub-module H of M .

Definition 1.34 [9]
Let X be an $\mathcal{F}$-module of an $\mathcal{R}$-module M. An $\mathcal{F}$-sub-module A of X is called essential if $A \cap B \neq 0_1$, for nontrivial $\mathcal{F}$-sub-module B of X.

Finally, (shortly fuzzy set, fuzzy sub-module, fuzzy ideal, fuzzy module and fuzzy singleton are $\mathcal{F}$-set, $\mathcal{F}$-sub-module, $\mathcal{F}$-ideal , $\mathcal{F}$-module and $\mathcal{F}$-singleton)."

$\mathcal{F}$-Soc-semi-prime sub-modules

In this section, we offer the concept of an $\mathcal{F}$-Soc-semi-prime sub-module as a generalization of ordinary concept(approximately semi-prime sub-module). Some characterizations of $\mathcal{F}$-Soc-prime sub-module are introduced.

Definition 2.1

Let r_b be an $\mathcal{F}$-singleton of $\mathcal{R}$ and m_t is an $\mathcal{F}$-singleton of X , then a proper $\mathcal{F}$-sub-module U of an $\mathcal{F}$-module X of an $\mathcal{R}$-module M is called an $\mathcal{F}$-Socle semi-prime (for short $\mathcal{F}$-Soc-semi-prime) sub-module(ideal) of X if whenever ${r_b}^n m_t \subseteq U$ with n $\in Z^+$ implies that $r_b m_t \subseteq U + \mathcal{F} - Soc(X)$ for each t, b $\in$ [0,1].

Furthermore, if $r_b$and s_h are $\mathcal{F}$-singletons of $\mathcal{R}$, then a proper $\mathcal{F}$-ideal L of $\mathcal{R}$ is called an $\mathcal{F}$-Socle semi-prime (for short $\mathcal{F}$-Soc-semi-prime) ideal of $\mathcal{R}$ if whenever ${r_b}^n s_h \subseteq L$ with n $\in Z^+$implies that $r_b s_h \subseteq L + \mathcal{F} - Soc(\mathcal{R})$ for each h, b $\in$ [0,1].

We will adopt the definition of an $\mathcal{F}$-socle of X in this research as follows:

such that:$\mathcal{F}-Soc(X): M \to [0,1]$

$$\mathcal{F}-Soc(X)(m)=\begin{cases}1 & if\ \ m\in Soc(M)\\ h & if\ \ m\notin Soc(M)\end{cases}\quad with\ 0<h<1$$

Lemma 2.2

for any $\mathcal{F}$-module X for each t ∈ (0,1] with $(\mathcal{F}-Soc(X))_t$ ≠$(\mathcal{F}-Soc(X))_t = Soc(X_t)$ X_t

Proof:

such that:$\mathcal{F}-Soc(X): M \to [0,1]$

$$\mathcal{F}-Soc(X)(m)=\begin{cases}1 & if\ \ m\in Soc(M)\\ h & if\ \ m\notin Soc(M)\end{cases}\quad with\ 0<h<1$$

Now, $(\mathcal{F}-Soc(X))_t=\{m\in M:\ (\mathcal{F}-Soc(X))(m)\geq t\}$

So, if $t=1$ then $(\mathcal{F}-Soc(X))_t = Soc(M) = Soc(X_t)$

If $0<t\leq h$ then $(\mathcal{F}-Soc(X))_t = M = X_t$ that is a contradiction

If $h<t<1$ then $(\mathcal{F}-Soc(X))_t = Soc(M) = Soc(X_t)$

Lemma 2.3

Let X be an $\mathcal{F}$-module of an $\mathcal{R}$-module M with X(m)=1 for each $m\in M$, if U is an $\mathcal{F}$-sub-module of X is defined by $U: M\to[0,1]$ such that:

$$U(m)=\begin{cases}1 & if\ \ m\in E\\ k & if\ \ m\notin E\end{cases}\quad with\ 0<k<1$$

Where E is a sub-module of M. Then U is an $\mathcal{F}$-Soc-semi-prime sub-module of X if and only if E is an app-semi-prime sub-module of M.

Proof:

First of all, we must define $U+\mathcal{F}-Soc(X)$.

$$(U+\mathcal{F}-Soc(X))(m)=\sup\{\min\big(U(y),\ \mathcal{F}-Soc(X)(z)\big), y+z=m\}$$

So, we have

$$\big(U+\mathcal{F}-Soc(X)\big)(m)=\begin{cases}1 & if\ \ m\in E+Soc(M)\\ s & if\ \ m\notin E+Soc(M)\end{cases}\quad with\ \ s=\max\{k,h\}$$

Where $\mathcal{F}-Soc(X): M\to[0,1]$ such that:

$$\mathcal{F}-Soc(X)(m)=\begin{cases}1 & if\ \ m\in Soc(M)\\ h & if\ \ m\notin Soc(M)\end{cases}\quad with\ 0<h<1$$

Now,

Suppose E is an app-semi-prime sub-module of M, to prove that U is an $\mathcal{F}$-Soc-semi-prime sub-module of X. Let $r_b\subseteq\mathcal{R}$ and $m_t\subseteq X$ for each $t, b\in[0,1]$ such that $(r_b)^n m_t\subseteq U$,thus $(r^n)_b m_t\subseteq U$ that is either $r^n m\in E$ or $r^n m\notin E$.

1) If $r^n m \in E$, then $rm \in E + Soc(M)$. Hence $\big(U + \mathcal{F} - Soc(X)\big)(rm) = 1$ this implies $r_b m_t = (rm)_t \subseteq (rm)_1 \subseteq U + \mathcal{F} - Soc(X)$.

2) If $r^n m \notin E$ then $U(r^n m) = k$ with $m \notin E$ thus $U(m) = k$. Since $(r_b)^n m_t \subseteq U$ then $(r^n m)_\lambda \subseteq U$ where $\lambda = \min\{b, t\}$, that is $U(r^n m) \geq \lambda$ thus $k \geq \lambda$. Now, if $\lambda = t$ this implies $m_t \subseteq m_k \subseteq U \subseteq U + \mathcal{F} - Soc(X)$. That is mean $r_b m_t \subseteq r_b m_k \subseteq U \subseteq U + \mathcal{F} - Soc(X)$ If $\lambda = b$, $U(h) \geq k$ for any $h \in M$, and:

$$({r^n}_b\, X_M)(h) = \begin{cases} b & if\ \ h = r^n a \\ 0 & otherwise \end{cases} \quad for\ some\ \ a \in M$$

Then we get $({r^n}_b\, X_M)(h) \leq U(h)$, hence ${r^n}_b X_M \subseteq U \subseteq U + \mathcal{F} - Soc(X)$

So, each case implies that $r_b m_t \subseteq U + \mathcal{F} - Soc(X)$

Therefore U is an $\mathcal{F}$-Soc-semi-prime sub-module of X.

Conversely

Suppose U is an $\mathcal{F}$-Soc-semi-prime of X. Let $a^n x \in U_t$, with $a \in \mathcal{R}$, n $\in Z^+$ and $x \in X_t$ it follows that $(a^n x)_t \subseteq U$, that is $(a^n)_t x_t = (a_t)^n x_t \subseteq U$. But U is an $\mathcal{F}$-Soc-semi-prime of X, then we get $a_t x_t = (ax)_t \subseteq U + \mathcal{F} - Soc(X)$. Thus we get $(U + \mathcal{F} - Soc(X))(ax) \geq t$, hence, by (Lemma 1.12) and (Lemma 2.2), we have $ax \in \big(U + \mathcal{F} - Soc(X)\big)_t = U_t + \big(\mathcal{F} - Soc(X)\big)_t = U_t + Soc(X_t)$. That is mean U_t is an app-semi-prime sub-module of X_t .

Hence $U_1 = E$ is an app-semi-prime sub-module of M.

The following example shows that the definition of an $\mathcal{F}$-socle of X that we adopt in this research is necessary to prove one side of above lemma.

Example 2.4

Let $M = Z_{12}$ as a Z-module and $X: M \to [0,1]$, $U: M \to [0,1]$ defined by:

$$X(m) = 1 \qquad if\ \ m \in Z_{12}$$

$$U(m) = \begin{cases} 1 & if\ \ m \in \langle \bar{0} \rangle \\ 1/4 & otherwise \end{cases}$$

And an $\mathcal{F}$-socle of X is defined by $\mathcal{F} - Soc(X): M \to [0,1]$ such that:

$$\mathcal{F} - Soc(X)(m) = \begin{cases} 1 & if\ x = \bar{0} \\ 2/3 & if\ \ m \in \langle \bar{2} \rangle - \{\bar{0}\} \\ 1/3 & otherwise \end{cases}$$

Where $Soc(M) = \langle \bar{2} \rangle$. That's clear X is an $\mathcal{F}$-module and U be an $\mathcal{F}$-sub-module of X.

We have U_t is an app-semi-prime sub-module of M for every $t > 0$.

Now,

$$\big(U+\mathcal{F}-Soc(X)\big)(m)=\begin{cases}1 & if\ x=\bar{0}\\ 2/3 & if\ \ m\in\langle\bar{2}\rangle-\{\bar{0}\}\\ 1/3 & otherwise\end{cases}$$

But, U is not an $\mathcal{F}$-Soc-semi-prime sub-module of X, since for an $\mathcal{F}$-singleton $\bar{3}_{\frac{3}{4}}\subseteq X$ and an $\mathcal{F}$-singleton $2_{\frac{3}{4}}$ of $\mathcal{R}$ such that $(2^2)_{\frac{3}{4}}\bar{3}_{\frac{3}{4}}=\bar{0}_{\frac{3}{4}}$, where $\bar{0}_{\frac{3}{4}}\subseteq U$ since $U(\bar{0})=1>\frac{3}{4}$. but $2_{\frac{3}{4}}\bar{3}_{\frac{3}{4}}=\bar{6}_{\frac{3}{4}}\nsubseteq U+\mathcal{F}-Soc(X)$ since $(U+\mathcal{F}-Soc(X))(\bar{6})=\frac{2}{3}\ngeq\frac{3}{4}$.

Hence, U is not an $\mathcal{F}$-Soc-semi-prime of sub-module of X.

Proposition 2.5

Let U and V are $\mathcal{F}$-sub-modules of an $\mathcal{F}$-module X of an $\mathcal{R}$-module M with V is an $\mathcal{F}$- semi-prime sub-module of X. Then $[U:_{\mathcal{R}} V]$ is an $\mathcal{F}$-Soc-semi-prime ideal of $\mathcal{R}$.

Proof :

Suppose that $r_b{}^n m_t\subseteq[U:_{\mathcal{R}} V]$,for $r_b\subseteq\mathcal{R}, m_t\subseteq X$,thus $r_b{}^n m_t V\subseteq U$.So we have $r_b{}^n(m_tV)\subseteq U$, but V is an $\mathcal{F}$-semi-prime sub-module of X. that is $r_b(m_tV)\subseteq U$, hence $r_b m_t V\subseteq U$ that is mean $r_b m_t\subseteq[U:_{\mathcal{R}} V]\subseteq[U:_{\mathcal{R}} V]+\mathcal{F}-Soc(\mathcal{R})$.

Proposition 2.6

Let U and V are $\mathcal{F}$-Soc-semi-prime sub-modules of an $\mathcal{F}$-module X of an $\mathcal{R}$-module M with $\mathcal{F}-Soc(X)\subseteq U$, Then U ∩V is an $\mathcal{F}$-Soc-semi-prime sub-module of X.

Proof :

Let $r_b{}^n m_t\subseteq$ U ∩V ,for $r_b\subseteq\mathcal{R}, m_t\subseteq X$, that is $r_b{}^n m_t\subseteq$ U and $r_b{}^n m_t\subseteq$ V. But U and V are $\mathcal{F}$-Soc-semi-prime sub-modules of X, this implies $r_b m_t\subseteq U+\mathcal{F}-Soc(X)$ and $r_b m_t\subseteq V+\mathcal{F}-Soc(X)$. That is mean $r_b m_t\subseteq(U+\mathcal{F}-Soc(X))\cap(V+\mathcal{F}-Soc(X))$, by using modular law we get $r_b m_t\subseteq(\mathrm{U}\cap\mathrm{V})+\mathcal{F}-Soc(X)$. Hence U ∩V is an $\mathcal{F}$-Soc-semi-prime sub-module of X.

Remark 2.7

Every $\mathcal{F}$-semi-prime sub-module is an $\mathcal{F}$-Soc-semi-prime sub-module , but the converse is not true .

Proof :

Suppose U be an $\mathcal{F}$-semi-prime sub-module of an $\mathcal{F}$-module X of an $\mathcal{R}$-module M and $r_b{}^n m_t\subseteq U$,for $r_b\subseteq R, m_t\subseteq X$. Since U is an $\mathcal{F}$-semi-prime sub-module, then we get $r_b m_t\subseteq U\subseteq U+\mathcal{F}-Soc(X)$, thus $r_b m_t\subseteq U+\mathcal{F}-Soc(X)$. Therefore U is an $\mathcal{F}$-Soc-prime sub-module.

The following example show that the converse is not true

Example 2.8

Consider $M=Z_{12}$ as a Z-module and $X: M\to[0,1]$, $U: M\to[0,1]$ defined by:

$$X(m)=1\qquad if\ \ m\in Z_{12}$$

$$U(m)=\begin{cases}1 & if\ \ m\in\langle\bar{0}\rangle\\ 1/5 & if\ \ m\notin\langle\bar{0}\rangle\end{cases}$$

And an $\mathcal{F}$-socle of X is defined by $\mathcal{F}-Soc(X): M\to[0,1]$ such that:

$$\mathcal{F}-Soc(X)(m)=\begin{cases}1 & if\ \ m\in\langle\bar{2}\rangle\\ 1/3 & if\ \ m\notin\langle\bar{2}\rangle\end{cases}$$

Where $Soc(M)=\langle\bar{2}\rangle$. That's clear X is an $\mathcal{F}$-module and U be an $\mathcal{F}$-sub-module of X.

From ([4] Remark 2.3.2) $\langle\bar{0}\rangle$ is an app-semi-prime sub-module of M, so by (Lemma 2.3) we get U is an $\mathcal{F}$-Soc-semi-prime sub-module of X.

But, U is not an $\mathcal{F}$-semi-prime sub-module of X, since for an $\mathcal{F}$-singleton $3_{\frac{1}{3}}\subseteq X$ and an $\mathcal{F}$-singleton $2_{\frac{1}{3}}$ of $\mathcal{R}$ such that $(2_{\frac{1}{3}})^2\,3_{\frac{1}{3}}=0_{\frac{1}{3}}$ where $0_{\frac{1}{3}}\subseteq U$ since $U(0)=1>\frac{1}{3}$. but $2_{\frac{1}{3}}3_{\frac{1}{3}}=6_{\frac{1}{3}}\nsubseteq U$ since $U(6)=\frac{1}{5}\ngtr\frac{1}{3}$.

Hence, U is not an $\mathcal{F}$-semi-prime sub-module of X.

Remark 2.9

Every completely $\mathcal{F}$-sub-module of an $\mathcal{F}$-module X of an $\mathcal{R}$-module M is an $\mathcal{F}$-Soc-semi-prime sub-module of X, but the converse is not true .

Proof :

We take U as a completely $\mathcal{F}$-sub-module of X with $r_b{}^n m_t\subseteq U$,for $r_b\subseteq R, m_t\subseteq X$.Now, if $r_b=0_1$ then $r_b\,m_t=0_t\subseteq 0_1\subseteq U$.we get U is an $\mathcal{F}$-Soc- semi-prime sub-module of X. If $r_b\neq 0_1$,thus $(r_b)^{n-1}\,(r_b\ m_t)\subseteq U$, we get $(r^{n-1})_b\ \ (rm)_d\subseteq U$ where $d=\min\{b,t\}$.Now, since U is a completely $\mathcal{F}$-sub-module of X, then we have $(rm)_d\subseteq U\subseteq U+\mathcal{F}-Soc(X)$, thus $r_b m_t\subseteq U+\mathcal{F}-Soc(X)$. Therefore U is an $\mathcal{F}$-Soc-semi-prime sub-module.

The following example show that the converse is not true

Example 2.10

Consider $M=Z$ as a Z-module and $X: M\to[0,1]$, $U: M\to[0,1]$ defined by:

$$X(m)=1\qquad if\ \ m\in Z$$

$$U(m)=\begin{cases}1 & if\ \ m\in 2Z\\ 1/4 & if\ \ m\notin 2Z\end{cases}$$

And an $\mathcal{F}$-socle of X is defined by $\mathcal{F}-Soc(X): M\to[0,1]$ such that:

$$\mathcal{F}-Soc(X)(m)=\begin{cases}1 & if\ \ m\in\{0\}\\ 1/3 & if\ \ m\notin\{0\}\end{cases}$$

$$\big(U+\mathcal{F}-Soc(X)\big)(m)=\begin{cases}1 & if\ \ m\in 2Z\\ 1/3 & if\ \ m\notin 2Z\end{cases}$$

Where $Soc(M)=\{0\}$. That's clear X is an $\mathcal{F}$-module and U be an $\mathcal{F}$-sub-module of X.

is an app-semi-prime sub-module of M, so by (Lemma 2.3) we get U is an $\mathcal{F}$-Soc-semi-2Z prime sub-module of X.

But U is not completely $\mathcal{F}$-sub-module of X, since for an $\mathcal{F}$-singleton $5_{\frac{1}{3}} \subseteq X$ and an $\mathcal{F}$-singleton $2_{\frac{1}{2}}$ of $\mathcal{R}$ such that $2_{\frac{1}{2}}\ 5_{\frac{1}{3}} = 10_{\frac{1}{3}}$ where $10_{\frac{1}{3}} \subseteq U$ since $U(10) = 1 > \frac{1}{3}$. but $5_{\frac{1}{3}} \nsubseteq U$ since $U(5) = \frac{1}{4} \ngtr \frac{1}{3}$.

Hence, U is not completely $\mathcal{F}$-sub-module of X.

Proposition 2.11

Let U be an $\mathcal{F}$-Soc-semi-prime sub-module of an $\mathcal{F}$-module X of an $\mathcal{R}$-module M, Then U is an $\mathcal{F}$-Soc-semi-prime sub-module of X if and only if $\forall$ $\mathcal{F}$-sub-module S of X and an $\mathcal{F}$-ideal J of $\mathcal{R}$ with $(J)^n$S ⊆U for $n \in Z^+$implies that JS ⊆ $U + \mathcal{F} - Soc(X)$

Proof:

(⇒) Assume that $(J)^n$S ⊆U , for S is an $\mathcal{F}$-sub-module of X and J is an $\mathcal{F}$-ideal of $\mathcal{R}$, let $x_t \subseteq JS$ with $t \in [0,1]$ then $x_t = (c_1)_{h1}(y_1)_{t1} + (c_2)_{h2}(y_2)_{t2} + \cdots + (c_n)_{hn}(y_n)_{tn}$, for every $(c_i)_{hi} \subseteq J$ and $(y_i)_{ti} \subseteq U$ where $hi , ti \in [0,1]$ for every i=1,2,…..,n. Now, we get $((c_i)_{hi})^n(y_i)_{ti} \subseteq (J)^n$S ⊆U hence $((c_i)_{hi})^n(y_i)_{ti} \subseteq$ U. But U is an $\mathcal{F}$-Soc-semi-prime sub-module of X implies that $(c_i)_{hi}(y_i)_{ti} \subseteq$ U+$\mathcal{F} - Soc(X)$ for each i=1,2,…..,n. So we have $x_t \subseteq$ U+$\mathcal{F} - Soc(X)$. it follows that JS ⊆ $U + \mathcal{F} - Soc(X)$.

(⇐) Let $(r_b)^n x_t \subseteq U$ for $r_b \subseteq \mathcal{R}$ and $n \in Z^+$ then $\langle {r_b}^n \rangle \langle x_t \rangle \subseteq U$, that is $\langle r_b \rangle^n \langle x_t \rangle \subseteq U$ then by hypothesis we get $\langle r_b \rangle \langle x_t \rangle \subseteq U$, hence $r_b x_t \subseteq U$. That is mean U is an $\mathcal{F}$-Soc-semi-prime sub-module of X.

Corollary 2.12

Let U be an $\mathcal{F}$-sub-module of an $\mathcal{F}$-module X of an $\mathcal{R}$-module M, Then U is an $\mathcal{F}$-Soc-semi-prime sub-module of X if and only if $\forall$ $\mathcal{F}$-sub-module S of X and every $\mathcal{F}$-singleton r_b of $\mathcal{R}$ with $(r_b)^n$ S ⊆U implies that r_bS ⊆ $U + \mathcal{F} - Soc(X)$.

Proof:

It is clear from (proposition 2.11).

Corollary 2.13

Let L be an $\mathcal{F}$- ideal of $\mathcal{R}$, Then L is an $\mathcal{F}$-Soc-semi-prime ideal of $\mathcal{R}$ if and only if $\forall$ $\mathcal{F}$-sub-ideal J of $\mathcal{R}$ and every $\mathcal{F}$-singleton r_b of $\mathcal{R}$ with $(r_b)^n$ J ⊆L implies that r_bJ ⊆ $L + \mathcal{F} - Soc(\mathcal{R})$.

Proof :

Clearly from (proposition 2.11).

Proposition 2.14 :

If ${r_b}^n\ U$ $\mathcal{F}$-Soc-semi-prime sub-module of cancellative $\mathcal{F}$-module X. Where U is an $\mathcal{F}$-sub-modules of X and r_b is an idempotent $\mathcal{F}$-singleton of R. Then $U \subseteq {r_b}^{n-1} U + F - Soc(X)$

Proof :

Let $a_t \subseteq U$ this implies ${r_b}^n a_t \subseteq {r_b}^n U$, for r_b is an $\mathcal{F}$-singleton of R. But, ${r_b}^n U$ is an $\mathcal{F}$-Soc-semi-prime sub-module of X with $a_t \subseteq X$,where $t, b \in [0,1]$. Therefore $r_b a_t \subseteq {r_b}^n U + F - Soc(X)$, that is ${r_b}^2\ a_t \subseteq {r_b}^{n+1} U + r_b\ F - Soc(X)$, thus ${r_b}^2 a_t \subseteq {r_b}^2 {r_b}^{n-1} U + r_b\ F - Soc(X)$) , but r_b is an idempotent $\mathcal{F}$-singleton of R. So we get $r_b a_t \subseteq {r_b}^n U + r_b F - Soc(X)$. But, X is a cancellative $\mathcal{F}$-module, we have $a_t \subseteq {r_b}^{n-1} U + F - Soc(X)$,that is mean $U \subseteq {r_b}^{n-1} U + F - Soc(X)$.

Remark 2.15

Every $\mathcal{F}$-semi-prime sub-module is an $\mathcal{F}$-Soc-semi-prime sub-module.

Proof:

It is Clear by definition of $\mathcal{F}$-semi-prime sub-module.

Remark 2.16

If U is an $\mathcal{F}$-Soc-semi-prime sub-module of $\mathcal{F}$-module X, with $\mathcal{F} - Soc(X) \subseteq U$. Then U is an $\mathcal{F}$-semi-prime sub-module.

Proof:

Assume that U is an $\mathcal{F}$-Soc-semi-prime sub-module of an $\mathcal{F}$-module X of an $\mathcal{R}$-module M. Let $(r^n)_b m_t = (r_b)^n m_t \subseteq U$, for $r_b \subseteq \mathcal{R}, m_t \subseteq X$,where $t, b \in [0,1]$. Since U is an $\mathcal{F}$-Soc-semi-prime sub-module, then $r_b m_t \subseteq U + \mathcal{F} - Soc(X) \subseteq U$ but $\mathcal{F} - Soc(X) \subseteq U$. Hence U is an $\mathcal{F}$-semi-prime sub-module.

Corollary 2.17

If U is an $\mathcal{F}$-Soc-semi-prime sub-module of $\mathcal{F}$-module X, with U be an $\mathcal{F}$-essential sub-module of X. Then U is an $\mathcal{F}$-semi-prime sub-module.

Proof:

Since U be an $\mathcal{F}$-essential sub-module of X, then by definition of $\mathcal{F}$-socle we have $\mathcal{F} - Soc(X) \subseteq U$ and by (Remark 2.16) that is complete the proof.

Corollary 2.18

If U is an $\mathcal{F}$-sub-module of $\mathcal{F}$-module X, with $\mathcal{F} - Soc(X) \subseteq U$. Then U is an $\mathcal{F}$-semi-prime sub-module of X if and only if U is an $\mathcal{F}$-Soc-prime sub-module of X.

Proof:

Consequently from (Remark 2.7) and (Remark 2.16).

Remark 2.19

Let U and V are $\mathcal{F}$-sub-modules of $\mathcal{F}$-module X. If U+V is an $\mathcal{F}$-semi-prime sub-module of X with $V \subseteq \mathcal{F} - Soc(X)$, then U is an $\mathcal{F}$-Soc-semi-prime sub-module of X.

Proof:

Let $(r^n)_b x_k = (r_b)^n x_k \subseteq U$, for $r_b \subseteq \mathcal{R}$, $x_k \subseteq X$,where $k, b \in [0,1]$. this implies $(r^n)_b x_k \subseteq U + V$. But U+V is an $\mathcal{F}$-semi-prime sub-module of X, hence $r_b x_k \subseteq U + V \subseteq U + \mathcal{F} - Soc(X)$ since $V \subseteq \mathcal{F} - Soc(X)$. That is U is an $\mathcal{F}$-Soc-semi-prime sub-module of X.

Theorem 2.20

Any $\mathcal{F}$-sub-module of semi-simple $\mathcal{F}$-module X is an $\mathcal{F}$-Soc-semi-prime sub-module of X.

Proof:

If U is an $\mathcal{F}$-sub-module of an $\mathcal{F}$-module X of an $\mathcal{R}$-module M. Let $(r^n)_b x_k = (r_b)^n x_k \subseteq U$, for $r_b \subseteq \mathcal{R}$, $x_k \subseteq X$,where $k, b \in [0,1]$. But, X is a semi-simple $\mathcal{F}$-module, thus $X = \mathcal{F} - Soc(X)$.We have $x_k \subseteq X = \mathcal{F} - Soc(X) \subseteq U + \mathcal{F} - Soc(X)$, this implies $r_b x_k \subseteq r_b X = r_b \mathcal{F} - Soc(X) \subseteq r_b\big(U + \mathcal{F} - Soc(X)\big) \subseteq U + \mathcal{F} - Soc(X)$ that is mean U is an $\mathcal{F}$-Soc-semi-prime sub-module of X.

Proposition 2.21 :

If U is a weakly pure $\mathcal{F}$-sub-module of $\mathcal{F}$-module X with $(r^n)_b$U is an $\mathcal{F}$-Soc-semi-prime sub-module of X for every non-empty $\mathcal{F}$-singleton r_b of R, then U is an $\mathcal{F}$-Soc-semi-prime sub-module of X.

Proof:

Suppose that $(r^n)_b x_t \subseteq U$,with r_b is an $\mathcal{F}$-singleton of R and $x_t \subseteq X$,where $t, b \in [0,1]$.Also $(r^n)_b x_t \subseteq (r^n)_b X$ this implies $(r^n)_b x_t \subseteq U \cap (r^n)_b X = (r^n)_b U$ since U is a weakly pure $\mathcal{F}$-sub-module of X, but $(r^n)_b U$ is an $\mathcal{F}$-Soc-semi-prime sub-module of X, hence $r_b x_t \subseteq (r^n)_b U + F - Soc(X) \subseteq U + F - Soc(X)$. Thus U is an $\mathcal{F}$-Soc-semi-prime sub-module of X.

Lemma 2.22 :

for every fuzzy sub-$(A \oplus B) + F - Soc(X \oplus Y) = \big(A + F - Soc(X)\big) \oplus \big(B + F - Soc(Y)\big)$ modules A and B of fuzzy modules X and Y respectively.

Proof:

From (Lemma 2.2) we get $(F - Soc(X \oplus Y))_t = Soc((X \oplus Y)_t)$

For each $t \in (0,1]$. But, $Soc((X \oplus Y)_t) = Soc(X_t \oplus Y_t)$ and we have $Soc(X_t \oplus Y_t) = Soc(X_t) \oplus Soc(Y_t)$

That is $(F - Soc(X \oplus Y))_t = Soc(X_t) \oplus Soc(Y_t) = (F - Soc(X))_t \oplus (F - Soc(Y))_t$

Thus $(F - Soc(X \oplus Y))_t = [(F - Soc(X)) \oplus (F - Soc(Y)]_t$

Hence from (Remark 1.5) then $F - Soc(X \oplus Y) = F - Soc(X) \oplus F - Soc(Y)$

Proposition 2.23 :

If U and V are $\mathcal{F}$-sub-modules of $\mathcal{F}$-modules X and Y respectively, then

1) If $U\oplus Y$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $X\oplus Y$ thus U is an $\mathcal{F}$-Soc-semi-prime sub-module of X.

2) if $X\oplus V$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $X\oplus Y$ thus V is an $\mathcal{F}$-Soc-semi-prime sub-module of X.

Proof :

1) Suppose that $U\oplus Y$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $X\oplus Y$ and r_b is an $\mathcal{F}$-singleton of R and $x_t \subseteq X$ such that $(r^n)_b x_t \subseteq U$. Then $(r^n)_b(x_t, y_p) = ((r^n)_b x_t, (r^n)_b y_p) \subseteq U\oplus Y$, for any $\mathcal{F}$-singleton $y_p \subseteq Y$, but $U\oplus Y$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $X\oplus Y$. Thus $(r_b x_t, r_b y_p) \subseteq (U\oplus Y) + F - Soc(X\oplus Y)$, by (Lemma 2.22) we get $(r_b x_t, r_b y_p) \subseteq \big(U + F - Soc(X)\big)\oplus\big(Y + F - Soc(Y)\big)$. That is $r_b x_t \subseteq U + F - Soc(X)$, therefore U is an $\mathcal{F}$-Soc-semi-prime sub-module of X.

2) Similarly as the idea in (1).

Lemma 2.24 :

If X is an $\mathcal{F}$-module of an $\mathcal{R}$-module M, and M be a faithful multiplication $\mathcal{R}$-module, then:

$.\mathcal{F} - Soc(X) = X\,\mathcal{F} - Soc(\mathcal{R})$

Proposition 2.25 :

Let X be a finitely generated multiplication and faithful $\mathcal{F}$-module of an $\mathcal{R}$-module M, if J is an $\mathcal{F}$-Soc-semi-prime ideal of $\mathcal{R}$ then JX is an $\mathcal{F}$-Soc-semi-prime sub-module of X.

Proof :

Assume that r_b is an $\mathcal{F}$-singleton of $\mathcal{R}$ and $x_t \subseteq X$ such that $(r^n)_b x_k = (r_b)^n x_k \subseteq$ JX,where $k, b \in [0,1]$.that is $(r^n)_b\langle x_t\rangle \subseteq$ JX. But X is a multiplication $\mathcal{F}$-module, thus there exists an $\mathcal{F}$-ideal L of $\mathcal{R}$ with $\langle x_t\rangle = LX$. Then we get $(r^n)_b LX \subseteq$ JX, so $(r^n)_b L \subseteq \mathrm{J} + \mathcal{F} -$ ann(X) = J since X is a faithful $\mathcal{F}$- module. But J is an $\mathcal{F}$-Soc-semi-prime ideal of $\mathcal{R}$, then by (Corollary 2.13) implies that $r_b L \subseteq J + \mathcal{F} - Soc(\mathcal{R})$.Now, by multiplying both sides with X and using (Lemma 2.24) we have $r_b LX \subseteq JX + \mathcal{F} - Soc(\mathcal{R})X = JX + \mathcal{F} - Soc(X)$. Therefore, JX is an $\mathcal{F}$-Soc-semi-prime sub-module of X.

Proposition 2.26

Suppose that U is an $\mathcal{F}$-Soc-semi-prime sub-module of an $\mathcal{F}$-module X and V is an $\mathcal{F}$-semi-prime sub-module of X with $\mathcal{F} - Soc(X) \subseteq V$. Then the intersection of U and V is an $\mathcal{F}$-Soc-semi-prime of X.

Proof :

If r_b is an $\mathcal{F}$-singleton of $\mathcal{R}$ and $x_t \subseteq X$ where $b, t \in [0,1]$, such that $(r^n)_b x_k = (r_b)^n x_k \subseteq U \cap V$. This implies $(r^n)_b x_t \subseteq U$ and $(r^n)_b x_t \subseteq V$, but U is an $\mathcal{F}$-Soc-semi-prime sub-module of X. So, we have $r_b x_k \subseteq U + \mathcal{F} - Soc(X)$.Now, since V is an $\mathcal{F}$-semi-prime sub-module of X then $r_b x_k \subseteq V$.We get $r_b x_k \subseteq [U + \mathcal{F} - Soc(X)] \cap V$, but $\mathcal{F} - Soc(X) \subseteq V$

then by using (Lemma 1.29) we have $r_b x_k \subseteq (U \cap V) + \mathcal{F} - Soc(X)$. That is mean $U \cap V$ is an $\mathcal{F}$-Soc-semi-prime of X.

Proposition 2.27

Let X be a faithful multiplication $\mathcal{F}$-module of an $\mathcal{R}$-module M, then a proper $\mathcal{F}$-sub-module U is an $\mathcal{F}$-Soc-semi-prime sub-module of if and only if $[U:_R X]$ is an $\mathcal{F}$-Soc-semi-prime ideal of $\mathcal{R}$.

Proof:

Let $(r^n)_b m_t = (r_b)^n m_t \subseteq [U:_R X]$ with m_t and r_b are $\mathcal{F}$-singletons of $\mathcal{R}$ where $b, t \in [0,1]$implies that $(r^n)_b (m_t X) \subseteq U$. But, U is an $\mathcal{F}$-Soc-semi-prime sub-module, so by (Corollary 2.13) then $r_b(m_t X) \subseteq U + \mathcal{F} - Soc(X)$. Since X is a multiplication $\mathcal{F}$-module, then by (Preposition 1.32) $U = [U:_R X]X$, and since X is a faithful multiplication, so by (Lemma 2.24) $F - Soc(X) = \mathcal{F} - Soc(\mathcal{R})X$. Therefore $r_b m_t X \subseteq [U:_R X]X + \mathcal{F} - Soc(\mathcal{R})X$, this implies $r_b m_t \subseteq [U:_R X] + \mathcal{F} - Soc(\mathcal{R})$. Thus $[U:_R X]$ is an $\mathcal{F}$-Soc-semi-prime ideal of $\mathcal{R}$.

Conversely

Let $(r^n)_b D = (r_b)^n D \subseteq U$ with r_b be an $\mathcal{F}$-singleton of $\mathcal{R}$ and D is an $\mathcal{F}$-sub-module of X. Since X is a multiplication $\mathcal{F}$-module, then $D = JX$ for some an $\mathcal{F}$-ideal of $\mathcal{R}$, we get $(r^n)_b JX \subseteq U$ that is mean $(r^n)_b J \subseteq [U:_R X]$,but $[U:_R X]$ is an $\mathcal{F}$-Soc-semi-prime ideal of $\mathcal{R}$, so by (Corollary 2.12) we have $r_b J \subseteq [U:_R X] + F - Soc(\mathcal{R})$, this implies $r_b JX \subseteq [U:_R X]X + \mathcal{F} - Soc(\mathcal{R})X$, then by (Lemma 2.25) we get $r_b JX \subseteq U + \mathcal{F} - Soc(X)$.

Lemma 2.28

Let $f\colon \mathrm{M} \to \bar{M}$ be isomorphism mapping from an $\mathcal{R}$-module M into an $\mathcal{R}$-module $\bar{M}$ .If X and $\bar{X}$ are $\mathcal{F}$-modules of an $\mathcal{R}$-modules M and $\bar{M}$ respectively. Then f$(\mathcal{F} - Soc(X)) \subseteq \mathcal{F} - Soc(\bar{X})$.

Proposition 2.29

Let $f\colon X \to \bar{X}$ be an $\mathcal{F}$-isomorphism from $\mathcal{F}$-module X into $\mathcal{F}$-module $\bar{X}$, with U is an $\mathcal{F}$-Soc-semi-prime sub-module of X, such that $\ker(f) \subseteq U$. Then $f(U)$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $\bar{X}$.

Proof :

is a proper $\mathcal{F}$-sub-module of $\bar{X}$. If not, then $f(U) = \bar{X}$. Let $x_t \subseteq X$, so $f(x_t) \subseteq \bar{X} = f(U)$ $f(U)$, that is there exists $y_s \subseteq U$ where $s, t \in [0,1]$ such that $f(x_t) = f(y_s)$implies that $f(x_t) - f(y_s) = 0_1$ then $f(x_t - y_s) = 0_1$, thus $x_t - y_s \subseteq \ker(f) \subseteq U$, it follows that $x_t \subseteq U$.Thus $U = X$ that is a contradiction. Now, Let $(r_b)^n z_c \subseteq f(U)$ with $r_b \subseteq \mathcal{R}$ and $z_c \subseteq \bar{X}$ with $b, c \in [0,1]$,but f is onto $f(x_t) = z_c$ for some $x_t \subseteq X$, therefore $(r^n)_b z_c = (r^n)_b f(x_t) = f((r^n)_b x_t) \subseteq f(U)$, this implies that there exists $k_h \subseteq U$ with $h \in [0,1]$ such that $f(k_h) = f((r^n)_b x_t)$, that is $f(k_h - (r^n)_b x_t) = 0_1$, so $k_h - (r^n)_b x_t \subseteq \ker(f) \subseteq U$. It follows that $(r_b)^n x_t \subseteq U$. But, U is an $\mathcal{F}$-Soc-semi-prime sub-module of X, thus $r_b x_t \subseteq U + \mathcal{F} - Soc(X)$. Then by (Lemma 2.28) we have $r_b z_c = r_b f(x_t) \subseteq f(U) + f(\mathcal{F} - Soc(X)) \subseteq f(U) + \mathcal{F} - Soc(\bar{X})$. Hence $f(U)$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $\bar{X}$.

2.Conclusion

Through this research, we were able to know some of the fuzzy algebraic properties of fuzzy socle semi-prime sub-modules and the relationship with other concepts .
The idea of fuzzy socle semi-prime sub-modules is dualized in this study by introducing several characteristics and properties of semi-prime fuzzy sub-modules. This approach has opened up new possibilities for studying the fuzzy dimension. Thus, socle semi-prime module and completely socle semi-prime sub-modules can be defined utilizing the concept of fuzzy socle semi-prime sub-modules.

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