Some Properties for the Restriction of $\mathcal{P}^*$–field of Sets

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Abstract

The restriction concept is a basic feature in the field of measure theory and has many important properties. This article introduces the notion of restriction of a non-empty class of subset of the power set on a nonempty subset of a universal set. Characterization and examples of the proposed concept are given, and several properties of restriction are investigated. Furthermore, the relation between the $\mathcal{P}^*$–field and the restriction of the $\mathcal{P}^*$–field is studied, explaining that the restriction of the $\mathcal{P}^*$–field is a $\mathcal{P}^*$–field too. In addition, it has been shown that the restriction of the $\mathcal{P}^*$–field is not necessarily contained in the $\mathcal{P}^*$–field, and the converse is true. We provide a necessary condition for the $\mathcal{P}^*$–field to obtain that the restriction of the $\mathcal{P}^*$–field is included in the $\mathcal{P}^*$–field. Finally, this article aims to study the restriction notion and give some propositions, lemmas, and theorems related to the proposed concept.

Keywords: $\sigma$–field, $\sigma$– ring, field, smallest $\sigma$–field and restriction.

1. Introduction

In the real analysis and probability, the $\sigma$–field concept is the class $\mathcal{M}$ for a subset of a universal set $\mathcal{U}$ such that $\mathcal{U} \subseteq \mathcal{M}$ and it is closed under the complement, countable union [1] and [2]. The main reason for $\sigma$–field is the idea of measure, which is substantial in the real analysis as the basis of Lebesgue integrals, where it exponent as a family of events which may
be assigned probability [3] and [4]. In the probability theory, a \( \sigma \)-field is essential in the conditional expected. Also, in statistics, sub \( \sigma \)-field is necessary for an official mathematical definition for sufficient statistic, where a statistic be a map or a random variable. A \( \sigma \)-ring idea was studied by [5] as a class \( \mathcal{M} \) such that \( B_1 \setminus B_2 \in \mathcal{M} \) and \( \bigcup_{n=1}^{\infty} B_n \in \mathcal{M} \) whenever \( B_1, B_2, ... \in \mathcal{M} \). Many authors were interested in studying \( \sigma \)-field and \( \sigma \)-ring; for example, see [6], [7], and [8]. In this work, we denote a universal set by \( \mathcal{U} \).

**Preliminaries**

In the following, we mention some basic definitions and notations in measure space that will be used in this paper.

**Definition 2.1 [9].**

Suppose \( \mathcal{M} \) is a class of subsets of \( \mathcal{U} \). Then, \( \mathcal{M} \) is the \( \mathcal{P}^* \)-field of \( \mathcal{U} \) if:

1. \( \Phi \in \mathcal{M} \).
2. \( N, M \in \mathcal{M} \); then, \( N \cap M \in \mathcal{M} \).
3. \( M_2, ... \in \mathcal{M} \); then, \( \bigcup_{i=1}^{\infty} M_i \in \mathcal{M} \).

**Example 2.2 [9].**

Let \( \mathcal{U} = \{1,2,3,4\} \). Consider \( \mathcal{M} = \{ \Phi, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\} \} \).

Then \( \mathcal{M} \) is a \( \mathcal{P}^* \)-field of \( \mathcal{U} \).

**Definition 2.3 [5].**

The family of all subsets of \( \mathcal{U} \) is called a power set and denoted by \( \mathcal{P}(\mathcal{U}) \), in symbols:

\[ \mathcal{P}(\mathcal{U}) = \{ B : B \text{ is a subset of } \mathcal{U} \} \]

**Proposition 2.4 [9].**

If \( \{\mathcal{M}_i\}_{i \in I} \) is a family of \( \mathcal{P}^* \)-field of \( \mathcal{U} \), then so is \( \bigcap_{i \in I} \mathcal{M}_i \).

**Definition 2.5 [9].**

If \( \mathcal{M} \) is \( \sigma \)-field, then \( \mathcal{M} \) is a \( \sigma \)-ring.

**Proposition 2.6 [9].**

If \( \mathcal{I} \subseteq \mathcal{P}(\mathcal{U}) \), then \( \mathcal{P}^*(\mathcal{I}) = \bigcap \{ \mathcal{M}_i : \mathcal{M}_i \text{ is a } \mathcal{P}^* \text{-field of } \mathcal{U} \text{ and } \mathcal{M}_i \supseteq \mathcal{I} \text{, } \forall i \in I \} \) is called the \( \mathcal{P}^* \)-field generated by \( \mathcal{I} \).

**Proposition 2.7 [9].**

If \( \mathcal{M} \) is \( \sigma \)-field, then \( \mathcal{M} \) is a \( \sigma \)-ring.

**Proposition 2.8 [9].**

Every \( \sigma \)-field is \( \mathcal{P}^* \)-field.
Proposition 2.9 [9].

Every \( \sigma \)-ring is \( \mathcal{P}^* \)-field.

2. The Main Results

In this section, the basic definitions and facts related to this work are recalled, starting with the following definition:

**Definition 3.1**

Suppose \( \mathcal{M} \) is a \( \mathcal{P}^* \)-field of \( \mathcal{U} \) and \( \Phi \neq \mathcal{B} \subseteq \mathcal{U} \), then a restriction of \( \mathcal{M} \) over \( \mathcal{B} \) is defined as:

\[
\mathcal{M}|_{\mathcal{B}} = \{ N : N = M \cap \mathcal{B}, \text{for some } M \in \mathcal{M}\}.
\]

**Proposition 3.2**

Suppose \( \mathcal{M} \) is \( \mathcal{P}^* \)-field of \( \mathcal{U} \) and \( \Phi \neq \mathcal{B} \subseteq \mathcal{U} \), then \( \mathcal{M}|_{\mathcal{B}} \) is \( \mathcal{P}^* \)-field on \( \mathcal{B} \).

**Proof.**

Since \( \Phi \in \mathcal{M} \) and \( \Phi = \Phi \cap \mathcal{B} \), then \( \Phi \in \mathcal{M}|_{\mathcal{B}} \).

Let \( N_1, N_2 \in \mathcal{M}|_{\mathcal{B}} \), then there is \( M_1, M_2 \in \mathcal{M} \) such that \( N_i = M_i \cap \mathcal{B} \) where \( i = 1, 2 \) which implies that

\[
N_1 \cap N_2 = (M_1 \cap \mathcal{B}) \cap (M_2 \cap \mathcal{B}) = (M_1 \cap M_2) \cap \mathcal{B}.
\]

Since \( \mathcal{M} \) is a \( \mathcal{P}^* \)-field of \( \mathcal{U} \), then \( M_1 \cap M_2 \in \mathcal{M} \). Thus \( N_1 \cap N_2 \in \mathcal{M}|_{\mathcal{B}} \).

Let \( N_1, N_2, \ldots \in \mathcal{M}|_{\mathcal{B}} \), then there is \( M_1, M_2, \ldots \in \mathcal{M} \) such that \( N_i = M_i \cap \mathcal{B} \) where \( i = 1, 2 \ldots \) which implies that \( \bigcup_{i=1}^{\infty} N_i = \bigcup_{i=1}^{\infty} (M_i \cap \mathcal{B}) = (\bigcup_{i=1}^{\infty} M_i) \cap \mathcal{B} \).

Since \( \mathcal{M} \) is a \( \mathcal{P}^* \)-field of a set \( \mathcal{U} \), then \( \bigcup_{i=1}^{\infty} M_i \in \mathcal{M} \) and hence \( \bigcup_{i=1}^{\infty} N_i \in \mathcal{M}|_{\mathcal{B}} \).

Thus, \( \mathcal{M}|_{\mathcal{B}} \) is a \( \mathcal{P}^* \)-field on \( \mathcal{B} \).

**Proposition 3.3**

If \( \mathcal{M} \) is \( \mathcal{P}^* \)-field of \( \mathcal{U} \) and \( C \subseteq \mathcal{B} \subseteq \mathcal{U} \) such that \( C \in \mathcal{M} \), then \( C \in \mathcal{M}|_{\mathcal{B}} \).

**Proof.**

Clearly.

The following examples explain that if \( \mathcal{M} \) is a \( \mathcal{P}^* \)-field of a set \( \mathcal{U} \), then it is not necessarily that:

1. \( \mathcal{M}|_{\mathcal{B}} \subseteq \mathcal{M} \).
2. \( \mathcal{M} \subseteq \mathcal{M}|_{\mathcal{B}} \).

**Example 3.4**

Let \( \mathcal{U} = \{1,2,3,4\} \) and \( \mathcal{M} = \{ \Phi, \{1,3\}, \{1,2,3\}, \{1,3,4\}, \mathcal{U} \} \). Then, \( \mathcal{M} \) is a \( \mathcal{P}^* \)-field of \( \mathcal{U} \). If \( \mathcal{B} = \{2,3,4\} \), then \( \mathcal{M}|_{\mathcal{B}} = \{ \Phi, \{3\}, \{2,3\}, \{3,4\}, \mathcal{B} \} \). It is clear that \( \mathcal{M}|_{\mathcal{B}} \not\subseteq \mathcal{M} \), since \( \{3\} \in \mathcal{M}|_{\mathcal{B}} \) but \( \{3\} \notin \mathcal{M} \).

**Example 3.5**

Let \( \mathcal{U} = \{1,2,3,4\} \) and \( \mathcal{M} = \{ \Phi, \{1,2\}, \{1,2,3\}, \{1,2,4\}, \mathcal{U} \} \). Then, \( \mathcal{M} \) is a \( \mathcal{P}^* \)-field of \( \mathcal{U} \). If \( \mathcal{B} = \{2,3,4\} \), then \( \mathcal{M}|_{\mathcal{B}} = \{ \Phi, \{2\}, \{2,3\}, \{2,4\}, \mathcal{B} \} \). It is clear that \( \mathcal{M} \not\subseteq \mathcal{M}|_{\mathcal{B}} \), since \( \{1,2\} \in \mathcal{M} \) but \( \{1,2\} \notin \mathcal{M}|_{\mathcal{B}} \).
Proposition 3.6
If $\mathcal{M}$ is $\mathcal{P}^*$-field on $\mathcal{U}$ and $\Phi \neq \mathcal{B} \subseteq \mathcal{U}$ such that $\mathcal{B} \in \mathcal{M}$. Then $\mathcal{M}|_{\mathcal{B}} = \{ C \subseteq \mathcal{B} : C \in \mathcal{M} \}$.

Proof.
Assume that $N \in \mathcal{M}|_{\mathcal{B}}$, then $N = M \cap \mathcal{B}$, for some $M \in \mathcal{M}$ and thus $N \in \mathcal{M}$. Hence, $N \in \{ C \subseteq \mathcal{B} : C \in \mathcal{M} \}$. Therefore, $\mathcal{M}|_{\mathcal{B}} \subseteq \{ C \subseteq \mathcal{B} : C \in \mathcal{M} \}$. Let $D \in \{ C \subseteq \mathcal{B} : C \in \mathcal{M} \}$. Then $D \subseteq \mathcal{B}$ and $D \in \mathcal{M}$, hence $D = D \cap \mathcal{B}$, but $D \in \mathcal{M}$, then $D \in \mathcal{M}|_{\mathcal{B}}$. So, we get $\{ C \subseteq \mathcal{B} : C \in \mathcal{M} \} \subseteq \mathcal{M}|_{\mathcal{B}}$. Consequently, $\mathcal{M}|_{\mathcal{B}} = \{ C \subseteq \mathcal{B} : C \in \mathcal{M} \}$.

Corollary 3.7
If $\mathcal{M}$ is $\mathcal{P}^*$-field on $\mathcal{U}$ and $\Phi \neq \mathcal{B} \subseteq \mathcal{U}$ such that $\mathcal{B} \in \mathcal{M}$.
Then, $\mathcal{M}|_{\mathcal{B}} \subseteq \mathcal{M}$.

Proof.
The proof follows Proposition 3.6.

Definition 3.8
If $\mathcal{U}$ is a universal set and $\mathcal{I} \subseteq P(\mathcal{U})$ and $\Phi \neq \mathcal{B} \subseteq \mathcal{U}$, then a restriction of $\mathcal{I}$ on $\mathcal{B}$ is defined as:
$\mathcal{I}|_{\mathcal{B}} = \{ N : N = M \cap \mathcal{B}, \text{ for some } M \in \mathcal{I} \}$.

Proposition 3.9
If $\mathcal{I} \subseteq P(\mathcal{U})$ and $\Phi \neq \mathcal{B} \subseteq \mathcal{U}$. Assume $\mathcal{M}$ is a $\mathcal{P}^*$-field of $\mathcal{U}$ that contains $\mathcal{I}$ and $\mathcal{B} \in \mathcal{M}$, then $\mathcal{P}^*(\mathcal{I})|_{\mathcal{B}}$ is a $\mathcal{P}^*$-field of $\mathcal{B}$.

Proof.
The proof is done by proposition 2.6 and 3.2.

Theorem 3.10
Assume $\mathcal{I} \subseteq P(\mathcal{U})$ and $\Phi \neq \mathcal{B} \subseteq \mathcal{U}$, then $\mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})$ is the smallest $\mathcal{P}^*$-field on $\mathcal{B}$ that contain $\mathcal{I}|_{\mathcal{B}}$, where
$\mathcal{P}^*(\mathcal{I}|_{\mathcal{B}}) = \cap \{ \mathcal{M}_i|_{\mathcal{B}} : \mathcal{M}_i|_{\mathcal{B}} \text{ is a } \mathcal{P}^* \text{-field of } \mathcal{B} \text{ and } \mathcal{M}_i|_{\mathcal{B}} \supseteq \mathcal{I}|_{\mathcal{B}}, \forall i \in I \}$.

Proof.
In the same way as in proposition 2.4, we can prove that $\mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})$ is a $\mathcal{P}^*$-field on $\mathcal{B}$. To prove that $\mathcal{P}^*(\mathcal{I}|_{\mathcal{B}}) \supseteq \mathcal{I}|_{\mathcal{B}}$, assume that $\mathcal{M}_i|_{\mathcal{B}}$ is a $\mathcal{P}^*$-field on $\mathcal{B}$ and $\mathcal{M}_i|_{\mathcal{B}} \supseteq \mathcal{I}|_{\mathcal{B}}$, $\forall i \in I$, then $\mathcal{I}|_{\mathcal{B}} \subseteq \cap_{i \in I} \mathcal{M}_i|_{\mathcal{B}}$; hence $\mathcal{I}|_{\mathcal{B}} \subseteq \mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})$. Now, let $\mathcal{M}^*|_{\mathcal{B}}$ be a $\mathcal{P}^*$-field on $\mathcal{B}$ such that $\mathcal{M}^*|_{\mathcal{B}} \supseteq \mathcal{I}|_{\mathcal{B}}$. Then, $\mathcal{M}^*|_{\mathcal{B}} \supseteq \mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})$.
Therefore, $\mathcal{P}(\mathcal{I}|_{\mathcal{B}})$ is the smallest $\mathcal{P}^*$-field on $\mathcal{B}$ containing $\mathcal{I}|_{\mathcal{B}}$.

Theorem 3.11
If $\mathcal{I} \subseteq P(\mathcal{U})$ and $\Phi \neq \mathcal{B} \subseteq \mathcal{U}$, define a class $\mathcal{M}$ by:
$\mathcal{M} = \{ M \subseteq \mathcal{U} : M \cap \mathcal{B} \in \mathcal{P}^*(\mathcal{I}|_{\mathcal{B}}) \}$. Then $\mathcal{M}$ is a $\mathcal{P}^*$-field on a set $\mathcal{U}$.
Proof.

By Theorem 3.10, we have \( P^* (J|_B) \) as a \( P^- \)-field on \( B \), so \( \Phi \in P^* (J|_B) \).
Since \( \Phi = \Phi \cap B \), then we get \( \Phi \in \mathcal{M} \).
Assume that \( M_1, M_2 \in \mathcal{M} \). Then \( (M_1 \cap B, M_2 \cap B) \) for each \( i = 1, 2 \).
Now, \( (M_1 \cap M_2) \cap B = (M_1 \cap B) \cap (M_2 \cap B) \). Since \( P^* (J|_B) \) \( \subseteq \) a \( P^- \)-field on \( B \), \( P^* (J|_B) \) \( \subseteq \) \( P^* (J|_B) \).
Let \( M_1, M_2, \ldots \in \mathcal{M} \). Then \( (M_i \cap B) \in P^* (J|_B) \), for \( i = 1, 2, \ldots \).
Since \( P^* (J|_B) \) \( \subseteq \) a \( P^- \)-field on \( B \), \( \cup_{i=1}^{\infty} (M_i \cap B) \in P^* (J|_B) \).
Now, \( \cup_{i=1}^{\infty} (M_i \cap B) = \cup_{i=1}^{\infty} (M_i \cap B) \in P^* (J|_B) \), thus \( \cup_{i=1}^{\infty} M_i \in \mathcal{M} \).
Therefore, \( \mathcal{M} \) \( \subseteq \) \( P^- \)-field on a universal set \( U \).

Theorem 3.12

If \( U \) is a universal set and \( J \subseteq P(U) \) such that \( \Phi \neq B \subseteq U \), then \( P^*(J|_B) = P^*(J|_B) \).

Proof.

By proposition 2.6, we have \( P^*(J) \) \( \subseteq \) \( P^- \)-field on \( U \). So, we get \( P^* (J|_B) \) \( \subseteq \) a \( P^- \)-field on \( B \) by proposition 3.2. Assume that \( N \in \mathcal{I} |_B \). Then \( N = M \cap B \) for some \( M \in J \).
But \( J \subseteq P^*(J) \), so we have \( \Phi \in P^*(J) \) and thus \( N \in P^*(J|_B) \).
Hence \( J|_B \subseteq P^*(J)|_B \). Therefore, \( P^*(J)|_B \) \( \subseteq \) \( P^*(J|_B) \), which implies that \( P^*(J|_B) \subseteq P^*(J)|_B \).
Now, if we define a class \( \mathcal{M} \) by \( \mathcal{M} = \{ C \subseteq U : C \cap B \in P^*(J|_B) \} \), then in Theorem 3.11, we have \( \mathcal{M} \) as a \( P^- \)-field on \( U \). Let \( C \in J \), then \( (C \cap B) \in P^*(J|_B) \), but \( J|_B \subseteq P^*(J|_B) \) implies that \( (C \cap B) \in P^*(J|_B) \), hence \( C \subseteq \mathcal{M} \) and \( J \subseteq \mathcal{M} \).
Now, if we assume that \( N \in P^*(J|_B) \), then \( N = M \cap B \), for some \( M \in P(J) \). But \( P^*(J) \subseteq \mathcal{M} \), then \( M \subseteq \mathcal{M} \), hence \( N \in P^*(J|_B) \). Consequently, \( P^*(J)|_B \subseteq P^*(J|_B) \).
This completes the proof.

3. Conclusions

We tried to define the concept of measure relative to the \( P^- \)-field \( \mathcal{M} \) of \( U \) and also define the idea of the restriction of measure on \( \mathcal{M}|_B \) of a set \( B \). Also, we discuss many properties of these notions. In this article, the idea of \( P^- \)-field is given to refer to the generalization of each \( \sigma^- \)-field and \( \sigma^- \)-ring. Furthermore, some properties of the purposed notion are proven as explained below:

1. Let \( \mathcal{M} \) be a \( P^- \)-field of a set \( U \) and let \( B \) be a nonempty subset of \( U \). Then, \( \mathcal{M}|_B \) is a \( P^- \)-field of a set \( B \).
2. Assume that \( \mathcal{M} \) is a \( P^- \)-field on \( U \) and \( A \subseteq B \subseteq U \). If \( A \in \mathcal{M} \), then \( A \in \mathcal{M}|_B \).
3. If \( \mathcal{M} \) is a \( P^- \)-field and \( B \) be a nonempty subset of \( U \) such that \( B \in \mathcal{M} \). Then \( \mathcal{M}|_B = \{ A \subseteq B : A \in \mathcal{M} \} \).
4. Suppose that \( \mathcal{M} \) is a \( P^- \)-field and \( B \subseteq U \) such that \( B \in \mathcal{M} \). Then \( \mathcal{M}|_B \subseteq \mathcal{M} \).
5. If \( J \subseteq P(U) \) and \( \Phi \neq B \subseteq U \) and \( P^*(J)|_B \) is a \( P^- \)-field on \( B \). Then, \( P^*(J|_B) = P^*(J)|_B \).
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