The Classical Continuous Optimal Control for Quaternary Nonlinear Parabolic Boundary Value Problems with State Vector Constraints

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Abstract

This paper aims to study the quaternary classical continuous optimal control problem consisting of the quaternary nonlinear parabolic boundary value problem, the cost function, and the equality and inequality constraints on the state and the control. Under appropriate hypotheses, it is demonstrated that the quaternary classical continuous optimal control ruling by the quaternary nonlinear parabolic boundary value problem has a quaternary classical continuous optimal control vector that satisfies the equality constraint and inequality state and control constraint. Moreover, mathematical formulation of the quaternary adjoint equations related to the quaternary state equations is discovered, and then the weak form of the quaternary adjoint equations is obtained. Lastly, both the necessary conditions for optimality and sufficient conditions for optimality of the proposed problem are stated and proved. The derivation for the Fréchet derivative of the Hamiltonian is attained.

Keywords: Quaternary Classical Optimal Control, Quaternary Nonlinear Parabolic Boundary Value Problems, Necessary and Sufficient for Optimality Theorems.

1. Introduction

It is a well-known fact that optimal control problems (OCPs) are widely used in a variety of scientific fields, including biology [1], economics [2], robotics [3], Aircraft [4], and many others. OCPs are typically ruled by nonlinear ODEs (NLODEs) [5] or nonlinear PDEs (NLPDEs) [6]. During the last decade, great attention has been made to studying OCPs for system ruling by NLPDEs of elliptic, hyperbolic, and parabolic types [7-9]. Later, the study of this subject is expanded to include classical continuous optimal control problem (CCOCP) for systems ruling by couple NLPDEs and then recently by triple NLPDEs for the above three
indicated types of NLPDEs [10 - 15]. As a result, these concerns made us study the quaternary classical continuous optimal control problem (QCCOCP) ruling by quaternary nonlinear parabolic boundary value problems (QNLBPVPs) with equality constraint (EQC) and inequality constraint (INEQC).

This paper is concerned with studying the QCCOCP ruling by a QNLBPVP; it begins with stating and demonstrating the existence theorem of a quaternary classical continuous optimal control vector (QCCOCV) ruling by the QNLBPVP with EQC and INEQC under suitable hypotheses. In addition, the mathematical formulation of the quaternary adjoint equations (QAES) related to the quaternary state equations (QSEs) is discovered so as the weak form (WF). Moreover, the Fréchet derivative (FrD) of the Hamiltonian is attained. Lastly, both the necessary conditions for optimality (NCsTh) and sufficient conditions (SCsTh) for optimality are stated and demonstrated.

**Description of the problem**

Let Ω ⊂ ℝ² be an open and bounded region with boundary Γ = ∂Ω, x = (x₁, x₂), Q = I × Ω, I = [0, T], Γ = ∂Ω, Σ = Γ × I.

The QCCOC consists of the continuous quaternary state vector solution (QSVS), which is expressed by the following QNLBPVP:

\[
y_{1t} - \Delta y_{1} + y_{1} - y_{2} + y_{3} + y_{4} = f_{1}(x, t, y_{1}, u_{1}), \quad \text{in } Q
\]

\[
y_{2t} - \Delta y_{2} + y_{1} + y_{2} - y_{3} - y_{4} = f_{2}(x, t, y_{2}, u_{2}), \quad \text{in } Q
\]

\[
y_{3t} - \Delta y_{3} - y_{1} + y_{2} + y_{3} + y_{4} = f_{3}(x, t, y_{3}, u_{3}), \quad \text{in } Q
\]

\[
y_{4t} - \Delta y_{4} - y_{1} - y_{2} - y_{3} + y_{4} = f_{4}(x, t, y_{4}, u_{4}), \quad \text{in } Q
\]

With the following boundary conditions (BCs) and initial conditions (ICs):

\[
y_{i}(x, 0) = y_{i}^{0}(x), \quad \forall \ i = 1, 2, 3, 4. \quad \text{on } \Sigma
\]

\[
y_{i}(x, 0) = y_{i}^{0}(x), \quad \forall \ i = 1, 2, 3, 4. \quad \text{on } \Omega
\]

Where \(\bar{y} = (y_{1}, y_{2}, y_{3}, y_{4}) = (y_{1}(x, t), y_{2}(x, t), y_{3}(x, t), y_{4}(x, t)) \in (H^2(\Omega))^4\) is the quaternary state vector solution (QSVS), \(\bar{u} = (u_{1}, u_{2}, u_{3}, u_{4}) = (u_{1}(x, t), u_{2}(x, t), u_{3}(x, t), u_{4}(x, t)) \in (L^2(\Omega))^4\) is the QCCCV, and \((f_{1}, f_{2}, f_{3}, f_{4}) = (f_{1}(x, t), f_{2}(x, t), f_{3}(x, t), f_{4}(x, t)) \in (L^2(\Omega))^4\) is given, for all \(x = (x_{1}, x_{2}) \in \Omega\).

**The set of admissible control (SAC) is:**

\[\bar{W}_a = \left\{ \bar{w} \in (L^2(\Omega))^4 \bigg| \bar{w} \in \bar{U} \subset \mathbb{R}^4 \ \text{a.e. in } Q, G_{1}(\bar{w}) = 0, G_{2}(\bar{w}) \leq 0 \right\}\]

**The CF is:**

\[
G_{0}(\bar{u}) = \int_{Q} g_{01}(x, t, y_{1}, u_{1})dxdt + \int_{Q} g_{02}(x, t, y_{2}, u_{2})dxdt + \int_{Q} g_{03}(x, t, y_{3}, u_{3})dxdt + \int_{Q} g_{04}(x, t, y_{4}, u_{4})dxdt
\]

\[\text{(7.a)}\]

**The constraints on the state and the control (CSSC) are:**

\[
G_{1}(\bar{u}) = \int_{Q} g_{11}(x, t, y_{1}, u_{1})dxdt + \int_{Q} g_{12}(x, t, y_{2}, u_{2})dxdt + \int_{Q} g_{13}(x, t, y_{3}, u_{3})dxdt + \int_{Q} g_{14}(x, t, y_{4}, u_{4})dxdt = 0,
\]

\[\text{(7.b)}\]

\[
G_{2}(\bar{u}) = \int_{Q} g_{21}(x, t, y_{1}, u_{1})dxdt + \int_{Q} g_{22}(x, t, y_{2}, u_{2})dxdt + \int_{Q} g_{23}(x, t, y_{3}, u_{3})dxdt + \int_{Q} g_{24}(x, t, y_{4}, u_{4})dxdt \leq 0,
\]

\[\text{(7.c)}\]

Where \((y_{1}, y_{2}, y_{3}, y_{4}) = (y_{u1}, y_{u2}, y_{u3}, y_{u4})\) is the QSVS of ((1) – (6)) corresponding to the QCCCV \((u_{1}, u_{2}, u_{3}, u_{4})\).
Let $\vec{V} = V_1 \times V_2 \times V_3 \times V_4 = (\mathcal{H}_0^1(\Omega))^4$ and $\vec{v} = (v_1, v_2, v_3, v_4) = (v_1(x), v_2(x), v_3(x), v_4(x))$.

$\vec{V} = \{ \vec{v}; \vec{v} \in (\mathcal{H}_0^1(\Omega))^4, \text{with } v_1 = v_2 = v_3 = v_4 = 0 \text{ on } \partial \Omega \}$.

The WF of the QSVEs:

Thewf of $\{ (1) - (6) \}$ with $\vec{y} \in (\mathcal{H}_0^1(\Omega))^4$ is given by

$(y_{1t}, y_1) + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1) = (f_1(y_1, u_1), v_1)$, \hspace{1cm} (8.a)

$(y_{2t}, v_2) = (y_1(0), v_2)$, \hspace{1cm} (8.b)

$(y_{3t}, v_3) + (\nabla y_3, \nabla v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3) = (f_3(y_3, u_3), v_3)$, \hspace{1cm} (9.a)

$(y_{4t}, v_4) = (y_4(0), v_4)$, \hspace{1cm} (9.b)

$(y_{4t}, v_4) + (\nabla y_4, \nabla v_4) - (y_1, v_4) + (y_2, v_4) - (y_3, v_4) + (y_4, v_4) = (f_4(y_4, u_4), v_4)$, \hspace{1cm} (10.a)

$(y_{4t}, v_4) = (y_4(0), v_4)$, \hspace{1cm} (11.b)

The following hypotheses are important to study the QCCOC.

**Hypotheses (A):** Assume $\forall i = 1, 2, 3, 4$ that:

(i) $f_i$ is Carathéodory type (Cara. T.) on $Q \times (\mathbb{R})^4$, and satisfies the following conditions w.r.t. $y_i$ & $u_i$, i.e.:

$|f_i(x, t, y_i, u_i)| \leq \eta_i(x, t) + c_i |y_i| + \dot{c}_i |u_i|$, where $(x, t) \in Q$, $c_i, \dot{c}_i > 0$ and $\eta_i \in L^2(Q, \mathbb{R})$.

(ii) $f_i$ satisfies Lipschitz condition (LC) w.r.t. $y_i$, i.e.:

$|f_i(x, t, y_i, u_i) - f_i(x, t, \tilde{y}_i, u_i)| \leq L_i |y_i - \tilde{y}_i|$. Where $(x, t) \in Q$, $y_i, \tilde{y}_i, u_i \in \mathbb{R}$ and $L_i > 0$.

**Theorem (2.1) [13]:** (EUTH for the w.f. of the QSVEs)

With hypotheses (A), for each given QCCCV $\vec{u} \in (L^2(Q))^4$, the wf $\{ (8) - (11) \}$ has a unique QSV $\vec{y} \in (L^2(L, V))^4$, with $\vec{y}_i \in (L^2(L, V))^4$.

**Hypotheses (B):**

Suppose that for each $l = 0, 1, 2$ and $i = 1, 2, 3, 4$, that $g_{li}$ is of Cara. T. on $Q \times (\mathbb{R})^4$ and satisfies the following conditions w.r.t. $y_i$ and $u_i$:

$|g_{li}(x, t, y_i, u_i)| \leq \eta_{li}(x, t) + c_{li1} |y_i|^2 + c_{li2} |u_i|^2$. Where $y_i, u_i \in \mathbb{R}$ with $\eta_{li} \in L^1(Q)$.

**Lemma (2.1):**

With hypotheses (B), for each $l = 0, 1, 2$, the functional $\vec{u} \mapsto G_l(\vec{u})$ is cont. on $(L^2(Q))^4$.

**Proof:** The requirement result is gotten ($\forall l = 0, 1, 2$) directly from hypotheses (B) and Lemma 1.12 in [16].

**Theorem (2.2) [16]:** Consider the set $\mathbb{W}_A \neq \emptyset$, for each $i = 1, 2, 3, 4$, the functions $f_i$ has the form $f_i(x, t, y_i, u_i) = f_{i1}(x, t, y_i) + f_{i2}(x, t)u_i$

With $|f_{i1}(x, t, y_i)| \leq \eta_i(x, t) + c_i |y_i|$ where $\eta_i \in L^2(Q)$ and $|f_{i2}(x, t)| \leq k_i$.

If $\forall i = 1, 2, 3, 4$, $g_{0i}$ is convex (CO) w.r.t. $u_i$ for fixed $(x, t, y_i)$. Then there is a QCCOCV.

**Hypotheses (C):**

Assume that, for $l = 0, 2$ and $i = 1, 2, 3, 4$ $g_{li\dot{y}_i}$ and $g_{li\dot{u}_i}$ are of Cara. T. on $\times (\mathbb{R})^4$,

$|g_{li\dot{y}_i}(x, t, y_i, u_i)| \leq \eta_{li5}(x, t) + c_{li5} |y_i| + \dot{c}_{li5} |u_i|,$

and $|g_{li\dot{u}_i}(x, t, y_i, u_i)| \leq \eta_{li6}(x, t) + c_{li6} |y_i| + \dot{c}_{li6} |u_i|.$

Where $(x, t) \in Q$, $y_i, u_i \in \mathbb{R}$, $\eta_{li5}, \eta_{li6} \in L^2(Q)$. 

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Theorem (2.3) [16]: In addition to hypotheses (A), if \( \tilde{y} \) and \( \tilde{y} + \delta \tilde{y} \) are the QSVS corresponding to the QCCCV \( \tilde{u}, \tilde{u} + \delta \tilde{u} \in (L^2(Q))^4 \), resp., then
\[
\| \tilde{y} \|_{L^\infty(I;L^2(\Omega))} \leq M \| \delta \tilde{u} \|_{L^2(Q)}, \quad \| \delta \tilde{y} \|_{L^2(Q)} \leq M \| \delta \tilde{u} \|_{L^2(Q)}, \quad \| \delta \tilde{y} \|_{L^2(\Omega)} \leq M \| \delta \tilde{u} \|_{L^2(Q)}.
\]

Theorem (2.4) (The Kuhn-Tucker-Lagrange conditions (KTL)) [10]:
Let \( U \) be a nonempty CO subset of a vector space \( X \), \( K \) be a nonempty CO positive cone in a normed space \( Z \), and \( W = \{ u \in U | G_1(u) = 0, G_2(u) \in -K \} \).

The functional \( G_0: U \to \mathbb{R}, G_1: U \to \mathbb{R}^m, G_2: U \to \mathbb{Z} \) are \((m+1)\) locally continuous at \( u \in U \), and have \((m+1)\) derivatives at \( u \) where \( m \neq 0 \). And if \( m = 0 \), we assume that \( DG_1(u), l = 0,1,2 \) are \( K \)-linear at the point \( u \). If \( G_0(u) \) has a minimum at \( u \) in \( W \), then it satisfies the following KUTULA conditions, \( \forall w \in W \):
There exists \( \lambda_0 \in \mathbb{R}, \lambda_1 \in \mathbb{R}^m, \lambda_2 \in \mathbb{Z} \), with \( \lambda_0 \geq 0, \lambda_2 \geq 0, \sum_{i=0}^2 |\lambda_i| = 1 \) s.t
\( \lambda_0 DG_0(u,w-u) + \lambda_1^{-1} DG_1(u,w-u) + \langle \lambda_2, DG_2(u,w-u) \rangle \geq 0 \), \( \langle \lambda_2, G_2(u) \rangle = 0 \).

Main Results
3. Existence of the QCCOCV and the FrD:
This section deals with the existence theorem of the QCCOCV, the discovery of the mathematical formulation for the QAEs and their WF is obtained, and the derivation of the FrD is derived under some appropriate hypotheses.

Theorem (3.1): Consider the set \( W_A \neq \emptyset \), the functions \( f_i, \forall i = 1,2,3,4 \), has the form
\( f_i(x,t,y_i,u_i) = f_{i1}(x,t,y_i) + f_{i2}(x,t)u_i \)
With \( |f_{i1}(x,t,y_i)| \leq \eta_i(x,t) + c_i |y_i| \) and \( |f_{i2}(x,t)| \leq k_i \), where \( \eta_i \in L^2(Q) \).
If \( \forall i = 1,2,3,4 \), \( g_{i1} \) is independent of \( u_i \), \( g_{i0} \) and \( g_{21} \) are convex w.r.t. \( u_i \) for fixed \( (x,t,y_i) \).
Then there is a QCCOCV.

Proof: From the hypotheses on \( W_t \) and \( g_{i1} (\forall i = 1,2,3,4) \), with using lemma (2.1) and the.
2.2, one can get that there is a QCCOCV with the EQC and INEQC.

Theorem (3.2): We drop the index \( i \) in \( g_i \) and \( G_i \) . In addition to hypotheses (A), (B), and (C), the following adjoint (\( z_1, z_2, z_3, z_4 \)) \( \mathbf{(z_{u_1}, z_{u_2}, z_{u_3}, z_{u_4})} \) equations corresponding to the state (\( y_1, y_2, y_3, y_4 \)) \( \mathbf{(y_{u_1}, y_{u_2}, y_{u_3}, y_{u_4})} \) equations (1) – (6) are expressed by:
\[
\begin{align*}
- z_{1t} - z_1 + z_2 - z_3 - z_4 &= z_1 f_{y_1}(x,t,y_1,u_1) + g_{y_1}(x,t,y_1,u_1), \\
- z_{2t} - z_2 + z_3 - z_4 &= z_2 f_{y_2}(x,t,y_1,u_1) + g_{y_2}(x,t,y_1,u_1), \\
- z_{3t} - z_3 + z_1 - z_2 - z_4 &= z_3 f_{y_3}(x,t,y_1,u_1) + g_{y_3}(x,t,y_1,u_1), \\
- z_{4t} - z_4 + z_2 - z_1 - z_2 &= z_4 f_{y_4}(x,t,y_1,u_1) + g_{y_4}(x,t,y_1,u_1), \\
\end{align*}
\]
\( z_i(x,t) = 0, \forall i = 1,2,3,4 \), on \( \Sigma \),\( z_i(T) = 0, \forall i = 1,2,3,4 \), on \( \Gamma \).

Also, the Hamiltonian is defined: \( H(x,t,y_i,z_i,u_i) = \sum_{i=1}^4 z_i f_{i}(x,t,y_i,u_i) + g_i(x,t,y_i,u_i) \).

Then the FrD of \( G \) is given by \( \hat{G}(\tilde{u}) = \int_I \begin{pmatrix} z_1 f_{u_1} + g_{u_1} \\ z_2 f_{u_2} + g_{u_2} \\ z_3 f_{u_3} + g_{u_3} \\ z_4 f_{u_4} + g_{u_4} \end{pmatrix} \cdot \begin{pmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \\ \delta u_4 \end{pmatrix} dx. \)
Proof:
Firstly, let $\vec{u}$ be a QCCCV, and $\vec{y}$ be its QSVS, and let $G(\vec{u}) = \sum_{l=1}^{t} \int_{Q} g_{l}(x, t, y_{i}, u_{i}) dx dt$.
From the hypotheses on $g_{l}$ ($l = 1, 2, 3, 4$), the FrD definition, the result of The. 2.3, and then using the Minkowski's inequality (MGIN), one can get that:

$$G(\vec{u} + \vec{d}u) - G(\vec{u}) = \int_{Q} (g_{y_{1}} \delta y_{1} + g_{u_{1}} \delta u_{1}) dx dt + \int_{Q} (g_{y_{2}} \delta y_{2} + g_{u_{2}} \delta u_{2}) dx dt + \int_{Q} (g_{y_{3}} \delta y_{3} + g_{u_{3}} \delta u_{3}) dx dt + \int_{Q} (g_{y_{4}} \delta y_{4} + g_{u_{4}} \delta u_{4}) dx dt +$$

$$\epsilon_{6}(\vec{d}u) \| \vec{d}u \|_{L^{2}(Q)}$$

(18)

Where $\epsilon_{6}(\vec{d}u) = \epsilon_{2}(\vec{d}u) + \epsilon_{3}(\vec{d}u) + \epsilon_{4}(\vec{d}u) + \epsilon_{5}(\vec{d}u) \to 0$ as $\| \vec{d}u \|_{L^{2}(Q)} \to 0$.

On the other hand, the wf of the QAEs for $v_{i} \in V, \forall i = 1, 2, 3, 4$ is given by:

$$- (z_{1t}, v_{1}) + (\nabla z_{1}, \nabla v_{1}) + (z_{1}, v_{1}) + (z_{2}, v_{1}) - (z_{3}, v_{1}) - (z_{4}, v_{1}) = (z_{1f_{1y_{1}}, v_{1}}) + (g_{1y_{1}, v_{1}}),$$

$$- (z_{2t}, v_{2}) + (\nabla z_{2}, \nabla v_{2}) + (z_{2}, v_{2}) - (z_{1}, v_{2}) + (z_{3}, v_{2}) + (z_{4}, v_{2}) = (z_{2f_{2y_{2}}, v_{2}}) + (g_{2y_{2}, v_{2}}),$$

$$- (z_{3t}, v_{3}) + (\nabla z_{3}, \nabla v_{3}) + (z_{3}, v_{3}) + (z_{1}, v_{3}) - (z_{2}, v_{3}) - (z_{4}, v_{3}) = (z_{3f_{3y_{3}}, v_{3}}) + (g_{3y_{3}, v_{3}}),$$

$$- (z_{4t}, v_{4}) + (\nabla z_{4}, \nabla v_{4}) + (z_{4}, v_{4}) + (z_{1}, v_{4}) - (z_{2}, v_{4}) + (z_{3}, v_{4}) = (z_{4f_{4y_{4}}, v_{4}}) + (g_{4y_{4}, v_{4}}).$$

(19) (20) (21) (22)

The existence of a unique solution of ((19) – (22)) can be proved by the same manner which is used in the proof of the unique of the QSVS.

Now, substituting $v_{i} = \delta y_{i}, \forall i = 1, 2, 3, 4$ in ((19) – (22)) resp., then taking the integrating both sides (IBS) from 0 to $T$, lastly, applying integration by part (IBP) for the 1st terms of each resulting equation, to get that:

$$\int_{0}^{T} (\delta y_{1}, z_{1}) dt + \int_{0}^{T} [(\nabla z_{1}, \nabla \delta y_{1}) + (z_{1}, \delta y_{1}) + (z_{2}, \delta y_{1}) - (z_{3}, \delta y_{1}) - (z_{4}, \delta y_{1})] dt =$$

$$\int_{0}^{T} [(z_{1f_{1y_{1}}, \delta y_{1}}) + (g_{1y_{1}, \delta y_{1}})] dt,$$

$$\int_{0}^{T} (\delta y_{2}, z_{2}) dt + \int_{0}^{T} [(\nabla z_{2}, \nabla \delta y_{2}) + (z_{2}, \delta y_{2}) - (z_{1}, \delta y_{2}) + (z_{3}, \delta y_{2}) + (z_{4}, \delta y_{2})] dt =$$

$$\int_{0}^{T} [(z_{2f_{2y_{2}}, \delta y_{2}}) + (g_{2y_{2}, \delta y_{2}})] dt,$$

$$\int_{0}^{T} (\delta y_{3}, z_{3}) dt + \int_{0}^{T} [(\nabla z_{3}, \nabla \delta y_{3}) + (z_{3}, \delta y_{3}) + (z_{1}, \delta y_{3}) - (z_{2}, \delta y_{3}) - (z_{4}, \delta y_{3})] dt =$$

$$\int_{0}^{T} [(z_{3f_{3y_{3}}, \delta y_{3}}) + (g_{3y_{3}, \delta y_{3}})] dt,$$

$$\int_{0}^{T} (\delta y_{4}, z_{4}) dt + \int_{0}^{T} [(\nabla z_{4}, \nabla \delta y_{4}) + (z_{4}, \delta y_{4}) + (z_{1}, \delta y_{4}) - (z_{2}, \delta y_{4}) + (z_{3}, \delta y_{4})] dt =$$

$$\int_{0}^{T} [(z_{4f_{4y_{4}}, \delta y_{4}}) + (g_{4y_{4}, \delta y_{4}})] dt,$$

(23) (24) (25) (26)

Also, substituting $v_{i} = z_{i}, \forall i = 1, 2, 3, 4$ in ((8.a) – (11.a)) resp., then IBS w.r.t. $t$ from 0 to $T$, to obtain:

$$\int_{0}^{T} (\delta y_{1t}, z_{1}) dt + \int_{0}^{T} [(\nabla \delta y_{1}, \nabla z_{1}) + (\delta y_{1}, z_{1}) - (\delta y_{2}, z_{1}) + (\delta y_{3}, z_{1}) + (\delta y_{4}, z_{1})] dt =$$

$$\int_{0}^{T} (f_{1}(y_{1} + \delta y_{1}, u_{1} + \delta u_{1}), z_{1}) dt - \int_{0}^{T} (f_{1}(y_{1}, u_{1}), z_{1}) dt,$$

$$\int_{0}^{T} (\delta y_{2t}, z_{2}) dt + \int_{0}^{T} [(\nabla \delta y_{2}, \nabla z_{2}) + (\delta y_{2}, z_{2}) + (\delta y_{2}, z_{2}) - (\delta y_{3}, z_{2}) - (\delta y_{4}, z_{2})] dt =$$

$$\int_{0}^{T} (f_{2}(y_{2} + \delta y_{2}, u_{2} + \delta u_{2}), z_{2}) dt - \int_{0}^{T} (f_{2}(y_{2}, u_{2}), z_{2}) dt,$$

(27) (28)

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Now, by substituting (35) in (18)
\[ \int_0^T (\delta y_{3t}, z_3) dt + \int_0^T [(\nabla \delta y_3, \nabla z_3) - (\delta y_1, z_3) + (\delta y_2, z_3) + (\delta y_4, z_3)] dt = \int_0^T (f_3(y_3 + \delta y_3, u_3), z_3) dt - \int_0^T (f_3(y_3, u_3), z_3) dt, \]
(29)
\[ \int_0^T (\delta y_{4t}, z_4) dt + \int_0^T [(\nabla \delta y_4, \nabla z_4) - (\delta y_1, z_4) + (\delta y_2, z_4) - (\delta y_3, z_4) + (\delta y_4, z_4)] dt = \int_0^T (f_4(y_4 + \delta y_4, u_4 + \delta u_4), z_4) dt - \int_0^T (f_4(y_4, u_4), z_4) dt, \]
(30)
Using the hypotheses on \( f_i \) (for each \( i = 1,2,3,4 \)), the FrD of it exists, then from the result of The. 2.3, and the MKIN, the followings are yielded:
\[ \int_0^T (\delta y_{1t}, z_1) dt + \int_0^T [(\nabla \delta y_1, \nabla z_1) - (\delta y_2, z_1) + (\delta y_3, z_1) + (\delta y_4, z_1)] dt = \int_0^T (f_1 y_1, \delta y_1 + f_1 u_1, \delta u_1, z_1) dt + \varepsilon_{12}(\delta \bar{u}) \| \delta \bar{u} \|_{L^2(Q)}, \]
(31)
\[ \int_0^T (\delta y_{2t}, z_2) dt + \int_0^T [(\nabla \delta y_2, \nabla z_2) + (\delta y_1, z_2) + (\delta y_2, z_2) - (\delta y_3, z_2) - (\delta y_4, z_2)] dt = \int_0^T (f_2 y_2, \delta y_2 + f_2 u_2, \delta u_2, z_2) dt + \varepsilon_{22}(\delta \bar{u}) \| \delta \bar{u} \|_{L^2(Q)}, \]
(32)
\[ \int_0^T (\delta y_{3t}, z_3) dt + \int_0^T [(\nabla \delta y_3, \nabla z_3) - (\delta y_1, z_3) + (\delta y_2, z_3) + (\delta y_3, z_3) + (\delta y_4, z_3)] dt = \int_0^T (f_3 y_3, \delta y_3 + f_3 u_3, \delta u_3, z_3) dt + \varepsilon_{32}(\delta \bar{u}) \| \delta \bar{u} \|_{L^2(Q)}, \]
(33)
\[ \int_0^T (\delta y_{4t}, z_4) dt + \int_0^T [(\nabla \delta y_4, \nabla z_4) - (\delta y_1, z_4) + (\delta y_2, z_4) - (\delta y_3, z_4) + (\delta y_4, z_4)] dt = \int_0^T (f_4 y_4, \delta y_4 + f_4 u_4, \delta u_4, z_4) dt + \varepsilon_{42}(\delta \bar{u}) \| \delta \bar{u} \|_{L^2(Q)}, \]
(34)
By subtracting ((31) – (34)) from ((23) – (26)) resp., and adding the attained equations, one obtains:
\[ \int_0^T [\left((f_{1u_1}, \delta u_1, z_1) + (f_{2u_2}, \delta u_2, z_2) + (f_{3u_3}, \delta u_3, z_3) + (f_{4u_4}, \delta u_4, z_4)\right)] dt + \varepsilon_5(\delta \bar{u}) \| \delta \bar{u} \|_{L^2(Q)} = \int_0^T [\left((g_{1y_1}, \delta y_1) + (g_{2y_2}, \delta y_2) + (g_{3y_3}, \delta y_3) + (g_{4y_4}, \delta y_4)\right)] dt, \]
(35)
Where \( \varepsilon_5(\delta \bar{u}) = \varepsilon_{12}(\delta \bar{u}) + \varepsilon_{22}(\delta \bar{u}) + \varepsilon_{32}(\delta \bar{u}) + \varepsilon_{42}(\delta \bar{u}) \to 0 \), as \( \| \delta \bar{u} \|_{L^2(Q)} \to 0 \).

Now, by substituting (35) in (18), one gets:
\[ G(\bar{u} + \delta \bar{u}) - G(\bar{u}) = \int_Q \left[ \left(z_1 f_{1u_1} + g_{1u_1}\right) \delta u_1 + \left(z_2 f_{2u_2} + g_{2u_2}\right) \delta u_2 + \left(z_3 f_{3u_3} + g_{3u_3}\right) \delta u_3 + \left(z_4 f_{4u_4} + g_{4u_4}\right) \delta u_4 \right] dx dt + \varepsilon_7(\delta \bar{u}) \| \delta \bar{u} \|_{L^2(Q)}, \]
(36)
Where \( \varepsilon_7(\delta \bar{u}) = \varepsilon_5(\delta \bar{u}) + \varepsilon_6(\delta \bar{u}) \to 0 \), as \( \| \delta \bar{u} \|_{L^2(Q)} \to 0 \).

Using the FrD definition of \( G \), one gets:
\[ G(\bar{u} + \delta \bar{u}) - G(\bar{u}) = (\hat{G}(\bar{u}), \delta \bar{u}) + \varepsilon_7(\delta \bar{u}) \| \delta \bar{u} \|_{L^2(Q)} \]
(37)
From (36) and (37), one can get:
\[ \left( \hat{G}(\bar{u}), \delta \bar{u} \right) = \int_Q \left( \begin{array}{c} z_1 f_{1u_1} + g_{1u_1} \\ z_2 f_{2u_2} + g_{2u_2} \\ z_3 f_{3u_3} + g_{3u_3} \\ z_4 f_{4u_4} + g_{4u_4} \end{array} \right) \cdot \left( \begin{array}{c} \delta u_1 \\ \delta u_2 \\ \delta u_3 \\ \delta u_4 \end{array} \right) dx. \]

4. The NCsTh and The SCsTh for Optimality:

This section deals with the state and demonstration of the NCsTh, so as the SCsTh, under some additional hypotheses.
Theorem (4.1): NCsTh for Optimality:

(a) In addition to hypotheses (A), (B) and (C), if \( \bar{u} \in \bar{W}_A \) is QCCOCV, then there is "multiplier" \( \lambda_l \in \mathbb{R}, l = 0,1,2 \), with \( \lambda_0 \geq 0, \lambda_2 \geq 0, \sum_{l=0}^{2}|\lambda_l| = 1 \), s.t. the following Lagrange -Kuhn-Tucker conditions (LKT) conditions are held:

\[
\int_Q H_{\bar{u}}(x,t,\bar{y},\bar{z},\bar{u}) \cdot \delta \bar{u} dx dt \geq 0, \forall \bar{w} \in \bar{W}, \quad \delta \bar{u} = \bar{w} - \bar{u} \tag{38.a}
\]

Where \( g_i = \sum_{l=0}^{2}\lambda_l g_{li}, \forall i = 1,2,3,4 \) in the definition of \( H \), and also \( \lambda_2 G_2(\bar{u}) = 0 \) \( \tag{38.b} \)

(b) (Minimum principle in weak form) If \( \bar{W} \) is of the form

\[
\bar{W} = \{ \bar{w} \in (L^2(Q))^{4}|\bar{w}(x,t) \in \bar{U}a.e. on Q \} , \text{ with } \bar{U} \subset \mathbb{R}^2.
\]

Then, (38.a) is equivalent to the minimum principle in point-wise form (MPPWF)

\[
H_{\bar{u}}(x,t,\bar{y},\bar{z},\bar{u})\bar{u}(t) = \min_{\bar{w} \in \bar{U}} H_{\bar{u}}(x,t,\bar{y},\bar{z},\bar{u})\bar{w} \quad \text{a.e. on } Q \tag{39}
\]

Proof: (a) The functional \( G_t(\bar{u}) \) is \( \rho \) -loccall cont. at each \( \bar{u} \in \bar{W} \), \forall l = 0,1,2 and for every \( \rho \) (by hypotheses (A), (B) and (C) and Lemma 2.1), and \( G_t(\bar{u}) \) is \( \rho \) -differentiable at each \( \bar{u} \in \bar{W} \) \( \mu \) (by hypotheses (A), (B) and (C) and The. 3.2) and since \( \bar{W} \in (L^2(Q)^4 \}, L^2(Q) \) is open, then \( DG_t(\bar{u},\bar{w} - \bar{u}) = \hat{G}_t(\bar{u})(\bar{w} - \bar{u}) \), \( l = 0,1,2 \).

And since \( \bar{u} \in \bar{W}_A \) is QCCOCV, by The. 2.4, there is \( \lambda_l \in \mathbb{R}, l = 0,1,2 \), with \( \lambda_0 \geq 0, \lambda_2 \geq 0 \), \( \sum_{l=0}^{2}|\lambda_l| = 1 \) s.t. (38a&b) are held.

Again by The. 3.2, setting \( \delta \bar{u} = \bar{w} - \bar{u} \) and substituting the FrD of \( \hat{G}_t, l = 0,1,2 \) in (38.a), one has:

\[
\sum_{l=1}^{2} \int_Q [(\lambda_0 z_0i_l + \lambda_1 z_1i_l + \lambda_2 z_2i_l)f_{iu_l} + (\lambda_0 g_{0iu_l} + \lambda_1 g_{1iu_l} + \lambda_2 g_{2iu_l})]u_l dx dt \geq 0.
\]

\[
\Rightarrow \sum_{l=0}^{2} \int_Q [(z_if_{iu_l} + g_{iu_l})] \delta u_l dx dt \geq 0, \forall i = 1,2,3,4.
\]

(ii) To prove that (38.a) is equivalent to (39):

Let \( \bar{W}_{\bar{U}} = \{ \bar{w} \in (L^2(Q))^{4}|\bar{w}(x,t) \in \bar{U}a.e. in Q \} \) with \( \bar{U} \subset \mathbb{R}^2 \), let \( \{ \bar{w}_k \} \) dense in a set \( \bar{W}_{\bar{U}} \), \( \mu \) is "Lebesgue measure" on \( Q \) and let \( S \subset Q \) be a measurable set s.t.:

\[
\begin{align*}
\bar{w}(x,t) & = \begin{cases} 
\bar{w}_k(x,t) & \text{if } (x,t) \in S \\
\bar{u}(x,t) & \text{if } (x,t) \in \bar{S} 
\end{cases} 
\end{align*}
\]

Therefore (38.a) becomes:

\[
\int_S H_{\bar{u}}(x,t,\bar{y},\bar{z},\bar{u})(\bar{w}_k - \bar{u}) \geq 0, \quad \forall S,
\]

Using the 3.2, to obtain: \( H_{\bar{u}}(x,t,\bar{y},\bar{z},\bar{u})(\bar{w}_k - \bar{u}) \geq 0 \), a.e. in \( Q \),

\[
H_{\bar{u}}(x,t,\bar{y},\bar{z},\bar{u})(\bar{w}_k - \bar{u}) \geq 0 \quad \text{in } P = \bigcap_k P_k \text{, where } P_k = Q - Q_k \text{ with } \mu(Q_k) = 0 \quad \forall k \text{ ,}
\]

since \( P \) is independent of \( k \), hence \( \mu(Q - P) = \mu(\bigcup_k Q_k) = 0 \), from the density of \( \{ \bar{w}_k \} \) in \( \bar{W}_{\bar{U}} \), one has \( H_{\bar{u}}(x,t,\bar{y},\bar{z},\bar{u})(\bar{w} - \bar{u}) \geq 0 \), a.e. in \( Q \).

\[
\Rightarrow H_{\bar{u}}(x,t,\bar{y},\bar{z},\bar{u})\bar{u} = \min_{\bar{w} \in \bar{U}} H_{\bar{u}}(x,t,\bar{y},\bar{z},\bar{u})\bar{w} , \forall \bar{w} \in \bar{W} \text{, a.e. in } Q.
\]

Conversely, if

\[
H_{\bar{u}}(x,t,\bar{y},\bar{z},\bar{u})\bar{u} = \min_{\bar{w} \in \bar{U}} H_{\bar{u}}(x,t,\bar{y},\bar{z},\bar{u})\bar{w} \quad \text{a.e. on } Q.
\]
⇒ \( H_{\bar{u}}(x, t, \bar{y}, \bar{z}, \bar{u})(\bar{w} - \bar{u}) \geq 0 \), \( \forall \bar{w} \in \bar{W} \), a.e. on \( Q \)
⇒ \( \int_Q H_{\bar{u}}(x, t, \bar{y}, \bar{z}, \bar{u}) \delta u \, dx \, dt \geq 0 \), \( \forall \bar{w} \in \bar{W} \).

**Theorem (4.2) (SCsTh for Optimality)**

In addition to hypotheses (A), (B) and (C), suppose that \( \bar{W} = \bar{W}_Q \) is CO, \( f_i \), \( \forall i = 1, 2, 3, 4 \) and \( g_{1i} \) are affine w.r.t. \((y_i, u_i)\) in \( Q \), and \( g_{2i} \) and \( g_{2i} \) are CO w.r.t. \((y_i, u_i)\) in \( Q \) \( \forall i = 1, 2, 3, 4 \). Then the NCs in The. 4.1 with \( \lambda_0 \geq 0 \) are also sufficient.

**Proof:** From The. 4.1, \( DG_i(\bar{u}, \bar{w} - \bar{u}) = \dot{G}_i(\bar{u})(\bar{w} - \bar{u}) \), \( l = 0, 1, 2, \)
Assume that \( \bar{u} \) satisfies (38) and \( \bar{u} \in \bar{W}_A \), i.e.:  
\[
\int_Q H_{\bar{u}}(x, t, \bar{y}, \bar{z}, \bar{u}) \delta u \, dx \, dt \geq 0, \forall \bar{w} \in \bar{W}.
\]
And \( \lambda_2 G_2(\bar{u}) = 0 \). Let \( \bar{u} = \sum_{i=0}^{\alpha_2} \lambda_i G_i(\bar{u}) \), then
\[
\dot{G}(\bar{u}) \cdot \delta \bar{u} = \sum_{i=0}^{\alpha_2} \lambda_i \dot{G}_i(\bar{u}) \cdot \delta \bar{u} = \lambda_0 \int_Q \sum_{i=1}^{\alpha_4} (z_{0i} f_{iu_i} + g_{0iu_i}) \delta u_i \, dx \, dt + \lambda_1 \int_Q \sum_{i=1}^{\alpha_4} (z_{1i} f_{iu_i} + g_{1iu_i}) \delta u_i \, dx \, dt + \lambda_2 \int_Q \sum_{i=1}^{\alpha_4} (z_{2i} f_{iu_i} + g_{2iu_i}) \delta u_i \, dx \, dt,
\]
\[
= \int_Q H_{\bar{u}}(x, t, \bar{y}, \bar{z}, \bar{u}) \delta \bar{u} \, dx \, dt \geq 0.
\]
Now, to demonstrate \( \bar{u} \mapsto \bar{y}_i \) is convex – linear (COL), since \( \forall i = 1, 2, 3, 4, f_i \) is affine from the HYPOTHESES on \( f_i \), \( \forall i = 1, 2, 3, 4 \): \( f_i(x, t, y_i, u_i) = f_i(x, t) y_i + f_i(x, t) u_i + f_i(x, t), \)
\( \forall i = 1, 2, 3, 4 \). Let \( \bar{u} = (u_1, u_2, u_3, u_4) \) & \( \bar{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4) \) be two given QCCCVs and from The. 2.1, \( \bar{y} = \left(y_{u_1}, y_{u_2}, y_{u_3}, y_{u_4}\right) \) & \( \bar{y} = \left(\bar{y}_{u_1}, \bar{y}_{u_2}, \bar{y}_{u_3}, \bar{y}_{u_4}\right) \) are their corresponding QSVS, precisely from (1),
\[
y_{1t} - \Delta y_1 + y_1 - y_2 + y_3 + y_4 = f_{11}(x, t) y_1 + f_{12}(x, t) u_1 + f_{13}(x, t),
\]
\[
y_1(x, 0) = y_1^0(x),
\]
\[
y_{1t} - \Delta \bar{y}_1 + \bar{y}_1 - \bar{y}_2 + \bar{y}_3 + \bar{y}_4 = f_{11}(x, t) \bar{y}_1 + f_{12}(x, t) \bar{u}_1 + f_{13}(x, t),
\]
\[
\bar{y}_1(x, 0) = y_1^0(x).
\]
By MBS the 1st above equation and its IC by \( \alpha \in [0, 1] \), and the 2nd equation and its IC by \((1 - \alpha)\), and adding the attained equations and their attained ICs, one gets that:
\[
(ay_1 + (1 - \alpha)\bar{y}_1)_{t} - \Delta(ay_1 + (1 - \alpha)\bar{y}_1) + (ay_1 + (1 - \alpha)\bar{y}_1) - (ay_2 + (1 - \alpha)\bar{y}_2) + (ay_3 + (1 - \alpha)\bar{y}_3) + (ay_4 + (1 - \alpha)\bar{y}_4) = f_{21}(x, t)(ay_1 + (1 - \alpha)\bar{y}_1) + f_{22}(x, t)(au_2 + (1 - \alpha)\bar{u}_2) + f_{23}(x, t) \tag{40.a}
\]
\[
ay_1(x, 0) + (1 - \alpha)\bar{y}_1(x, 0) = y_1^0(x) \tag{40.b}
\]
By the same way, one obtains that:
\[
(ay_2 + (1 - \alpha)\bar{y}_2)_{t} - \Delta(ay_2 + (1 - \alpha)\bar{y}_2) + (ay_2 + (1 - \alpha)\bar{y}_2) + (ay_1 + (1 - \alpha)\bar{y}_1) - (ay_3 + (1 - \alpha)\bar{y}_3) - (ay_4 + (1 - \alpha)\bar{y}_4) = f_{21}(x, t)(ay_2 + (1 - \alpha)\bar{y}_2) + f_{22}(x, t)(au_2 + (1 - \alpha)\bar{u}_2) + f_{23}(x, t) \tag{41.a}
\]
\[
ay_2(x, 0) + (1 - \alpha)\bar{y}_2(x, 0) = y_2^0(x) \tag{41.b}
\]
\[
(ay_3 + (1 - \alpha)\bar{y}_3)_{t} - \Delta(ay_3 + (1 - \alpha)\bar{y}_3) + (ay_3 + (1 - \alpha)\bar{y}_3) - (ay_1 + (1 - \alpha)\bar{y}_1) + (ay_2 + (1 - \alpha)\bar{y}_2) + (ay_4 + (1 - \alpha)\bar{y}_4) = f_{31}(x, t)(ay_3 + (1 - \alpha)\bar{y}_3) + f_{32}(x, t)(au_3 + (1 - \alpha)\bar{u}_3) + f_{33}(x, t) \tag{42.a}
\]
\[
ay_3(x, 0) + (1 - \alpha)\bar{y}_3(x, 0) = y_3^0(x) \tag{42.b}
\]
\[
(ay_4 + (1 - \alpha)\bar{y}_4)_{t} - \Delta(ay_4 + (1 - \alpha)\bar{y}_4) + (ay_4 + (1 - \alpha)\bar{y}_4) - (ay_1 + (1 - \alpha)\bar{y}_1) + (ay_2 + (1 - \alpha)\bar{y}_2) - (ay_3 + (1 - \alpha)\bar{y}_3) = f_{41}(x, t)(ay_4 + (1 - \alpha)\bar{y}_4) + f_{42}(x, t)(au_4 + (1 - \alpha)\bar{u}_4) + f_{43}(x, t) \tag{43.a}
\]
\[
\alpha y_4(x,0) + (1-\alpha)\bar{y}_4(x,0) = y^0_4(x) \quad (43.b)
\]
From equations (40) – (43), the QCCCV \( \bar{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4) \), with \( \bar{u} = \alpha \bar{u} + (1-\alpha) \bar{u} \) has the corresponding QSVS, \( \bar{y}_1 = (\bar{y}_{11}, \bar{y}_{12}, \bar{y}_{13}, \bar{y}_{14}) \), \( \bar{y}_2 = \alpha \bar{y}_2 + (1-\alpha) \bar{y}_2 \), i.e.:  
\[
\begin{align*}
\bar{y}_{1t} - \Delta \bar{y}_1 + \bar{y}_3 - \bar{y}_2 + \bar{y}_4 &= f_{11}(x,t)\bar{y}_1 + f_{12}(x,t)\bar{u}_1 + f_{13}(x,t)\bar{u}_1, \\
\bar{y}_2(t,0) &= y^0_2(x), \\
\bar{y}_{3t} - \Delta \bar{y}_3 + \bar{y}_2 - \bar{y}_1 + \bar{y}_4 &= f_{31}(x,t)\bar{y}_3 + f_{32}(x,t)\bar{u}_3 + f_{33}(x,t), \\
\bar{y}_4(t,0) &= y^0_4(x).
\end{align*}
\]
Therefore \( \bar{u} \mapsto \bar{y}_\bar{u} \) is COL w.r.t. \( (\bar{y}, \bar{u}) \) in \( Q \).

From hypotheses on \( g_{1i}, g_{2i} \) in \( Q \) for each \( i = 1, 2, 3, 4 \):  
\[
g_{1i}(x,t, y_i, u_i) = h_{1i}(x,t) y_i + h_{2i}(x,t) u_i + h_{3i}(x,t).
\]
Now, to show \( g_{1i} \) is COL w.r.t. \( (y_i, u_i) \), in \( Q \), since  
\[
G_{1t}(\bar{u} + (1-\alpha)\bar{u}) = \Sigma_{i=1}^4 \left[ \int_Q g_{1i}(x,t,y_i, u_i + (1-\alpha)\bar{u}_i) \right] dxdt \\
= \Sigma_{i=1}^4 \left[ \int_Q h_{1i}(x,t,y_i, u_i) \right] dxdt + \Sigma_{i=1}^4 \left[ \int_Q h_{2i}(x,t,u_i) \right] dxdt + \Sigma_{i=1}^4 \left[ \int_Q h_{3i}(x,t) \right] dxdt,
\]
Since \( \bar{u} \mapsto \bar{y}_\bar{u} \) is COL. Then \( G_{1i}(\bar{u}) \) is COL w.r.t. \( (\bar{y}, \bar{u}) \), in \( Q \), i.e.:  
\[
G_{1t}(\bar{u} + (1-\alpha)\bar{u}) = \Sigma_{i=1}^4 \left[ \int_Q h_{1i}(x,t, (\alpha y_i + (1-\alpha)\bar{y}_i)) \right] dxdt + \Sigma_{i=1}^4 \left[ \int_Q h_{2i}(x,t) (\alpha u_i + (1-\alpha)\bar{u}_i) \right] dxdt + \Sigma_{i=1}^4 \left[ \int_Q h_{3i}(x,t) \right] dxdt, \\
= \alpha \Sigma_{i=1}^4 \left[ \int_Q h_{1i}(x,t) y_i + h_{2i}(x,t) u_i + h_{3i}(x,t) \right] dxdt + (1-\alpha) \Sigma_{i=1}^4 \left[ \int_Q h_{1i}(x,t) \bar{y}_i + h_{2i}(x,t) \bar{u}_i + h_{3i}(x,t) \right] dxdt, \\
= \alpha G_{1t}(\bar{u}) + (1-\alpha)G_{1t}(\bar{u}).
\]
Since \( g_{oi} \) & \( g_{2i} \) are CO w.r.t. \( (y_i, u_i) \), in \( Q \), \( \forall i = 1, 2, 3, 4 \), then  
\[
\Sigma_{i=1}^4 \int_Q g_{oi} \right] dxdt \text{ and } \Sigma_{i=1}^4 \int_Q g_{2i} \right] dxdt \text{ are CO w.r.t. } (y_i, u_i), \text{ in } Q, \forall i = 1, 2, 3, 4, \text{ and then } G_{0}(\bar{u}) \text{ and } G_{2}(\bar{u}) \text{ are CO w.r.t. } (\bar{y}, \bar{u}) \text{, in } Q, \text{ i.e. } G(\bar{u}) \text{ is CO w.r.t. } (\bar{y}, \bar{u}) \text{, in } Q \text{. On the other hand, since } \bar{W} = \bar{W}_\bar{u} \text{ is CO } \text{ and the FrD of } G_{l}(\bar{u}) \text{, } (l = 0, 1, 2) \text{ exists for each } \bar{u} \in \bar{W} \text{ and its cont. (By The. 3.2 and hypotheses (A), (B) and (C)), then } G(\bar{u}) \text{ is CO w.r.t. } (\bar{y}, \bar{u}) \text{, in the CO set } \bar{W} \text{ and it has a cont. FrD, and satisfies } G(\bar{u}) \bar{\delta} \bar{u} \geq 0 \text{, which means } G(\bar{u}) \text{ has a minimum at } \bar{u}, \text{ i.e.: } G(\bar{u}) \leq \sup_{\bar{W}}(\bar{w}) , \forall \bar{w} \in \bar{W}. \\
\Rightarrow \lambda_o G_{0}(\bar{u}) + \lambda_1 G_{1}(\bar{u}) + \lambda_2 G_{2}(\bar{u}) \leq \lambda_o G_{0}(\bar{w}) + \lambda_1 G_{1}(\bar{w}) + \lambda_2 G_{2}(\bar{w}) \\
(44)
\]
Let \( \bar{W} = \bar{W}_A, \text{ with } \lambda_2 \geq 0, \text{ then from (38) and (44) gives: } \\
\lambda_o G_{0}(\bar{u}) \leq \lambda_o G_{0}(\bar{w}), \forall \bar{w} \in \bar{W} \Rightarrow G_{0}(\bar{u}) \leq G_{0}(\bar{w}), \forall \bar{w} \in \bar{W}, \text{ since } (\lambda_o > 0). \\
\Rightarrow \text{ Then } \bar{u} \text{ is QCCV.}
\]
Quaternary classical continuous optimal control consists of a quaternary nonlinear parabolic boundary value problem with a cost function and the constraints on state and control (equality constraint and inequality constraint). Under appropriate hypotheses, the quaternary classical continuous optimal control ruling by the quaternary nonlinear parabolic boundary value problem is demonstrated as a quaternary classical continuous optimal control vector that
satisfies the equality constraint and inequality constraint. Moreover, mathematical formulation of the quaternary adjoint equations related to the quaternary state equations is discovered so as their weak form. The derivation for the Fréchet derivative of the Hamiltonian is attained. Lastly, both the necessary conditions for optimality and sufficient conditions for optimality of the proposed problem are stated and proved.

5. Conclusion

This work studies the quaternary classical continuous optimal control ruling by a quaternary nonlinear parabolic boundary value problem. The existence of a quaternary classical continuous optimal control vector ruling by the considered the quaternary nonlinear parabolic boundary value problem satisfies the equality constraint, and inequality constraint is proved under appropriate hypotheses. Moreover, mathematical formulation of the quaternary adjoint equations related to the quaternary state equations has been discovered. The derivation of the Fréchet derivative is attained. Lastly, the necessary (conditions) theorem for optimality and the sufficient (conditions) for optimality of the proposed problem are stated and demonstrated.

References

