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The Classical Continuous Optimal Control for Quaternary Nonlinear Parabolic Boundary Value Problems with State Vector Constraints

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Abstract

This paper aims to study the quaternary classical continuous optimal control problem consisting of the quaternary nonlinear parabolic boundary value problem, the cost function, and the equality and inequality constraints on the state and the control. Under appropriate hypotheses, it is demonstrated that the quaternary classical continuous optimal control ruling by the quaternary nonlinear parabolic boundary value problem has a quaternary classical continuous optimal control vector that satisfies the equality constraint and inequality state and control constraint. Moreover, mathematical formulation of the quaternary adjoint equations related to the quaternary state equations is discovered, and then the weak form of the quaternary adjoint equations is obtained. Lastly, both the necessary conditions for optimality and sufficient conditions for optimality of the proposed problem are stated and proved. The derivation for the Fréchet derivative of the Hamiltonian is attained.

Keywords: Quaternary Classical Optimal Control, Quaternary Nonlinear Parabolic Boundary Value Problems, Necessary and Sufficient for Optimality Theorems.

1. Introduction

It is a well-known fact that optimal control problems (OCPs) are widely used in a variety of scientific fields, including biology [1], economics [2], robotics [3], Aircraft [4], and many others. OCPs are typically ruled by nonlinear ODEs (NLODEs) [5] or nonlinear PDEs (NLPDEs) [6]. During the last decade, great attention has been made to studying OCPs for system ruling by NLPDEs of elliptic, hyperbolic, and parabolic types [7-9]. Later, the study of this subject is expanded to include classical continuous optimal control problem (CCOCP) for systems ruling by couple NLPDEs and then recently by triple NLPDEs for the above three indicated types of NLPDEs [10 - 15]. As a result, these concerns made us study the quaternary classical continuous optimal control problem (QCCOCP) ruling by quaternary nonlinear parabolic boundary value problems (QNLPBVPs) with equality constraint (EQC) and inequality constraint (INEQC).

This paper is concerned with studying the QCCOCP ruling by a QNLPBVP; it begins with stating and demonstrating the existence theorem of a quaternary classical continuous optimal control vector (QCCOCV) ruling by the QNLPBVP with EQC and INEQC under suitable hypotheses. In addition, the mathematical formulation of the quaternary adjoint equations (QAEs) related to the quaternary state equations (QSEs) is discovered so as the weak form (WF). Moreover, the Fréchet derivative (FrD) of the Hamiltonian is attained. Lastly, both the necessary conditions for optimality (NCsTh) and sufficient conditions (SCsTh)) for optimality are stated and demonstrated.

Description of the problem

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded region with boundary $\Gamma = \partial\Omega,\ x = (x_1, x_2),\ Q = I \times \Omega,\ I = [0, T]\ ,\ \Gamma = \partial\Omega\ ,\ \Sigma = \Gamma \times \mathrm{I}$.

The QCCOC consists of the continuous quaternary state vector solution (CQSVS), which is expressed by the following QNLPBVP:

$$y_{1t} - \Delta y_1 + y_1 - y_2 + y_3 + y_4 = f_1(x, t, y_1, u_1)\ , \quad \text{in } Q \qquad (1)$$

$$y_{2t} - \Delta y_2 + y_1 + y_2 - y_3 - y_4 = f_2(x, t, y_2, u_2), \quad \text{in } Q \qquad (2)$$

$$y_{3t} - \Delta y_3 - y_1 + y_2 + y_3 + y_4 = f_3(x, t, y_3, u_3)\ , \quad \text{in } Q \qquad (3)$$

$$y_{4t} - \Delta y_4 - y_1 + y_2 - y_3 + y_4 = f_4(x, t, y_4, u_4)\ , \quad \text{in } Q \qquad (4)$$

With the following boundary conditions (BCs) and initial conditions (ICs):

$$y_i(x, t) = 0\ , \qquad \forall\ i = 1,2,3,4. \qquad \text{on } \Sigma \qquad (5)$$

$$y_i(x, 0) = y_i^0(x)\ , \qquad \forall\ i = 1,2,3,4. \qquad \text{on } \Omega \qquad (6)$$

Where $\vec{y} = (y_1, y_2, y_3, y_4) = \big(y_1(x, t), y_2(x, t), y_3(x, t), y_4(x, t)\big) \in \big(H^2(\overline{\mathrm{Q}})\big)^4$ is the quaternary state vector solution (QSVS), $\vec{u} = (u_1, u_2, u_3, u_4) = \big(u_1(x, t), u_2(x, t), u_3(x, t), u_4(x, t)\big) \in \big(L^2(\mathrm{Q})\big)^4$ is the QCCCV and $(f_1, f_2, f_3, f_4) = (f_1(x, t), f_2(x, t), f_3(x, t), f_4(x, t)) \in \big(L^2(\mathrm{Q})\big)^4$ is given, for all $x = (x_1, x_2) \in \Omega$.

The set of admissible control (SAC) is:

$$\vec{W}_A = \left\{\vec{w} \in \big(L^2(\mathrm{Q})\big)^4 \middle| \vec{w} \in \vec{U} \subset \mathbb{R}^4 \text{ a.e. in Q}\ , G_1(\vec{w}) = 0\ , G_2(\vec{w}) \le 0 \right\}.$$

The CF is:

$$G_0(\vec{u}) = \int_Q g_{01}(x, t, y_1, u_1)dxdt + \int_Q g_{02}(x, t, y_2, u_2)dxdt + \int_Q g_{03}(x, t, y_3, u_3)dxdt + \int_Q g_{04}(x, t, y_4, u_4)dxdt, \qquad (7.a)$$

The constraints on the state and the control (CSSC) are:

$$G_1(\vec{u}) = \int_Q g_{11}(x, t, y_1, u_1)dxdt + \int_Q g_{12}(x, t, y_2, u_2)dxdt + \int_Q g_{13}(x, t, y_3, u_3)dxdt + \int_Q g_{14}(x, t, y_4, u_4)dxdt = 0, \qquad (7.b)$$

$$G_2(\vec{u}) = \int_Q g_{21}(x, t, y_1, u_1)dxdt + \int_Q g_{22}(x, t, y_2, u_2)dxdt + \int_Q g_{23}(x, t, y_3, u_3)dxdt + \int_Q g_{24}(x, t, y_4, u_4)dxdt \le 0, \qquad (7.c)$$

Where $(y_1, y_2, y_3, y_4) = (y_{u1}, y_{u2}, y_{u3}, y_{u4})$ is the QSVS of ((1) – (6)) corresponding to the QCCCV (u_1, u_2, u_3, u_4).

Let $\vec{V} = V_1 \times V_2 \times V_3 \times V_4 = \left(\mathcal{H}_0^1(\Omega)\right)^4$ and $\vec{v} = (v_1, v_2, v_3, v_4) = (v_1(x), v_2(x), v_3(x), v_4(x))$.

$\vec{V} = \left\{\vec{v}: \vec{v} \in \left(\mathcal{H}_0^1(\Omega)\right)^4, \text{with } v_1 = v_2 = v_3 = v_4 = 0 \text{ on } \partial\Omega\right\}$.

The WF of the QSVEs:

The wf of $((1) - (6))$ with $\vec{y} \in \left(\mathcal{H}_0^1(\Omega)\right)^4$ is given by

$\langle y_{1t}, v_1\rangle + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1) = (f_1(y_1, u_1), v_1),$ (8.a)

$(y_1^0, v_1) = (y_1(0), v_1),$ (8.b)

$\langle y_{2t}, v_2\rangle + (\nabla y_2, \nabla v_2) + (y_1, v_2) + (y_2, v_2) - (y_3, v_2) - (y_4, v_2) = (f_2(y_2, u_2), v_2),$ (9.a)

$(y_2^0, v_2) = (y_2(0), v_2),$ (9.b)

$\langle y_{3t}, v_3\rangle + (\nabla y_3, \nabla v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3) = (f_3(y_3, u_3), v_3),$ (10.a)

$(y_3^0, v_3) = (y_3(0), v_3),$ (10.b)

$\langle y_{4t}, v_4\rangle + (\nabla y_4, \nabla v_4) - (y_1, v_4) + (y_2, v_4) - (y_3, v_4) + (y_4, v_4) = (f_4(y_4, u_4), v_4),$ (11.a)

$(y_4^0, v_4) = (y_4(0), v_4),$ (11.b)

The following hypotheses are important to study the QCCOC.

Hypotheses (A): Assume $\forall \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ that:

(i) $\boldsymbol{f_i}$ is Carathéodory type (Cara. T.) on $\boldsymbol{Q} \times (\mathbb{R})^{\mathbf{4}}$, and satisfies the following conditions w.r.t. $\boldsymbol{y_i}$ & $\boldsymbol{u_i}$, i.e.:

$|\boldsymbol{f_i}(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{y_i}, \boldsymbol{u_i})| \leq \boldsymbol{\eta_i}(\boldsymbol{x}, \boldsymbol{t}) + \boldsymbol{c_i}|\boldsymbol{y_i}| + \boldsymbol{\acute{c}_i}|\boldsymbol{u_i}|$, where $(\boldsymbol{x}, \boldsymbol{t}) \in \boldsymbol{Q}$, $\boldsymbol{c_i}, \boldsymbol{\acute{c}_i} > \mathbf{0}$ and $\boldsymbol{\eta_i} \in \boldsymbol{L^2}(\boldsymbol{Q}, \mathbb{R})$.

(ii) f_i satisfies Lipschitz condition (LC) w.r.t. y_i, i.e.:

$|f_i(x, t, y_i, u_i) - f_i(x, t, \bar{y}_i, u_i)| \leq L_i|y_i - \bar{y}_i|$. Where $(x, t) \in Q$, $y_i, \bar{y}_i, u_i \in \mathbb{R}$ and $L_i > 0$.

Theorem (2.1) [13]: (EUTh for the wf of the QSVEs)

With hypotheses (A), for each given QCCCV $\vec{u} \in \left(L^2(Q)\right)^4$, the wf ((8) – (11)) has a unique QSVS $\vec{y} \in \left(L^2(I, V)\right)^4$, with $\vec{y}_t \in \left(L^2(I, V^*)\right)^4$.

Hypotheses (B):

Suppose that for each $l = 0,1,2$ and $i = 1,2,3,4$, that g_{li} is of Cara. T. on $Q \times (\mathbb{R})^4$ and satisfies the following conditions w.r.t. y_i and u_i:

$|g_{li}(x, t, y_i, u_i)| \leq \eta_{li}(x, t) + c_{li1}(y_i)^2 + c_{li2}(u_i)^2$. Where $y_i, u_i \in \mathbb{R}$ with $\eta_{li} \in L^1(Q)$.

Lemma (2.1):

With hypotheses (B), for each $l = 0,1,2$, the functional $\vec{u} \longmapsto G_l(\vec{u})$ is cont. on $(L^2(Q))^4$.

Proof: The requirement result is gotten ($\forall l = 0,1,2$) directly from hypotheses (B) and Lemma 1.12 in [16].

Theorem (2.2) [16]: Consider the set $\overrightarrow{\mathrm{W}}_{\mathrm{A}} \neq \emptyset$, for each $i = 1,2,3,4$, the functions f_i has the form $f_i(x, t, y_i, u_i) = f_{i1}(x, t, y_i) + f_{i2}(x, t)u_i$

With $|f_{i1}(x, t, y_i)| \leq \eta_i(x, t) + c_i|y_i|$ where $\eta_i \in L^2(Q)$ and $|f_{i2}(x, t)| \leq k_i$,

If $\forall i = 1,2,3,4$, g_{0i} is convex (CO) w.r.t. u_i for fixed (x, t, y_i). Then there is a QCCOCV.

Hypotheses (C):

Assume that, for $l = 0,2$ and $i = 1,2,3,4$ $g_{l_i y_i}$ and $g_{l_i u_i}$ are of Cara. T. on $\times (\mathbb{R})^4$,

$\left|g_{l_i y_i}(x, t, y_i, u_i)\right| \leq \eta_{l_{i5}}(x, t) + c_{l_{i5}}|y_i| + \acute{c}_{l_{i5}}|u_i|$,

and $\left|g_{l_i u_i}(x, t, y_i, u_i)\right| \leq \eta_{l_{i6}}(x, t) + c_{l_{i6}}|y_i| + \acute{c}_{l_{i6}}|u_i|$.

Where $(x, t) \in Q$, $y_i, u_i \in \mathbb{R}$, $\eta_{l_{i5}}, \eta_{l_{i6}} \in L^2(Q)$.

Theorem (2.3) [16]: In addition to hypotheses (A), if $\vec{y}$ and $\vec{y}+\overrightarrow{\delta y}$ are the QSVS corresponding to the QCCCV $\vec{u}$, $\vec{u}+\overrightarrow{\delta u} \in (L^2(Q))^4$, resp., then
$\|\overrightarrow{\delta y}\|_{L^\infty(I,L^2(\Omega))} \leq$ M $\|\overrightarrow{\delta u}\|_{L^2(Q)}$, $\|\overrightarrow{\delta y}\|_{L^2(Q)} \leq$ M $\|\overrightarrow{\delta u}\|_{L^2(Q)}$, $\|\overrightarrow{\delta y}\|_{L^2(I,V)} \leq$ M $\|\overrightarrow{\delta u}\|_{L^2(Q)}$.

Theorem (2.4) (The Kuhn-Tucker-Lagrange conditions (KTL)) [10]:
Let U be a nonempty CO subset of a vector space X, K be a nonempty CO positive cone in a normed space Z, and $W=\{u\in U|G_1(u)=0, G_1(u)\in -K\}$.
The functional $G_0: U\to\mathbb{R}, G_1: U\to\mathbb{R}^m$, $G_2: U\to Z$ are $(m+1)-$ locally continuous at $u\in U$, and have $(m+1)-$ derivatives at u where $m\neq 0$. And if $m=0$, we assume that $DG_l(u), l=0{,}1{,}2$, are K -linear at the point u. If $G_0(u)$ has a minimum at u in W, then it satisfies the following KUTULA conditions, $\forall w\in W$:
There exists $\lambda_0\in\mathbb{R}, \lambda_1\in\mathbb{R}^m, \lambda_2\in\mathbb{Z}^*$, with $\lambda_0\geq 0, \lambda_2\geq 0$, $\sum_{l=0}^{2}|\lambda_l|=1$ s.t
$\lambda_0 DG_0(u,w-u)+\lambda_1^T DG_1(u,w-u)+\langle\lambda_2, DG_2(u,w-u)\rangle\geq 0,\ \langle\lambda_2, G_2(u)\rangle=0.$

Main Results

3. Existence of the QCCOCV and the FrD:

This section deals with the existence theorem of the QCCOCV, the discovery of the mathematical formulation for the QAEs and their WF is obtained, and the derivation of the FrD is derived under some appropriate hypotheses.

Theorem (3.1): Consider the set $\overrightarrow{W}_A\neq\emptyset$, the functions f_i, $\forall i=1{,}2{,}3{,}4$, has the form
$f_i(x,t,y_i,u_i)=f_{i1}(x,t,y_i)+f_{i2}(x,t)u_i$
With $|f_{i1}(x,t,y_i)|\leq \eta_i(x,t)+c_i|y_i|$ and $|f_{i2}(x,t)|\leq k_i$, where $\eta_i\in L^2(Q)$.
If $\forall i=1{,}2{,}3{,}4$, g_{1i} is independent of u_i, g_{0i} and g_{2i} are convex w.r.t. u_i for fixed (x,t,y_i).
Then there is a QCCOCV.

Proof: From the hypotheses on W_i and g_{1i} ($\forall i=1{,}2{,}3{,}4$), with using lemma (2.1) and the. 2.2, one can get that there is a QCCOCV with the EQC and INEQC.

Theorem (3.2):
We drop the index l in g_l and G_l . In addition to hypotheses (A), (B), and (C), the following adjoint $(z_1, z_2, z_3, z_4)=(z_{u_1}, z_{u_2}, z_{u_3}, z_{u_4})$ equations corresponding to the state $(y_1, y_2, y_3, y_4)=(y_{u_1}, y_{u_2}, y_{u_3}, y_{u_4})$ equations ((1) – (6)) are expressed by:

$$-z_{1t}-\Delta z_1+z_1+z_2-z_3-z_4=z_1 f_{y_1}(x,t,y_1,u_1)+g_{y_1}(x,t,y_1,u_1)\,, \quad (12)$$
$$-z_{2t}-\Delta z_2+z_2-z_1+z_3+z_4=z_1 f_{y_1}(x,t,y_1,u_1)+g_{y_1}(x,t,y_1,u_1)\,, \quad (13)$$
$$-z_{3t}-\Delta z_3+z_3+z_1-z_2-z_4=z_1 f_{y_1}(x,t,y_1,u_1)+g_{y_1}(x,t,y_1,u_1)\,, \quad (14)$$
$$-z_{4t}-\Delta z_4+z_4+z_1-z_2+z_3=z_1 f_{y_1}(x,t,y_1,u_1)+g_{y_1}(x,t,y_1,u_1)\,, \quad (15)$$
$$z_i(x,t)=0,\quad \forall i=1{,}2{,}3{,}4, \quad \text{on } \Sigma\,, \quad (16)$$
$$z_i(T)=0,\quad \forall i=1{,}2{,}3{,}4, \quad \text{on } \Gamma\,, \quad (17)$$

Also, the Hamiltonian is defined: $H(x,t,y_i,z_i,u_i)=\sum_{i=1}^{4} z_i f_i(x,t,y_i,u_i)+g_i(x,t,y_i,u_i)$,

Then the FrD of G is given by $\acute{G}(\vec{u})\cdot\overrightarrow{\delta u}=\int_Q \begin{pmatrix} z_1 f_{u_1}+g_{u_1} \\ z_2 f_{u_2}+g_{u_2} \\ z_3 f_{u_3}+g_{u_3} \\ z_4 f_{u_4}+g_{u_4} \end{pmatrix}\cdot\begin{pmatrix}\delta u_1\\ \delta u_2\\ \delta u_3\\ \delta u_4\end{pmatrix} dx.$

Proof:

Firstly, let $\vec{u}$ be a QCCCV, and $\vec{y}$ be its QSVS, and let $G(\vec{u})=\sum_{i=1}^{4}\int_Q g_i(x,t,y_i,u_i)dxdt$,

From the hypotheses on g_l ($l=1,2,3,4$), the FrD definition, the result of The. 2.3, and then using the Minkowski's inequality (MKIN), one can get that:

$$G(\vec{u}+\overrightarrow{\delta u})-G(\vec{u})=\int_Q(g_{y_1}\delta y_1+g_{u_1}\delta u_1)\,dxdt+\int_Q(g_{y_2}\delta y_2+g_{u_2}\delta u_2)\,dxdt+\int_Q(g_{y_3}\delta y_3+g_{u_3}\delta u_3)\,dxdt+\int_Q(g_{y_4}\delta y_4+g_{u_4}\delta u_4)\,dxdt+\varepsilon_6(\overrightarrow{\delta u})\|\overrightarrow{\delta u}\|_{L^2(Q)} \quad (18)$$

Where $\varepsilon_6(\overrightarrow{\delta u})=\varepsilon_2(\overrightarrow{\delta u})+\varepsilon_3(\overrightarrow{\delta u})+\varepsilon_4(\overrightarrow{\delta u})+\varepsilon_5(\overrightarrow{\delta u})\longrightarrow 0$ as $\|\overrightarrow{\delta u}\|_{L^2(Q)}\longrightarrow 0$.

On the other hand, the wf of the QAEs for $v_i\in V$, $\forall\, i=1,2,3,4$ is given by:

$$-\langle z_{1t},v_1\rangle+(\nabla z_1,\nabla v_1)+(z_1,v_1)+(z_2,v_1)-(z_3,v_1)-(z_4,v_1)=(z_1f_{1y_1},v_1)+(g_{1y_1},v_1), \quad (19)$$

$$-\langle z_{2t},v_2\rangle+(\nabla z_2,\nabla v_2)+(z_2,v_2)-(z_1,v_2)+(z_3,v_2)+(z_4,v_2)=(z_2f_{2y_2},v_2)+(g_{2y_2},v_2), \quad (20)$$

$$-\langle z_{3t},v_3\rangle+(\nabla z_3,\nabla v_3)+(z_3,v_3)+(z_1,v_3)-(z_2,v_3)-(z_4,v_3)=(z_3f_{3y_3},v_3)+(g_{3y_3},v_3), \quad (21)$$

$$-\langle z_{4t},v_4\rangle+(\nabla z_4,\nabla v_4)+(z_4,v_4)+(z_1,v_4)-(z_2,v_4)+(z_3,v_4)=(z_4f_{4y_4},v_4)+(g_{4y_4},v_4), \quad (22)$$

The existence of a unique solution of ((19) – (22)) can be proved by the same manner which is used in the proof of the unique of the QSVS.

Now, substituting $v_i=\delta y_i$, $\forall i=1,2,3,4$ in ((19) $-$ (22)) resp., then taking the integrating both sides (IBS) from 0 to T, lastly, applying integration by part (IBP) for the 1st terms of each resulting equation, to get that:

$$\int_0^T\langle\delta y_1,z_1\rangle\,dt+\int_0^T[(\nabla z_1,\nabla\delta y_1)+(z_1,\delta y_1)+(z_2,\delta y_1)-(z_3,\delta y_1)-(z_4,\delta y_1)]\,dt=\int_0^T[(z_1f_{1y_1},\delta y_1)+(g_{1y_1},\delta y_1)]dt, \quad (23)$$

$$\int_0^T\langle\delta y_2,z_2\rangle\,dt+\int_0^T[(\nabla z_2,\nabla\delta y_2)+(z_2,\delta y_2)-(z_1,\delta y_2)+(z_3,\delta y_2)+(z_4,\delta y_2)]dt=\int_0^T[(z_2f_{2y_2},\delta y_2)+(g_{2y_2},\delta y_2)]dt, \quad (24)$$

$$\int_0^T\langle\delta y_3,z_3\rangle\,dt+\int_0^T[(\nabla z_3,\nabla\delta y_3)+(z_3,\delta y_3)+(z_1,\delta y_3)-(z_2,\delta y_3)-(z_4,\delta y_3)]dt=\int_0^T[(z_3f_{3y_3},\delta y_3)+(g_{3y_3},\delta y_3)]dt, \quad (25)$$

$$\int_0^T\langle\delta y_4,z_4\rangle\,dt+\int_0^T[(\nabla z_4,\nabla\delta y_4)+(z_4,\delta y_4)+(z_1,\delta y_4)-(z_2,\delta y_4)+(z_3,\delta y_4)]dt=\int_0^T[(z_4f_{4y_4},\delta y_4)+(g_{4y_4},\delta y_4)]dt, \quad (26)$$

Also, substituting $v_i=z_i$, $\forall i=1,2,3,4$ in ((8.a) $-$ (11.a)) resp., then IBS w.r.t. t from 0 to T, to obtain:

$$\int_0^T\langle\delta y_{1t},z_1\rangle\,dt+\int_0^T[(\nabla\delta y_1,\nabla z_1)+(\delta y_1,z_1)-(\delta y_2,z_1)+(\delta y_3,z_1)+(\delta y_4,z_1)]\,dt=\int_0^T(f_1(y_1+\delta y_1,u_1+\delta u_1),z_1)dt-\int_0^T(f_1(y_1,u_1),z_1)dt, \quad (27)$$

$$\int_0^T\langle\delta y_{2t},z_2\rangle\,dt+\int_0^T[(\nabla\delta y_2,\nabla z_2)+(\delta y_1,z_2)+(\delta y_2,z_2)-(\delta y_3,z_2)-(\delta y_4,z_2)]\,dt=\int_0^T(f_2(y_2+\delta y_2,u_2+\delta u_2),z_2)dt-\int_0^T(f_2(y_2,u_2),z_2)dt, \quad (28)$$

$\int_0^T \langle \delta y_{3t}, z_3 \rangle \, dt + \int_0^T [(\nabla \delta y_3, \nabla z_3) - (\delta y_1, z_3) + (\delta y_2, z_3) + (\delta y_3, z_3) + (\delta y_4, z_3)] \, dt =$
$\int_0^T (f_3(y_3 + \delta y_3, u_3 + \delta u_3), z_3) dt - \int_0^T (f_3(y_3, u_3), z_3) dt,$ (29)
$\int_0^T \langle \delta y_{4t}, z_4 \rangle \, dt + \int_0^T [(\nabla \delta y_4, \nabla z_4) - (\delta y_1, z_4) + (\delta y_2, z_4) - (\delta y_3, z_4) + (\delta y_4, z_4)] \, dt =$
$\int_0^T (f_4(y_4 + \delta y_4, u_4 + \delta u_4), z_4) dt - \int_0^T (f_4(y_4, u_4), z_4) dt,$ (30)
Using the hypotheses on f_i (for each $i = 1{,}23{,}4$) , the FrD of it exists, then from the result of The. 2.3, and the MKIN, the followings are yielded:
$\int_0^T \langle \delta y_{1t}, z_1 \rangle \, dt + \int_0^T [(\nabla \delta y_1, \nabla z_1) + (\delta y_1, z_1) - (\delta y_2, z_1) + (\delta y_3, z_1) + (\delta y_4, z_1)] \, dt =$
$\int_0^T (f_{1y_1} \delta y_1 + f_{1u_1} \delta u_1, z_1) dt + \varepsilon_{12}(\overrightarrow{\delta u}) \|\overrightarrow{\delta u}\|_{L^2(Q)},$
(31)
$\int_0^T \langle \delta y_{2t}, z_2 \rangle \, dt + \int_0^T [(\nabla \delta y_2, \nabla z_2) + (\delta y_1, z_2) + (\delta y_2, z_2) - (\delta y_3, z_2) - (\delta y_4, z_2)] \, dt =$
$\int_0^T (f_{2y_2} \delta y_2 + f_{2u_2} \delta u_2, z_2) dt + \varepsilon_{22}(\overrightarrow{\delta u}) \|\overrightarrow{\delta u}\|_{L^2(Q)},$ (32)
$\int_0^T \langle \delta y_{3t}, z_3 \rangle \, dt + \int_0^T [(\nabla \delta y_3, \nabla z_3) - (\delta y_1, z_3) + (\delta y_2, z_3) + (\delta y_3, z_3) + (\delta y_4, z_3)] \, dt =$
$\int_0^T (f_{3y_3} \delta y_3 + f_{3u_3} \delta u_3, z_3) dt + \varepsilon_{32}(\overrightarrow{\delta u}) \|\overrightarrow{\delta u}\|_{L^2(Q)},$ (33)
$\int_0^T \langle \delta y_{4t}, z_4 \rangle \, dt + \int_0^T [(\nabla \delta y_4, \nabla z_4) - (\delta y_1, z_4) + (\delta y_2, z_4) - (\delta y_3, z_4) + (\delta y_4, z_4)] \, dt =$
$\int_0^T (f_{4y_4} \delta y_4 + f_{4u_4} \delta u_4, z_4) dt + \varepsilon_{42}(\overrightarrow{\delta u}) \|\overrightarrow{\delta u}\|_{L^2(Q)},$ (34)
By subtracting ((31) – (34)) from ((23) − (26)) resp., and adding the attained equations, one obtains:
$\int_0^T [(f_{1u_1} \delta u_1, z_1) + (f_{2u_2} \delta u_2, z_2) + (f_{3u_3} \delta u_3, z_3) + (f_{4u_4} \delta u_4, z_4)] dt + \varepsilon_5(\overrightarrow{\delta u}) \|\overrightarrow{\delta u}\|_{L^2(Q)} =$
$\int_0^T [(g_{1y_1}, \delta y_1) + (g_{2y_2}, \delta y_2) + (g_{3y_3}, \delta y_3) + (g_{4y_4}, \delta y_4)] dt,$ (35)
Where $\varepsilon_5(\overrightarrow{\delta u}) = \varepsilon_{12}(\overrightarrow{\delta u}) + \varepsilon_{22}(\overrightarrow{\delta u}) + \varepsilon_{32}(\overrightarrow{\delta u}) + \varepsilon_{42}(\overrightarrow{\delta u}) \longrightarrow 0$, as $\|\overrightarrow{\delta u}\|_{L^2(Q)} \longrightarrow 0$,
Now, by substituting (35) in (18)**,** one gets:

$$G(\vec{u} + \overrightarrow{\delta u}) - G(\vec{u}) = \int_Q [(z_1 f_{1u_1} + g_{1u_1}) \delta u_1 + (z_2 f_{2u_2} + g_{2u_2}) \delta u_2 + (z_3 f_{3u_3} + g_{3u_3}) \delta u_3 + (z_4 f_{4u_4} + g_{4u_4}) \delta u_4] dx dt + \varepsilon_7(\overrightarrow{\delta u}) \|\overrightarrow{\delta u}\|_{L^2(Q)} \quad 36)$$

Where $\varepsilon_7(\overrightarrow{\delta u}) = \varepsilon_5(\overrightarrow{\delta u}) + \varepsilon_6(\overrightarrow{\delta u}) \longrightarrow 0$, as $\|\overrightarrow{\delta u}\|_{L^2(Q)} \longrightarrow 0$,
Using the FrD definition of G, one gets:

$$G(\vec{u} + \overrightarrow{\delta u}) - G(\vec{u}) = (\acute{G}(\vec{u}), \overrightarrow{\delta u}) + \varepsilon_7(\overrightarrow{\delta u}) \|\overrightarrow{\delta u}\|_{L^2(Q)} \quad (37)$$

From (36) and (37), one can get:

$$(\acute{G}(\vec{u}), \overrightarrow{\delta u}) = \int_Q \begin{pmatrix} z_1 f_{1u_1} + g_{1u_1} \\ z_2 f_{2u_2} + g_{2u_2} \\ z_3 f_{3u_3} + g_{3u_3} \\ z_4 f_{4u_4} + g_{4u_4} \end{pmatrix} \cdot \begin{pmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \\ \delta u_4 \end{pmatrix} dx \, .$$

4. The NCsTh and The SCsTh for Optimality:

This section deals with the state and demonstration of the NCsTh, so as the SCsTh, under some additional hypotheses.

Theorem (4.1): NCsTh for Optimality:

(a) In addition to hypotheses (A), (B) and (C), if $\vec{u} \in \vec{W}_A$ is QCCOCV, then there is ″multiplier ″$\lambda_l \in \mathbb{R}, l = 0,1,2$, with $\lambda_0 \geq 0$, $\lambda_2 \geq 0$, $\sum_{l=0}^{2}|\lambda_l| = 1$, s.t. the following Lagrange -Kuhn-Tucker conditions (LKT) conditions are held:

$$\int_Q H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u}) \cdot \overrightarrow{\delta u} dxdt \geq 0, \forall \vec{w} \in \vec{W},\ \overrightarrow{\delta u} = \vec{w} - \vec{u} \quad (38.a)$$

Where $g_i = \sum_{l=0}^{2} \lambda_l g_{li}$, $\forall i = 1,2,3,4$ in the definition of H, and also $\lambda_2 G_2(\vec{u}) = 0$, (38.b)

(b) (Minimum principle in weak form) If $\vec{W}$ is of the form

$\vec{W} = \{\vec{w} \in (L^2(Q,\mathbb{R}))^4 | \vec{w}(x,t) \in \vec{U} \text{a.e. on } Q\}$, with $\vec{U} \subset \mathbb{R}^2$.

Then, (37.a) is equivalent to the minimum principle in point-wise form (MPPWF)

$$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{u}(t) = \min_{\vec{w}\in\vec{U}} H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{w} \text{ a.e. on } Q \quad (39)$$

Proof: (a) The functional $G_l(\vec{u})$ is ρ −locall cont. at each $\vec{u} \in \vec{W}$, $\forall l = 0,1,2$ and for every ρ (by hypotheses (A) , (B) and (C) and Lemma 2.1), and $G_l(\vec{u})$ is ρ −differentiable at each $\vec{u} \in \vec{W}$, $\forall \rho$ (by hypotheses (A) , (B) and (C) and The. 3.2) and since $\vec{W} \subset \left(L^2(Q)\right)^4$, $L^2(Q)$ is open, then $DG_l(\vec{u}, \vec{w} - \vec{u}) = \acute{G}_l(\vec{u})(\vec{w} - \vec{u})$, $l = 0,1,2$,

And since $\vec{u} \in \vec{W}_A$ is QCCOCV, by The. 2.4, there is $\lambda_l \in \mathbb{R}$, $l = 0,1,2$, with $\lambda_0 \geq 0$, $\lambda_2 \geq 0$, $\sum_{l=0}^{2}|\lambda_l| = 1$ s.t. (38a&b) are held.

Again by The. 3.2, setting $\overrightarrow{\delta u} = \vec{w} - \vec{u}$ and substituting the FrD of $\acute{G}_l$, $l = 0,1,2$ in (38.a), one has:

$$\sum_{i=1}^{4} \int_Q [(\lambda_0 z_{0i} + \lambda_1 z_{1i} + \lambda_2 z_{2i}) f_{iu_i} + \left(\lambda_0 g_{0iu_i} + \lambda_1 g_{1iu_i} + \lambda_2 g_{2iu_i}\right)] \delta u_i dxdt \geq 0,$$

$$\Rightarrow \sum_{i=1}^{4} \int_Q [\left(z_i f_{iu_i} + g_{iu_i}\right)] \delta u_i dxdt \geq 0,$$

Where $g_i = \sum_{l=0}^{2} \lambda_l g_{li}$, $z_i = \sum_{l=0}^{2} \lambda_l z_{li}$, $\forall i = 1,2,3,4$.

$$\Rightarrow \quad \int_Q \begin{pmatrix} z_1 f_{1u_1} + g_{1u_1} \\ z_2 f_{2u_2} + g_{2u_2} \\ z_3 f_{3u_3} + g_{3u_3} \\ z_4 f_{4u_4} + g_{4u_4} \end{pmatrix} \cdot \begin{pmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \\ \delta u_4 \end{pmatrix} dxdt \geq 0 \text{ , or } \int_Q H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u}) \cdot \overrightarrow{\delta u} dxdt \geq 0.$$

(ii) To prove that (38.a) is equivalent to (39):

Let $\vec{W}_{\vec{U}} = \{\vec{w} \in (L^2(Q,\mathbb{R}))^4 | \vec{w}(x,t) \in \vec{U} \text{a.e. in } Q\}$ with $\vec{U} \subset \mathbb{R}^2$, let $\{\vec{w}_k\}$ dense in a set $\vec{W}_{\vec{U}}$, μ is "Lebesgue measure" on Q and let $S \subset Q$ be a measurable set s.t.:

$$\vec{w}(x,t) = \begin{cases} \vec{w}_k(x,t) & , \text{ if } (x,t) \in S \\ \vec{u}(x,t) & , \text{ if } (x,t) \notin S \end{cases}$$

Therefore (38.a) becomes:

$\int_S H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\,(\vec{w}_k - \vec{u}) \geq 0$, $\forall S$.

Using the 3.2, to obtain: $H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})(\vec{w}_k - \vec{u}) \geq 0$, a.e. in Q,

$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})(\vec{w}_k - \vec{u}) \geq 0$, in $P = \bigcap_k P_k$, where $P_k = Q - Q_k$ with $\mu(Q_k) = 0$, $\forall k$, since P is independent of k , hence $\mu(Q - P) = \mu(\bigcup_k Q_k) = 0$, from the density of $\{\vec{w}_k\}$ in $\vec{W}_{\vec{U}}$, one has $H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})(\vec{w} - \vec{u}) \geq 0$, a.e. in Q.

$\Rightarrow H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{u} = \min_{\vec{w}\in\vec{U}} H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{w}$, $\forall \vec{w} \in \vec{W}$, a.e. in Q.

Conversely, if

$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{u} = \min_{\vec{w}\in\vec{U}} H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{w}$, a.e. on Q

$\Rightarrow H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})(\vec{w}-\vec{u}) \geq 0$, $\forall \vec{w} \in \vec{W}$, a.e. on Q

$\Rightarrow \int_Q H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\overrightarrow{\delta u}\,dxdt \geq 0$, $\forall \vec{w} \in \vec{W}$.

Theorem (4.2) :(SCsTh for Optimality)

In addition to hypotheses (A), (B) and (C), suppose that $\vec{W} = \vec{W}_{\vec{U}}$ is CO, f_i , $\forall i = 1,2,3,4$ and g_{1i} are affine w.r.t. (y_i, u_i) in Q, and g_{0i} and g_{2i} are CO w.r.t. (y_i, u_i) in Q $\forall i = 1,2,3,4$. Then the NCs in The. 4.1 with $\lambda_0 > 0$ are also sufficient.

Proof: From The. 4.1, $DG_l(\vec{u}, \vec{w}-\vec{u}) = \acute{G}_l(\vec{u})(\vec{w}-\vec{u})$, $l = 0,1,2$,

Assume that $\vec{u}$ satisfies (38) and $\vec{u} \in \vec{W}_A$, i.e.:

$\int_Q H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\overrightarrow{\delta u}\,dxdt \geq 0$, $\forall \vec{w} \in \vec{W}$. And $\lambda_2 G_2(\vec{u}) = 0$. Let $(\vec{u}) = \sum_{l=0}^{2} \lambda_l G_l(\vec{u})$, then

$$\acute{G}(\vec{u}) \cdot \overrightarrow{\delta u} = \sum_{l=0}^{2} \lambda_l \acute{G}_l(\vec{u}) \cdot \overrightarrow{\delta u}$$

$$= \lambda_0 \int_Q \sum_{i=1}^{4} \left(z_{0i} f_{iu_i} + g_{0iu_i}\right) \delta u_i\,dxdt + \lambda_1 \int_Q \sum_{i=1}^{4} \left(z_{1i} f_{iu_i} + g_{1iu_i}\right) \delta u_i\,dxdt + \lambda_2 \int_Q \sum_{i=1}^{4} \left(z_{2i} f_{iu_i} + g_{2iu_i}\right) \delta u_i\,dxdt,$$

$$= \int_Q H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\overrightarrow{\delta u}\,dxdt \geq 0,$$

Now, to demonstrate $\vec{u} \longmapsto \vec{y}_{\vec{u}}$ is convex – linear (COL), since $\forall i = 1,2,3,4$, f_i is affine from the HYPOTHESES on f_i , $\forall i = 1,2,3,4$: $f_i(x,t,y_i,u_i) = f_{i1}(x,t)y_i + f_{i2}(x,t)u_i + f_{i3}(x,t)$, $\forall i = 1,2,3,4$.

Let $\vec{u} = (u_1,u_2,u_3,u_4)$ & $\vec{\bar{u}} = (\bar{u}_1,\bar{u}_2,\bar{u}_3,\bar{u}_4)$ be two given QCCCVs and from The. 2.1, $\vec{y} = \left(y_{u_1}, y_{u_2}, y_{u_3}, y_{u_4}\right) = (y_1,y_2,y_3,y_4)$ & $\vec{\bar{y}} = \left(\bar{y}_{\bar{u}_1}, \bar{y}_{\bar{u}_2}, \bar{y}_{\bar{u}_3}, \bar{y}_{\bar{u}_4}\right) = (\bar{y}_1,\bar{y}_2,\bar{y}_3,\bar{y}_4)$are their corresponding QSVS, precisely from (1),

$y_{1t} - \Delta y_1 + y_1 - y_2 + y_3 + y_4 = f_{11}(x,t)y_1 + f_{12}(x,t)u_1 + f_{13}(x,t)$,

$y_1(x,0) = y_1^0(x)$,

$\bar{y}_{1t} - \Delta \bar{y}_1 + \bar{y}_1 - \bar{y}_2 + \bar{y}_3 + \bar{y}_4 = f_{11}(x,t)\bar{y}_1 + f_{12}(x,t)\bar{u}_1 + f_{13}(x,t)$,

$\bar{y}_1(x,0) = y_1^0(x)$,

By MBS the 1^{st} above equation and its IC by $\alpha \in [0,1]$, and the 2^{nd} equation and its IC by $(1-\alpha)$, and adding the attained equations and their attained ICs , one gets that:

$$(\alpha y_1 + (1-\alpha)\bar{y}_1)_t - \Delta(\alpha y_1 + (1-\alpha)\bar{y}_1) + (\alpha y_1 + (1-\alpha)\bar{y}_1) - (\alpha y_2 + (1-\alpha)\bar{y}_2) + (\alpha y_3 + (1-\alpha)\bar{y}_3) + (\alpha y_4 + (1-\alpha)\bar{y}_4) = f_{11}(x,t)(\alpha y_1 + (1-\alpha)\bar{y}_1) + f_{12}(x,t)(\alpha u_1 + (1-\alpha)\bar{u}_1) + f_{13}(x,t) \quad (40.a)$$

$$\alpha y_1(x,0) + (1-\alpha)\bar{y}_1(x,0) = y_1^0(x) \quad (40.b)$$

By the same way, one obtains that:

$$(\alpha y_2 + (1-\alpha)\bar{y}_2)_t - \Delta(\alpha y_2 + (1-\alpha)\bar{y}_2) + (\alpha y_2 + (1-\alpha)\bar{y}_2) + (\alpha y_1 + (1-\alpha)\bar{y}_1) - (\alpha y_3 + (1-\alpha)\bar{y}_3) - (\alpha y_4 + (1-\alpha)\bar{y}_4) = f_{21}(x,t)(\alpha y_2 + (1-\alpha)\bar{y}_2) + f_{22}(x,t)(\alpha u_2 + (1-\alpha)\bar{u}_2) + f_{23}(x,t) \quad (41.a)$$

$$\alpha y_2(x,0) + (1-\alpha)\bar{y}_2(x,0) = y_2^0(x) \quad (41.b)$$

$$(\alpha y_3 + (1-\alpha)\bar{y}_3)_t - \Delta(\alpha y_3 + (1-\alpha)\bar{y}_3) + (\alpha y_3 + (1-\alpha)\bar{y}_3) - (\alpha y_1 + (1-\alpha)\bar{y}_1) + (\alpha y_2 + (1-\alpha)\bar{y}_2) + (\alpha y_4 + (1-\alpha)\bar{y}_4) = f_{31}(x,t)(\alpha y_3 + (1-\alpha)\bar{y}_3) + f_{32}(x,t)(\alpha u_3 + (1-\alpha)\bar{u}_3) + f_{33}(x,t) \quad (42.a)$$

$$\alpha y_3(x,0) + (1-\alpha)\bar{y}_3(x,0) = y_3^0(x) \quad (42.b)$$

$$(\alpha y_4 + (1-\alpha)\bar{y}_4)_t - \Delta(\alpha y_4 + (1-\alpha)\bar{y}_4) + (\alpha y_4 + (1-\alpha)\bar{y}_4) - (\alpha y_1 + (1-\alpha)\bar{y}_1) + (\alpha y_2 + (1-\alpha)\bar{y}_2) - (\alpha y_3 + (1-\alpha)\bar{y}_3) = f_{41}(x,t)(\alpha y_4 + (1-\alpha)\bar{y}_4) + f_{42}(x,t)(\alpha u_4 + (1-\alpha)\bar{u}_4) + f_{43}(x,t) \quad (43.a)$$

$\alpha y_4(x,0) + (1-\alpha)\bar{y}_4(x,0) = y_4^0(x)$ (43.b)

From equations ((40) − (43)), the QCCCV $\vec{\tilde{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4)$, with $\vec{\tilde{u}} = \alpha\vec{u} + (1-\alpha)\vec{\bar{u}}$ has the corresponding QSVS, $\vec{\tilde{y}} = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$, $\vec{\tilde{y}} = \alpha\vec{y} + (1-\alpha)\vec{\bar{y}}$, i.e.:

$\tilde{y}_{1t} - \Delta\tilde{y}_1 + \tilde{y}_1 - \tilde{y}_2 + \tilde{y}_3 + \tilde{y}_4 = f_{11}(x,t)\tilde{y}_1 + f_{12}(x,t)\tilde{u}_1 + f_{13}(x,t),$

$\tilde{y}_1(x,0) = y_1^0(x),$

$\tilde{y}_{2t} - \Delta\tilde{y}_2 + \tilde{y}_2 + \tilde{y}_1 - \tilde{y}_3 - \tilde{y}_4 = f_{21}(x,t)\tilde{y}_2 + f_{22}(x,t)\tilde{u}_2 + f_{23}(x,t),$

$\tilde{y}_2(x,0) = y_2^0(x),$

$\tilde{y}_{3t} - \Delta\tilde{y}_3 + \tilde{y}_3 - \tilde{y}_1 + \tilde{y}_2 + \tilde{y}_4 = f_{31}(x,t)\tilde{y}_3 + f_{32}(x,t)\tilde{u}_3 + f_{33}(x,t),$

$\tilde{y}_3(x,0) = y_3^0(x),$

$\tilde{y}_{4t} - \Delta\tilde{y}_4 + \tilde{y}_4 - \tilde{y}_1 + \tilde{y}_2 - \tilde{y}_3 = f_{41}(x,t)\tilde{y}_4 + f_{42}(x,t)\tilde{u}_4 + f_{43}(x,t),$

$\tilde{y}_4(x,0) = y_4^0(x),$

Therefore $\vec{u} \longmapsto \vec{y}_{\vec{u}}$ is COL w.r.t. $(\vec{y}, \vec{u})$ in Q.

From hypotheses on g_{1i} in Q for each $i = 1,2,3,4$: $g_{1i}(x,t,y_i,u_i) = h_{1i}(x,t)y_i + h_{2i}(x,t)u_i + h_{3i}(x,t)$.

Now, to show g_{1i} is COL w.r.t. (y_i, u_i) , in Q, since

$G_1\big(\vec{u} + (1-\alpha)\vec{\bar{u}}\big) = \sum_{i=1}^{4}[\int_Q g_{1i}\big(x,t,y_{i(\alpha u_i+(1-\alpha)\bar{u}_i)}, \alpha u_i + (1-\alpha)\bar{u}_i\big)\,dxdt]$

$= \sum_{i=1}^{4}\{\int_Q[h_{1i}(x,t)y_{i(\alpha u_i+(1-\alpha)\bar{u}_i)}]dxdt + \int_Q[h_{2i}(x,t)(\alpha u_i + (1-\alpha)\bar{u}_i) + h_{3i}(x,t)]dxdt\},$

Since $\vec{u} \longmapsto \vec{y}_{\vec{u}}$ is COL. Then $G_1(\vec{u})$ is COL w.r.t. $(\vec{y}, \vec{u})$, in Q, i. e.:

$$G_1\big(\vec{u} + (1-\alpha)\vec{\bar{u}}\big) = \sum_{i=1}^{4}\{\int_Q[h_{1i}(x,t)(\alpha y_i + (1-\alpha)\bar{y}_i)]dxdt + \int_Q[h_{2i}(x,t)(\alpha u_i + (1-\alpha)\bar{u}_i) + h_{3i}(x,t)]dxdt\},$$

$$= \alpha\sum_{i=1}^{4}\int_Q[h_{1i}(x,t)y_i + h_{2i}(x,t)u_i + h_{3i}(x,t)]dxdt + (1-\alpha)\sum_{i=1}^{4}\int_Q[h_{1i}(x,t)\bar{y}_i + h_{2i}(x,t)\bar{u}_i + h_{3i}(x,t)]dxdt,$$

$$= \alpha G_1(\vec{u}) + (1-\alpha)G_1\big(\vec{\bar{u}}\big),$$

Since g_{0i} &g_{2i} are CO w.r.t.(y_i, u_i) , in Q , $\forall i = 1,2,3,4$, then $\sum_{i=1}^{4}\int_Q g_{0i}\,dxdt$ and $\sum_{i=1}^{4}\int_Q g_{2i}\,dxdt$ are CO w.r.t. (y_i, u_i), in $Q, \forall i = 1,2,3,4$, and then $G_0(\vec{u})$ and $G_2(\vec{u})$ are CO w.r.t. $(\vec{y}, \vec{u})$, in Q, i.e. $G(\vec{u})$ is CO w.r.t. $(\vec{y}, \vec{u})$, in Q . On the other hand, since $\vec{W} = \vec{W}_{\vec{U}}$ is CO , and the FrD of $G_l(\vec{u})$, $(l = 0,1,2)$ exists for each $\vec{u} \in \vec{W}$ and its cont. (By The. 3.2 and hypotheses (A),(B) and (C)), then $G(\vec{u})$ is CO w.r.t. $(\vec{y}, \vec{u})$, in the CO set $\vec{W}$ and it has a cont. FrD, and satisfies $\acute{G}(\vec{u})\overrightarrow{\delta u} \geq 0$, which means $G(\vec{u})$ has a minimum at $\vec{u}$, i.e.: $G(\vec{u}) \leq G(\vec{w})$, $\forall \vec{w} \in \vec{W}$,

$$\Rightarrow \lambda_0 G_0(\vec{u}) + \lambda_1 G_1(\vec{u}) + \lambda_2 G_2(\vec{u}) \leq \lambda_0 G_0(\vec{w}) + \lambda_1 G_1(\vec{w}) + \lambda_2 G_2(\vec{w}) \quad (44)$$

Let $\vec{w} \in \vec{W}_A$, with $\lambda_2 \geq 0$, then from (38) and (44) gives:

$\lambda_0 G_0(\vec{u}) \leq \lambda_0 G_0(\vec{w}), \forall \vec{w} \in \vec{W} \Rightarrow G_0(\vec{u}) \leq G_0(\vec{w}), \forall \vec{w} \in \vec{W}$, since $(\lambda_0 > 0)$.

$\Longrightarrow$ Then $\vec{u}$ is QCCOC.

Quaternary classical continuous optimal control consists of a quaternary nonlinear parabolic boundary value problem with a cost function and the constraints on state and control (equality constraint and inequality constraint). Under appropriate hypotheses, the quaternary classical continuous optimal control ruling by the quaternary nonlinear parabolic boundary value problem is demonstrated as a quaternary classical continuous optimal control vector that satisfies the equality constraint and inequality constraint. Moreover, mathematical formulation of the quaternary adjoint equations related to the quaternary state equations is discovered so as their weak form. The derivation for the Fréchet derivative of the Hamiltonian is attained. Lastly, both the necessary conditions for optimality and sufficient conditions for optimality of the proposed problem are stated and proved.

5. Conclusion

This work studies the quaternary classical continuous optimal control ruling by a quaternary nonlinear parabolic boundary value problem. The existence of a quaternary classical continuous optimal control vector ruling by the considered the quaternary nonlinear parabolic boundary value problem satisfies the equality constraint, and inequality constraint is proved under appropriate hypotheses. Moreover, mathematical formulation of the quaternary adjoint equations related to the quaternary state equations has been discovered. The derivation of the Fréchet derivative is attained. Lastly, the necessary (conditions) theorem for optimality and the sufficient (conditions) for optimality of the proposed problem are stated and demonstrated.

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