The Optimal Classical Continuous Control Quaternary Vector of Quaternary Nonlinear Hyperbolic Boundary Value Problem

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Abstract
This work is concerned with studying the optimal classical continuous control quaternary vector problem. It is consisted of; the quaternary nonlinear hyperbolic boundary value problem and the cost functional. At first, the weak form of the quaternary nonlinear hyperbolic boundary value problem is obtained. Then under suitable hypotheses, the existence theorem of a unique state quaternary vector solution for the weak form where the classical continuous control quaternary vector is considered known is stated and demonstrated by employing the method of Galerkin and the compactness theorem. In addition, the continuity operator between the state quaternary vector solution of the weak form and the corresponding classical continuous control quaternary vector is demonstrated in three different infinite dimensional spaces (Hilbert spaces). Furthermore, with suitable hypotheses, the existence theorem of an optimal classical continuous control quaternary vector dominated by the weak form of the quaternary nonlinear hyperbolic boundary value problem is stated and demonstrated.

Keywords: Optimal Classical Continuous Control Quaternary Vector, Quaternary Nonlinear Hyperbolic Boundary Value Problem, Weak form.

1. Introduction
Different applications in real-life are classified as optimal control problems (OCPs). For example, in medicine [1], economics [2], robotics [3], Aircraft [4], and many other fields. Usually, this importance encouraged many researchers to be interested in studying OCPs in general and optimal classical continuous control problems (OCCC) in particular. During the last decade, great attention has been made to studying the subject of OCCC for a system dominated by nonlinear PDEs (NLPDEs).
of the three types elliptic [5], hyperbolic [6], and parabolic [7]. Latter, the study of this subject expanded to include OCCCP for systems dominated by a couple of NLPDEs of their three types [8-10]; through recent years, these studies for these three types expanded to deal with OCCCP for systems dominated by triple NLPDEs [11-13]. All these studies encouraged us to investigate the OCCCP dominated by QNLHBVP.

This article first concerns the mathematical formulation for the optimal classical continuous control quaternary vector problem. Then the existence theorem (ETH) of a unique state quaternary vector solution (SQVS) for the weak form (WF) “of the quaternary nonlinear hyperbolic boundary value problem (QNLHBVP)” is stated and demonstrated using the method of Galerkin (MGA) and the Aubin compactness theorem (ACTH) when the classical continuous control quaternary vector (CCCQV) is fixed under suitable hypotheses. Furthermore, the continuity operator between the SQVS of the WF for the QNLHBVP and the corresponding CCCQV is demonstrated. Lastly, the ETH of an optimal classical continuous control quaternary vector (OCCCQV) is stated and demonstrated with suitable hypotheses.

2. Problem Description
Let \( I = [0, T] \), \( T < \infty, \Omega \subset \mathbb{R}^2 \), be an open bounded region with boundary \( \Gamma = \partial \Omega, Q = \Omega \times I, \Sigma = \Gamma \times I \). The OCCCQV includes the quaternary state equations (QSEs) which are considered by the following QNLHBVP:

\[
y_{1tt} - \Delta y_1 + y_1 - y_2 + y_3 + y_4 = f_1(x, t, y_1, u_1), \quad \text{in } Q \\
y_{2tt} - \Delta y_2 + y_1 - y_2 - y_3 - y_4 = f_2(x, t, y_2, u_2), \quad \text{in } Q \\
y_{3tt} - \Delta y_3 - y_1 + y_2 + y_3 + y_4 = f_3(x, t, y_3, u_3), \quad \text{in } Q \\
y_{4tt} - \Delta y_4 - y_1 + y_2 - y_3 + y_4 = f_4(x, t, y_4, u_4), \quad \text{in } Q
\]

With the following boundary conditions (BCs) and the initial conditions (ICs)

\[
y_i(x, 0) = y_i^0(x), \text{and } y_{it}(x, 0) = y_{i}^i(x), \quad \text{in } \Omega \text{ for } i = 1, 2, 3, 4 \\
y_i(x, t) = 0, \quad \text{on } \Sigma, \text{ for } i = 1, 2, 3, 4
\]

Where \( \tilde{y} = (y_1, y_2, y_3, y_4) \) belongs to the Hilbert space \( (H^2(\Omega))^4 \) is the SQVS, corresponding to the CCCQV \( \tilde{u} = (u_1, u_2, u_3, u_4) \in (L^2(Q))^4 \) and \( (f_1, f_2, f_3, f_4) \in (L^2(Q))^4 \) is a vector of a given function on \( (Q \times \mathbb{R} \times U_1) \times (Q \times \mathbb{R} \times U_2) \times (Q \times \mathbb{R} \times U_3) \times (Q \times \mathbb{R} \times U_4) \), with \( U_i \subset \mathbb{R}, \forall i = 1, 2, 3, 4 \).

The quaternary controls constraints (QCCs) are

\[
\mathcal{W}_A = \{ \tilde{w} \in \tilde{W} \subset (L^2(Q))^4 | \tilde{w} \in \tilde{U} \subset \mathbb{R}^4 \ a.e. \ in \ Q \}, \text{with } \tilde{U} \subset \mathbb{R}^4 \text{ is a convex.}
\]

The cost function will is considered as

\[
G_0(\tilde{u}) = \sum_{i=1}^{4} \int_{Q} g_{0i}(x, t, y_i, u_i)dxdt
\]

The OCCCP is to find \( \tilde{u} \in \mathcal{W}_A \), s.t. \( G_0(\tilde{u}) = \min_{\tilde{w} \in \mathcal{W}_A} G_0(\tilde{w}) \)

Let \( \bar{\mathcal{V}} = (V)^4; V = H_0^1(\Omega) \), and \( \bar{\mathcal{V}} = \{ \bar{v}; \bar{v} \in (H^1(\Omega))^4, v_1 = v_2 = v_3 = v_4 = 0 \text{ on } \partial \Omega \}. \bar{v} = (v_1, v_2, v_3, v_4) \), we denote by \( (\bar{v}, v) \) and \( \| v \|_0 \) the inner product (IP) and the norm in \( (L^2(\Omega))^4 \), by \( (\bar{v}, v) = \sum_{i=1}^{4} \| v \|_0^2 \text{ the IP and the norm in } \bar{\mathcal{V}}, \text{ and } \bar{\mathcal{V}}^* \text{ is the dual of } \bar{\mathcal{V}}. \)

The WF of ((1)-(6)) when \( \bar{\mathcal{V}} \in (H_0^1(\Omega))^4 \) is given a.e. on \( I \) and \( \forall v_i \in V_i (\forall i = 1, 2, 3, 4) \) by:

\[
(y_{1tt}, v_1) + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1) = (f_1, v_1) \\
(y_{1t}, v_1) = (y_1(0), v_1), \text{ and } (y_{1tt}, v_1) = (y_{1t}(0), v_1)
\]
\[ (y_{2t}, v_2) + (\Delta y_2, v_2) + (y_1, v_2) + (y_2, v_2) - (y_3, v_2) - (y_4, v_2) = (f_2, v_2) \]  
\[ (y_{2h}, v_2) = (y_2(0), v_2), \text{ and } (y_{2h}, v_2) = (y_{2h}(0), v_2) \]  
\[ (y'_{3tt}, v_3) + (\nabla y_3, v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3) = (f_3, v_3) \]  
\[ (y'_{3t}, v_3, y_3) + (\nabla y_3, v_3) - (y_1, v_3) - (y_2, v_3) = (y_3(0), v_3) \]  
\[ (y'_{4tt}, v_4) - (y_1, v_4) + (y_2, v_4) - (y_3, v_4) + (y_4, v_4) = (f_4, v_4) \]  
\[ (y'_{4t}, v_4) = (y_4(0), v_4), \text{ and } (y'_{4t}, v_4) = (y_{4t}(0), v_4) \]  

**2.1. Assumptions (A):** Suppose that \( f_i \) is of the Carathéodory type on \( Q \times (\mathbb{R} \times U_i) \) and satisfies for \( i = 1, 2, 3, 4 \):  
(i) \( |f_i(x, t, y_i, u_i)| \leq F_i(x, t) + |y_i| + |u_i| + \beta_i |y_i| \), where \( y_i, u_i \in \mathbb{R}, \beta_i, y_i > 0 \) and \( F_i \in L^2(Q) \).  
(ii) \( f_i \) is satisfied Lipschitz condition (LIPC) w.r.t. \( y_i \), i.e. \( |f_i(x, t, y_i, u_i) - f_i(x, t, \bar{y}_i, u_i)| \leq L_i |y_i - \bar{y}_i| \), with \( y_i, \bar{y}_i \in \mathbb{R}, L_i > 0 \), for \( (x, t) \in Q \).

**2.2. Lemma 1: (Gronwall inequality):** Let \( K \) be a nonnegative constant and let \( f \) and \( g \) be continuous nonnegative functions on \( [\alpha, \beta] \), satisfies: \( f(t) \leq K + \int_{\alpha}^{t} f(s) g(s) ds \). Then,  
\[ f(t) \leq Ke^{\int_{\alpha}^{t} g(s) ds}, \text{ for } \alpha \leq t \leq \beta. \]

**3. The Solution for the QSEs:**

**3.1 Proposition [14]:** Let \( D \) be a measurable subset of \( \mathbb{R}^d \) \( (d = 2, 3) \), \( f: D \times \mathbb{R}^n \rightarrow \mathbb{R}^m \) is of Carathéodory type satisfies \( ||f(v, x)|| \leq \zeta(v) + \eta(v)||x||^p \) \( \forall (v, x) \in D \times \mathbb{R}^n \), where \( x \in L^p(D \times \mathbb{R}^n), \zeta \in L^1(D \times \mathbb{R}), \eta \in L^{p-a}(D \times \mathbb{R}), \alpha \in [0, p], \) if \( p \neq \infty \). Then, the functional \( F(x) = \int_{D} f(v, x(v)) dv \) is continuous.

**3.2 Theorem (ETH of a Unique SQVS):** with Assumptions (A), for each given \( \bar{u} \in L^2(Q) \), the WF \( (8)-(15) \) has a unique solution \( \bar{y} = (y_1, y_2, y_3, y_4) \) \( \in (L^2(1 \times V))^4 \) and \( \bar{y}_t = \bar{y}_{tt} = (y_{1tt}, y_{2tt}, y_{3tt}, y_{4tt}) \in (L^2(Q))^4 \).

**Proof:** Let \( V_n = (V_{n, i})^4 \subset V \) (for each \( n \)) be the set of piecewise affine functions in \( \Omega \), let \( \{v_n\}_{n=1}^\infty \) be a sequence of subspaces of \( V_n \), s.t. \( \forall \bar{v} = (v_1, v_2, v_3, v_4) \in V \), there is a sequence \( \{v_n\} \) with \( \bar{v}_n = (v_{1n}, v_{2n}, v_{3n}, v_{4n}) \in V_n, \forall n \) and \( \bar{v}_n \rightarrow \bar{v} \) strongly (ST) in \( V \) then \( \bar{v}_n \rightarrow \bar{v} \) ST in \((L^2(\Omega))^4 \). Let \( \{\bar{v}_j = (v_{1j}, v_{2j}, v_{3j}, v_{4j}) : j = 1, 2, ..., M(n)\} \) be a finite basis of \( V_n \) (where \( \bar{v} \) is a piecewise affine function in the boundary) and let \( \bar{y}_n = (y_{1n}, y_{2n}, y_{3n}, y_{4n}) \) be the Galerkin approximate solution (GAS) to the exact solution \( \bar{y} = (y_1, y_2, y_3, y_4) \) s.t.:  
\[ y_{in} = \sum_{j=1}^{n} c_{ij}(t)v_{ij}(x) \]  
\[ z_{in} = \sum_{j=1}^{n} d_{ij}(t)v_{ij}(x) \]  

Where \( c_{ij}(t), d_{ij}(t) \) are unknown functions, \( \forall i = 1, 2, 3, 4, j = 1, 2, ..., n \). The MGA is utilized to approximate the WF \( (8), (10), (12), (14) \) w.r.t. \( x \), they become after substituting \( y_{int} = z_{in} \), \( \forall \bar{v}_n \in V_n, \forall i = 1, 2, 3, 4 \):  
\[ (z_{1tt}, v_1) + (\nabla y_{1n}, v_1) + (y_{1n}, v_1) - (y_{2n}, v_1) + (y_{3n}, v_1) + (y_{4n}, v_1) = (f_1, v_1) \]  
\[ (y_{1n}, v_1), \text{ and } (z_{1n}, v_1) = (y_{1n}, v_1) \]  
\[ (z_{2tt}, v_2) + (\nabla y_{2n}, v_2) + (y_{1n}, v_2) + (y_{2n}, v_2) - (y_{3n}, v_2) - (y_{4n}, v_2) = (f_2, v_2) \]  
\[ (y_{2n}, v_2) = (y_{2n}, v_2), \text{ and } (z_{2n}, v_2) = (y_{2n}, v_2) \]  
\[ (z_{3tt}, v_3) + (\nabla y_{3n}, v_3) - (y_{1n}, v_3) + (y_{2n}, v_3) + (y_{3n}, v_3) + (y_{4n}, v_3) = (f_3, v_3) \]  
\[ (y_{3n}, v_3) = (y_{3n}, v_3), \text{ and } (z_{3n}, v_3) = (y_{3n}, v_3) \]  

\[ (y_{4n}, v_3) = (y_{4n}, v_3), \text{ and } (z_{4n}, v_3) = (y_{4n}, v_3) \]
where \( y^0_{in} = y^1_{in}(x) \) be the projection of \( y^1_{int} \) onto \( V_n \) (be the projection of \( y^1_i \) onto \( L^2(\Omega) \), i.e.

\[
y^0_{in} \rightarrow y^1_{in} \text{ ST in } V, \text{ with } \| y^0_{in} \| \leq b_0 \quad \text{and} \quad \| y^1_{in} \| \leq b_0
\]

Substituting (16) & (17) with

\[
(I - \Delta) v_i = \frac{1}{\lambda_n} \langle (\nabla \chi_n, \nabla v) \rangle + \langle (\chi_n, v) \rangle_{x \in \Gamma}
\]

and setting \( v_i = v_{il} \). Then the obtained equations are equivalent to the following system of nonlinear ODEs of 1st order with ICs (which has a unique solution), i.e.

\[
A_1 D_1(t) + B_1 C_1(t) - EC_2(t) + FC_3(t) + KC_4(t) = b_1 (\bar{V}_i^T (x) C_1(t))
\]

\[
A_1 C_1(0) = b_1^0 \quad \text{and} \quad A_1 \bar{D}_1(0) = b_1^1
\]

\[
A_2 D_2(t) + B_2 C_2(t) + H C_3(t) - GC_3(t) + DC_4(t) = b_2 (\bar{V}_i^T (x) C_1(t))
\]

\[
A_2 C_2(0) = b_2^0 \quad \text{and} \quad A_2 \bar{D}_2(0) = b_2^1
\]

\[
A_3 D_3(t) + B_3 C_3(t) - RC_3(t) + WC_2(t) + Z C_4(t) = b_3 (\bar{V}_i^T (x) C_1(t))
\]

\[
A_3 C_3(0) = b_3^0 \quad \text{and} \quad A_3 \bar{D}_3(0) = b_3^1
\]

\[
A_4 D_4(t) + B_4 C_4(t) - TC_1(t) + MC_2(t) - NC_3(t) = b_4 (\bar{V}_i^T (x) C_1(t))
\]

\[
A_4 C_4(0) = b_4^0 \quad \text{and} \quad A_4 \bar{D}_4(0) = b_4^1
\]

Then corresponding to the sequence \( \{ \bar{V}_n \} \), the following approximation problems are held, i.e.

for each \( \bar{V}_n = (v_{1n}, v_{2n}, v_{3n}, v_{4n}) \subset V_n^1 \), and \( n = 1, 2, \ldots \)

\[
(y_{1ntrt}, v_{1n}) + (\bar{V} v_{1n}) + (y_{1n}, v_{1n}) - (y_{2n}, v_{1n}) + (y_{3n}, v_{1n}) + (y_{4n}, v_{1n}) = (f_1(y_{1n}, u_1), v_{1n})
\]

\[
y_{1n} = (y_{1n}^0, v_{1n}) \quad \text{and} \quad (y_{1n}^1, v_{1n}) = (y_{1n}^0, v_{1n})
\]

\[
y_{2ntrt}, v_{2n}) + (\bar{V} v_{2n}) + (y_{1n}, v_{2n}) + (y_{2n}, v_{2n}) - (y_{3n}, v_{2n}) - (y_{4n}, v_{2n}) = (f_2(y_{2n}, u_2), v_{2n})
\]

\[
y_{2n} = (y_{2n}^0, v_{2n}) \quad \text{and} \quad (y_{2n}^1, v_{2n}) = (y_{2n}^0, v_{2n})
\]

\[
y_{3ntrt}, v_{3n}) + (\bar{V} v_{3n}) - (y_{1n}, v_{3n}) + (y_{2n}, v_{3n}) + (y_{3n}, v_{3n}) + (y_{4n}, v_{3n}) = (f_3(y_{3n}, u_3), v_{3n})
\]

\[
y_{3n} = (y_{3n}^0, v_{3n}) \quad \text{and} \quad (y_{3n}^1, v_{3n}) = (y_{3n}^0, v_{3n})
\]

\[
y_{4ntrt}, v_{4n}) + (\bar{V} v_{4n}) - (y_{1n}, v_{4n}) + (y_{2n}, v_{4n}) - (y_{3n}, v_{4n}) + (y_{4n}, v_{4n}) = (f_4(y_{4n}, u_4), v_{4n})
\]

\[
y_{4n} = (y_{4n}^0, v_{4n}) \quad \text{and} \quad (y_{4n}^1, v_{4n}) = (y_{4n}^0, v_{4n})
\]
Which have a sequence of unique solutions \( \{ \tilde{y}_n \} \). Substituting \( v_{in} = y_{int} \) for \( i = 1, 2, 3, 4 \) in (28), (30), (32) and (34) resp., using Lemma 1.2 in [15] for the 1st terms of the LHS of each equality, then adding the resulting equation, to get

\[
\frac{d}{dt} \parallel \tilde{y}_{nt} \parallel_0^2 + \frac{d}{dt} \parallel \tilde{y}_n \parallel_1^2 = 2(\parallel y_{2n}, y_{1nt} \parallel - (y_{3n}, y_{1nt}) - (y_{4n}, y_{1nt}) + (y_{3n}, y_{2nt}) + (y_{4n}, y_{2nt}) + (y_{1n}, y_{3nt}) - (y_{2n}, y_{3nt}) - (y_{4n}, y_{3nt}) + (y_{1n}, y_{4nt}) - (y_{2n}, y_{4nt}) + (y_{3n}, y_{4nt}))(f_1(y_{1n}, u_1), y_{1nt}) + (f_2(y_{2n}, u_2), y_{2nt}) + (f_3(y_{3n}, u_3), y_{3nt}) + (f_4(y_{4n}, u_4), y_{4nt}) \]

Taking the absolute value for both sides, we get:

\[
\frac{d}{dt} \parallel \tilde{y}_{nt} \parallel_0^2 + \frac{d}{dt} \parallel \tilde{y}_n \parallel_1^2 \leq 2[\parallel y_{2n}, y_{1nt} \parallel + \parallel y_{3n}, y_{1nt} \parallel + \parallel y_{4n}, y_{1nt} \parallel + \parallel y_{1n}, y_{2nt} \parallel] + (y_{3n}, y_{2nt}) + (y_{4n}, y_{2nt}) + (y_{1n}, y_{3nt}) + (y_{2n}, y_{3nt}) + (y_{4n}, y_{3nt}) + (y_{1n}, y_{4nt}) + (y_{2n}, y_{4nt}) + (f_1(y_{1n}, u_1), y_{1nt}) + (f_2(y_{2n}, u_2), y_{2nt}) + (f_3(y_{3n}, u_3), y_{3nt}) + (f_4(y_{4n}, u_4), y_{4nt}) \]

Using Assumptions (A) for the R.H.L. of (37), integrating both sides (IBS) on \([0, t]\), using

\[
\parallel y_{in} \parallel_0 \leq \parallel y_{nt} \parallel_0 \leq \parallel y_{int} \parallel_1, \parallel y_{nt} \parallel_0 \leq \parallel y_{nt} \parallel_1, \parallel y_{nt} \parallel_0 \leq \parallel y_{nt} \parallel_1 \text{, and } t \geq 0 \]

\[
\parallel \tilde{y}_{nt} \parallel_0^2 + \parallel \tilde{y}_n \parallel_1^2 \leq 3 \int_0^t \parallel \tilde{y}_n \parallel_0^2 + \parallel y_{nt} \parallel_1^2 dt + (\parallel F_1 \parallel_Q + \parallel F_2 \parallel_Q + \parallel F_3 \parallel_Q + \parallel F_4 \parallel_Q) + \beta_5 \int_0^t \parallel y_{nt} \parallel_1^2 dt + (1 + \beta_5 + \gamma_5) \int_0^t \parallel y_{nt} \parallel_0^2 dt + (\gamma_5 + c_5) \parallel u \parallel_0^2 + b_0 + b_1 \leq d_6 + 3 \int_0^t \parallel y_{nt} \parallel_1^2 + \parallel y_{nt} \parallel_0^2 dt + \beta_5 \int_0^t \parallel y_{nt} \parallel_1^2 dt + \beta_6 \int_0^t \parallel y_{nt} \parallel_0^2 dt \]

Where \( \beta_5 = \max(\beta_1, \beta_2, \beta_3, \beta_4) \). \( \parallel u \parallel_0^2 \leq c_i (i = 1, 2, 3, 4) \). \( \parallel F_i \parallel_Q \leq d_i \). \( d_5 = \sum_{i=1}^{4} d_i \), \( d_5 = \max(c_1, c_2, c_3, c_4) \); \( d_6 = \gamma_5 + c_5 + d_5 + b_0 + b_1 \), \( \beta_5 = 1 + \beta_5 + \gamma_5 \beta_5 = \max(3, \beta_6) \).

Using Lemma 2.2, \( \forall t \in [0, t] \) to get

\[
\parallel \tilde{y}_{nt}(t) \parallel_0^2 + \parallel \tilde{y}_n(t) \parallel_1^2 \leq d_6 \int_0^t \beta_5 \parallel y_{nt} \parallel_1^2 dt + b_2 \parallel y_{nt} \parallel_1^2 \leq b^2 \parallel y_{nt} \parallel_1^2 \leq b^2 \parallel y_{nt} \parallel_1^2 \leq b^2 \parallel y_{nt} \parallel_1^2 \]

Easily once can obtain that \( \parallel \tilde{y}_{nt}(t) \parallel_0 \leq b_1 \parallel y_{nt} \parallel_1 \) and \( \parallel \tilde{y}_n(t) \parallel_1 \leq b_2 \parallel y_{nt} \parallel_1 \). Then, by applying the Alaoglu’s theorem (ALTh), there is a subsequence of \( \{ \tilde{y}_n \}_{n \in \mathbb{N}} \), for simplicity say \( \{ \tilde{y}_n \} \), s.t. \( \tilde{y}_{nt} \rightarrow \tilde{y} \) weakly (WK) in \((L^2(Q))^4 \) and \( \tilde{y}_n \rightarrow \tilde{y} \) WK in \((L^2(I, V))^4 \). but

\[
(L^2(\mathbb{R}, V))^4 \subset (L^2(\mathbb{R}, Q))^4 \subset (L^2(\mathbb{R}, V^*))^4 \subset (L^2(\mathbb{R}, V^*))^4 \]

Then the (ACTH)[15] can be employed here to get that \( \tilde{y}_n \rightarrow \tilde{y} \) ST in \((L^2(Q))^4 \). Now multiplying both sides (MBS) of ((28), (30), (32) & (34)) by \( \phi_1(T) \in C^2[0, T] \) s.t. \( \phi_1(T) = \phi_1'(T) = 0, \phi_1(0) \neq 0, \phi_1'(0) \neq 0, \forall i = 1, 2, 3, 4 \), IBS on \([0, T] \), finally integrating by parts twice (IBPs2) the 1st term in each equation yield to

\[
- \int_0^T \frac{d}{dt} (f_1(y_{1n}, u_1), v_{1n}) dt + \int_0^T (\nabla y_{1n}, \nabla v_{1n}) + (y_{1n}, v_{1n}) - (y_{2n}, v_{1n}) + (y_{3n}, v_{1n}) + (y_{4n}, v_{1n}) \phi_1(t) dt + (y_{1n}, v_{1n}) \phi_1(0) \]

\[
(40) \]
\[
\int_0^T (y_{1n}, v_{1n}) \phi_1''(t) dt + \int_0^T [(\nabla y_{1n}, \nabla v_{1n}) + (y_{1n}, v_{1n}) - (y_{2n}, v_{1n}) + (y_{3n}, v_{1n}) + (y_{4n}, v_{1n})] \phi_1(t) dt = \int_0^T (f_1(y_{1n}, u_1), v_{1n}) \phi_1(t) dt + (y_{1n}', v_{1n}) \phi_1(0) - (y_{1n}, v_{1n}) \phi_1'(0) \tag{41}
\]

\[
- \int_0^T \frac{d}{dt} (y_{2n}, v_{2n}) \phi_2'(t) dt + \int_0^T [(\nabla y_{2n}, \nabla v_{2n}) + (y_{1n}, v_{2n}) + (y_{2n}, v_{2n}) - (y_{3n}, v_{2n})] \phi_2(t) dt = \int_0^T (f_2(y_{2n}, u_2), v_{2n}) \phi_2(t) dt + (y_{2n}', v_{2n}) \phi_1(0) \tag{42}
\]

\[
\int_0^T (y_{2n}, v_{2n}) \phi_2''(t) dt + \int_0^T [(\nabla y_{2n}, \nabla v_{2n}) + (y_{1n}, v_{2n}) + (y_{2n}, v_{2n}) - (y_{3n}, v_{2n})] \phi_2(t) dt
\]

\[
- \int_0^T \frac{d}{dt} (y_{3n}, v_{3n}) \phi_3'(t) dt + \int_0^T [(\nabla y_{3n}, \nabla v_{3n}) - (y_{1n}, v_{3n}) + (y_{2n}, v_{3n}) + (y_{3n}, v_{3n}) + (y_{4n}, v_{3n})] \phi_3(t) dt = \int_0^T (f_3(y_{3n}, u_3), v_{3n}) \phi_3(t) dt + (y_{3n}', v_{3n}) \phi_3(0) \tag{44}
\]

\[
\int_0^T (y_{3n}, v_{3n}) \phi_3''(t) dt + \int_0^T [(\nabla y_{3n}, \nabla v_{3n}) - (y_{1n}, v_{3n}) + (y_{2n}, v_{3n}) + (y_{3n}, v_{3n}) + (y_{4n}, v_{3n})] \phi_3(t) dt
\]

\[
= \int_0^T (f_3(y_{3n}, u_3), v_{3n}) \phi_3(t) dt + (y_{3n}', v_{3n}) \phi_3(0) \tag{45}
\]

\[
- \int_0^T \frac{d}{dt} (y_{4n}, v_{4n}) \phi_4'(t) dt + \int_0^T [(\nabla y_{4n}, \nabla v_{4n}) - (y_{1n}, v_{4n}) + (y_{2n}, v_{4n}) - (y_{3n}, v_{4n}) + (y_{4n}, v_{4n})] \phi_4(t) dt = \int_0^T (f_4(y_{4n}, u_4), v_{4n}) \phi_4(t) dt + (y_{4n}', v_{4n}) \phi_4(0) \tag{46}
\]

\[
\int_0^T (y_{4n}, v_{4n}) \phi_4''(t) dt + \int_0^T [(\nabla y_{4n}, \nabla v_{4n}) - (y_{1n}, v_{4n}) + (y_{2n}, v_{4n}) - (y_{3n}, v_{4n}) + (y_{4n}, v_{4n})] \phi_4(t) dt
\]

\[
= \int_0^T (f_4(y_{4n}, u_4), v_{4n}) \phi_4(t) dt + (y_{4n}', v_{4n}) \phi_4(0) \tag{47}
\]

First, since \( v_{in} \rightarrow v_i \) ST in \( L^2(\Omega) \) \( \Rightarrow \) \( \{v_{in}\phi_i(t) \rightarrow v_i\phi_i(t) \} \) ST in \( L^2(I, V) \).

\( v_{in}\phi_i(0) \rightarrow v_i\phi_i(0) \) ST in \( L^2(\Omega) \) and \( v_{in}\phi_i'(0) \rightarrow v_i\phi_i'(0) \) ST in \( L^2(\Omega) \) for \( i = 1,2,3,4 \)

Also since \( v_{in} \rightarrow v_i \) ST in \( V \) \( \Rightarrow \) \( \{v_{in}\phi_i(t) \rightarrow v_i\phi_i(t) \} \) ST in \( L^2(Q) \).

Second, we have \( y_{int} \rightarrow y_{it} \) WK in \( L^2(Q) \) and \( y_{int} \rightarrow y_{it} \) WK in \( L^2(I,V) \) and ST in \( L^2(Q) \).

Third and on the other hand, let \( w_{in} = v_{in}\phi_i \) and \( w_i = v_i\phi_i \) then \( w_{in} \rightarrow w_i \) ST in \( L^2(Q) \) and then \( w_{in} \) is measurable w.r.t. \((x,t)\), so using Assm (A-(i)), employing Proposition 1.3, the \( (f_i(x,t,y_{in},u_i),w_{in})dxdt \) is cont. w.r.t. \((y_{in},u_i,w_{in})\), then

\[
\int_0^T (f_i(y_{in}, u_i), v_{in})\phi_i(t) dt \rightarrow \int_0^T (f_i(y_i, u_i), v_i)\phi_i(t) dt \forall i = 1,2,3,4
\]

From these convergences, (26) and (27) we can passage the limits in ((40)-(47)), to get
\[ -\int_0^T (y_1(t), v_1) \phi_1'(t) dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1)] \phi_1(t) dt \]

\[ = \int_0^T (f_1(y_1, u_1), v_1) \phi_1(t) dt + (y_1', v_1) \phi_1(0) \]  

(48)

\[ \int_0^T (y_1, v_1) \phi_2'(t) dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1)] \phi_1(t) dt \]

\[ = \int_0^T (f_1(y_1, u_1), v_1) \phi_1(t) dt + (y_1', v_1) \phi_1(0) - (y_1^0, v_1) \phi_1'(0) \]  

(49)

\[ \int_0^T (y_2(t), v_2) \phi_3'(t) dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2) + (y_2, v_2) - (y_3, v_2) - (y_4, v_2)] \phi_2(t) dt \]

\[ = \int_0^T (f_2(y_2, u_2), v_2) \phi_2(t) dt + (y_2, v_2) \phi_1(0) \]  

(50)

\[ \int_0^T (y_2, v_2) \phi_2''(t) dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2) + (y_2, v_2) - (y_3, v_2) - (y_4, v_2)] \phi_2(t) dt \]

\[ = \int_0^T (f_2(y_2, u_2), v_2) \phi_2(t) dt + (y_2, v_2) \phi_2(0) - (y_2^0, v_2) \phi_2'(0) \]  

(51)

\[ \int_0^T (y_3(t), v_3) \phi_3'(t) dt + \int_0^T [(\nabla y_3, \nabla v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3)] \phi_3(t) dt \]

\[ = \int_0^T (f_3(y_3, u_3), v_3) \phi_3(t) dt + (y_3, v_3) \phi_3(0) \]  

(52)

\[ \int_0^T (y_3, v_3) \phi_3''(t) dt + \int_0^T [(\nabla y_3, \nabla v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3)] \phi_3(t) dt \]

\[ = \int_0^T (f_3(y_3, u_3), v_3) \phi_3(t) dt + (y_3, v_3) \phi_3(0) - (y_3^0, v_3) \phi_3'(0) \]  

(53)

\[ \int_0^T (y_4(t), v_4) \phi_4'(t) dt + \int_0^T [(\nabla y_4, \nabla v_4) - (y_1, v_4) + (y_2, v_4) - (y_3, v_4) + (y_4, v_4)] \phi_4(t) dt \]

\[ = \int_0^T (f_4(y_4, u_4), v_4) \phi_4(t) dt + (y_4, v_4) \phi_4(0) \]  

(54)

\[ \int_0^T (y_4, v_4) \phi_4''(t) dt + \int_0^T [(\nabla y_4, \nabla v_4) - (y_1, v_4) + (y_2, v_4) - (y_3, v_4) + (y_4, v_4)] \phi_4(t) dt \]

\[ = \int_0^T (f_4(y_4, u_4), v_4) \phi_4(t) dt + (y_4, v_4) \phi_4(0) - (y_4^0, v_4) \phi_4'(0) \]  

(55)

**Case 1:** Choose \( \phi_i \in C^2[0, T] \) s.t. \( \phi_i(0) = \phi_i'(0) = \phi_i''(T) = \phi_i'(T) = 0, \quad \forall i = 1, 2, 3, 4 \) substituting in (49), (51), (53), (55) IBPs2 the 1st terms in the LHS, i.e.

\[ \int_0^T (y_{1tt}, v_1) \phi_1(t) dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1)] \phi_1(t) dt \]

\[ = \int_0^T (f_1(y_1, u_1), v_1) \phi_1(t) dt \]  

(56)

\[ \int_0^T (y_{2tt}, v_2) \phi_2(t) dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_1, v_2) + (y_2, v_2) - (y_3, v_2) - (y_4, v_2)] \phi_2(t) dt \]

\[ = \int_0^T (f_2(y_2, u_2), v_2) \phi_2(t) dt \]  

(57)

\[ \int_0^T (y_{3tt}, v_3) \phi_3(t) dt + \int_0^T [(\nabla y_3, \nabla v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3)] \phi_3(t) dt \]
\[\int_0^T (f_3(y_3, u_3), v_3) \phi_3(t)dt\] (58)
\[\int_0^T (y_{4tt}, v_4) \phi_4 dt + \int_0^T [(\nabla y_4, \nabla v_4) - (y_1, v_4) + (y_2, v_4) - (y_3, v_4) + (y_4, v_4)] \phi_4(t)dt\]
\[= \int_0^T (f_4(y_4, u_4), v_4) \phi_4(t)dt\] (59)

Hence \(\ddot{y}\) is a solution of (16),(18), (20) & (22) a.e. on I

**Case 2:** Choose \(\phi_i \in C^2[0,T] \) s.t. \(\phi_i(T) = 0, \& \phi_i(0) = 0, \forall i = 1,2,3,4\). MBS of (8), (10), (12) and (14) by \(\phi_1(t), \phi_2(t), \phi_3(t), \) and \(\phi_4(t)\) resp., IBS on \([0,T]\), then IBPs the 1st term in the LHS of each equation, then subtracting each one of these obtained equations from those equations in (48),(50),(52) & (54) resp. to get
\[(y_4(0), v_4) \phi_i(0) = (y_1(0), v_1) \phi_i(0), \forall i = 1,2,3,4.\]

**Case 3:** Choose \(\phi_i \in C^2[0,T] \) s.t. \(\phi_i(0) = \phi_i'(0) = \phi_i(T) = 0, \forall i = 1,2,3,4\). MBS of (8), (10), (12) and (14) by \(\phi_1(t), \phi_2(t), \phi_3(t), \) and \(\phi_4(t)\) resp., IBS on \([0,T]\) then IBPs2 the 1st term in the LHS of each equation, then subtracting each one of these obtained equations from those equations corresponding in (49),(51),(53), and (55) resp. to get
\[(y_i(0), v_i) \phi_i(0) = (y_i^0, v_i) \phi_i(0), \forall i = 1,2,3,4\]

In the last two cases, the ICs (19), (11), (13) and (14) are held.

To prove that \(\ddot{y_n} \rightarrow \ddot{y}\) ST in \(L^2(I, V)\), we start by IBS (36) on \([0,T]\), to get
\[\int_0^T \frac{d}{dt} \|\ddot{y_n}\|_0^2 dt + 2 \int_0^T \|\ddot{y_n}\|_1^2 dt = 2 \int_0^T [(y_{2n}, y_{1nt}) - (y_{3n}, y_{1nt}) - (y_{4n}, y_{1nt}) - (y_{1n}, y_{2nt}) + (y_{1n}, y_{2nt}) + (y_{3n}, y_{2nt}) + (y_{4n}, y_{2nt}) + (y_{1n}, y_{3nt}) - (y_{2n}, y_{3nt}) - (y_{3n}, y_{3nt}) - (y_{1n}, y_{4nt}) + (y_{1n}, y_{4nt}) + (y_{2n}, y_{4nt}) + (y_{3n}, y_{4nt})]dt + 2 \int_0^T [(f_1(y_1, u_1), y_{1nt}) + (f_2(y_2, u_2), y_{2nt}) + (f_3(y_3, u_3), y_{3nt}) + (f_4(y_4, u_4), y_{4nt})]dt\] (60)

The same way applied to acquire (36)&(60), can be utilize here to obtain, i.e.
\[\|y_t(T)\|_0^2 - \|y_t(0)\|_0^2 + 2 \int_0^T \|\ddot{y}(t)\|_0^2 dt = 2 \int_0^T [(y_{2t}, y_{1tt}) - (y_{3t}, y_{1tt}) - (y_{4t}, y_{1tt}) - (y_{1t}, y_{2tt}) + (y_{1t}, y_{2tt}) - (y_{3t}, y_{2tt}) + (y_{1t}, y_{3tt}) - (y_{2t}, y_{3tt}) + (y_{1t}, y_{4tt}) - (y_{2t}, y_{4tt}) - (y_{3t}, y_{4tt}) + (y_{1t}, y_{4tt}) + (y_{2t}, y_{4tt}) + (y_{3t}, y_{4tt})]dt + 2 \int_0^T [(f_1(y_1, u_1), y_{1tt}) + (f_2(y_2, u_2), y_{2tt}) + (f_3(y_3, u_3), y_{3tt}) + (f_4(y_4, u_4), y_{4tt})]dt\] (61)

Since
\[\|\ddot{y}_{nt}(T) - \ddot{y}_{nt}(T)\|_0^2 - \|\ddot{y}_{nt}(0) - \ddot{y}_{nt}(0)\|_0^2 + 2 \int_0^T \|\ddot{y}_{n}(t) - \ddot{y}(t)\|_1^2 dt = (a) - (b) - (c)\] (62)

(a)\(=\|\ddot{y}_{nt}(T)\|_0^2 - \|\ddot{y}_{nt}(0)\|_0^2 + 2 \int_0^T \|\ddot{y}_{n}(t) - \ddot{y}_n(t)\|_1^2 dt\)

(b)\(=\left(\ddot{y}_{nt}(T), \ddot{y}_n(T)\right) - \left(\ddot{y}_{nt}(0), \ddot{y}_n(0)\right) + 2 \int_0^T (\ddot{y}_n(t), \ddot{y}(t)) dt\)

(c)\(=\left(\ddot{y}_n(T), \ddot{y}_n(T)\right) - \left(\ddot{y}_n(0), \ddot{y}_n(0)\right) + 2 \int_0^T (\ddot{y}(t), \ddot{y}_n(t) - \ddot{y}(t)) dt\)

Since \(\ddot{y_n} \rightarrow \ddot{y}\) ST in \(L^2(Q)\), and \(\ddot{y}_{nt} \rightarrow \ddot{y}\) WK in \(L^2(Q)\), then from (60) and the Assm on \(f_i\), for \(i = 1,2,3,4\), we get
(a) = 2 \int_0^T [(y_{2n}, y_{1nt}) - (y_{3n}, y_{1nt}) - (y_{4n}, y_{1nt}) - (y_{1n}, y_{2nt}) + (y_{3n}, y_{2nt}) + (y_{4n}, y_{2nt}) + (y_{1n}, y_{3nt}) - (y_{2n}, y_{3nt}) - (y_{4n}, y_{3nt}) + (y_{1n}, y_{4nt}) - (y_{2n}, y_{4nt}) + (y_{3n}, y_{4nt}) + (y_{4n}, y_{4nt})]dt + \\
2 \int_0^T [(f_1(y_1, u_1), y_{1t}) + (f_2(y_2, u_2), y_{2nt}) + (f_3(y_3, u_3), y_{3nt}) + (f_4(y_4, u_4), y_{4nt})]dt → \\
2 \int_0^T [(y_2, y_{1t}) - (y_3, y_{1t}) - (y_4, y_{1t}) - (y_1, y_{2t}) + (y_3, y_{2t}) + (y_4, y_{2t}) + (y_1, y_{3t}) - (y_2, y_{3t}) - (y_4, y_{3t}) + (y_1, y_{4t}) - (y_2, y_{4t}) + (y_3, y_{4t})]dt + 2 \int_0^T [(f_1(y_1, u_1), y_{1t}) + \\
(f_2(y_2, u_2), y_{2t}) + (f_3(y_3, u_3), y_{3t}) + (f_4(y_4, u_4), y_{4t})]dt

By the same way that was employed to acquire (27), it used here to acquire

\( \tilde{y}_{nt}(T) \to \tilde{y}(T) \) ST in \( L^2(0) \) (63)

On the other hand, since \( \tilde{y}_n → \tilde{y} \) in \( L^2(I, V) \), then using (27) & (63), yield to

(b)→ \( \text{RHS of (61)} = 2 \int_0^T [(y_2, y_{1t}) - (y_3, y_{1t}) - (y_4, y_{1t}) - (y_1, y_{2t}) + (y_3, y_{2t}) + (y_4, y_{2t}) + (y_1, y_{3t}) - (y_2, y_{3t}) - (y_4, y_{3t}) + (y_1, y_{4t}) - (y_2, y_{4t}) + (y_3, y_{4t})]dt + \\
2 \int_0^T [(f_1(y_1, u_1), y_{1t}) + (f_2(y_2, u_2), y_{2t}) + (f_3(y_3, u_3), y_{3t}) + (f_4(y_4, u_4), y_{4t})]dt

All the term in (c) approach to zero, so as the 1st two terms in the LHS of (62), hence (62) gives

\( \int_0^T \| \tilde{y}_n(t) - \tilde{y}(t) \|_1^2 dt → 0 \) as \( n → ∞ \), therefore \( \tilde{y}_n → \tilde{y} \) ST in \( L^2(I, V) \)

Uniqueness of the Solution:

Let \( \tilde{y} = (y_3, y_2, y_3, y_4) \) and \( \tilde{y} = (\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4) \) be two solutions of the SQVS of the WF ((8), (10), (12), and (14)), subtracting each equation from the other and replace \( v_i = y_i - \bar{y}_i \) for each \( i = 1,2,3,4 \), i.e.

\[
\begin{align*}
(y_1 - \bar{y}_1)t + y_1 - \bar{y}_1 + \nabla(y_1 - \bar{y}_1, y_1 - \bar{y}_1) + (y_1 - \bar{y}_1, y_1 - \bar{y}_1) + (y_2 - \bar{y}_2, y_1 - \bar{y}_1) + (y_3 - \bar{y}_3, y_1 - \bar{y}_1) + (y_4 - \bar{y}_4, y_1 - \bar{y}_1) &= (f_1(y_1, u_1) - f_1(y_1, u_1), y_1 - \bar{y}_1) \\
(y_2 - \bar{y}_2)t + y_2 - \bar{y}_2 + \nabla(y_2 - \bar{y}_2, y_2 - \bar{y}_2) + (y_1 - \bar{y}_1, y_2 - \bar{y}_2) + (y_2 - \bar{y}_2, y_2 - \bar{y}_2) + (y_3 - \bar{y}_3, y_2 - \bar{y}_2) + (y_4 - \bar{y}_4, y_2 - \bar{y}_2) &= (f_2(y_2, u_2) - f_2(y_2, u_2), y_2 - \bar{y}_2) \\
(y_3 - \bar{y}_3)t + y_3 - \bar{y}_3 + \nabla(y_3 - \bar{y}_3, y_3 - \bar{y}_3) + (y_1 - \bar{y}_1, y_3 - \bar{y}_3) + (y_2 - \bar{y}_2, y_3 - \bar{y}_3) + (y_3 - \bar{y}_3, y_3 - \bar{y}_3) + (y_4 - \bar{y}_4, y_3 - \bar{y}_3) &= (f_3(y_3, u_3) - f_3(y_3, u_3), y_3 - \bar{y}_3) \\
(y_4 - \bar{y}_4)t + y_4 - \bar{y}_4 + \nabla(y_4 - \bar{y}_4, y_4 - \bar{y}_4) + (y_1 - \bar{y}_1, y_4 - \bar{y}_4) + (y_2 - \bar{y}_2, y_4 - \bar{y}_4) + (y_3 - \bar{y}_3, y_4 - \bar{y}_4) + (y_4 - \bar{y}_4, y_4 - \bar{y}_4) &= (f_4(y_4, u_4) - f_4(y_4, u_4), y_4 - \bar{y}_4)
\end{align*}
\]

Collecting the above equalities for \( i = 1,2,3,4 \) using Lemma 1.2 in ref. [15] for the 1st in LHS of above equations, to get

\[
\frac{1}{2} \int_0^T \| (\tilde{y} - \bar{y})_t (t) \|_0^2 + 2 \| \tilde{y} - \bar{y} \|_1^2 = (f_1(y_1, u_1) - f_1(y_1, u_1), y_1 - \bar{y}_1) + \\
(f_2(y_2, u_2) - f_2(y_2, u_2), y_2 - \bar{y}_2) + (f_3(y_3, u_3) - f_3(y_3, u_3), y_3 - \bar{y}_3) + \\
(f_4(y_4, u_4) - f_4(y_4, u_4), y_4 - \bar{y}_4)
\]

The LHS of (68) is positive, IBS of it w.r.t. \( t \) from 0 to \( t \), and using Assumptions (A-ii) of the RHS of it, yields to

\[
\int_0^t \frac{d}{dt} \| \tilde{y} - \bar{y} \|_0^2 ≤ 2 \int_0^t L \| \tilde{y} - \bar{y} \|_0^2 dt , \text{ where } L = \max(L_1, L_2, L_3, L_4)
\]
Then, \( \| \ddot{y} - \ddot{\gamma} \|_0^2 \leq 2 \int_0^T L \| \ddot{y} - \ddot{\gamma} \|_0^2 \). 

By using Lemma (1.2), to acquire
\[
\int_0^T \| \ddot{y} - \ddot{\gamma} \|_0^2 + 2 \| \ddot{y} - \ddot{\gamma} \|_1^2 \ dt \leq L \int_0^T \| (\ddot{y} - \ddot{\gamma}) \|_0^2 \ dt \Rightarrow \int_0^T \| (\ddot{y} - \ddot{\gamma}) (t) \|_2^2 \leq 0 \quad \Rightarrow \left\| (\ddot{y} - \ddot{\gamma}) \right\|_{L^2(0,T)} = 0 \Rightarrow \ddot{y} = \ddot{\gamma} \ i.e. 
\]

**3.1 Lemma:** In addition to Assumptions (A), if the functions \( f_i \) (for each \( i = 1,2,3,4 \)) is Lipschitz w.r.t. \( y_i \) & \( u_i \), and if CCCQV is bounded, then the operator \( \ddot{u} \rightarrow \ddot{\gamma} \) form \((L^2(Q))^4\) to \((L^\infty(I,L^2(\Omega)))^4\) or to \((L^2(Q))^4\) to \((L^2(Q))^4\) is continuous.

**Proof:** Let \( \ddot{u} = (u_1,u_2,u_3,u_4) \), \( \ddot{\gamma} = (\ddot{u_1},\ddot{u_2},\ddot{u_3},\ddot{u_4}) \in (L^2(Q))^4 \), \( \delta \ddot{u} = \ddot{u} - \ddot{\gamma} \in (L^2(Q))^4 \), \( \delta y = \ddot{y} - \ddot{\gamma} \), to obtain
\[
\delta y_{1x}(x,0) = 0 \quad \text{and} \quad \delta y_{1x}(x,0) = 0 
\]
\[
\delta y_{2x}(x,0) \quad \text{and} \quad \delta y_{2x}(x,0) = 0 
\]
\[
\delta y_{3x}(x,0) = 0 \quad \text{and} \quad \delta y_{3x}(x,0) = 0 
\]
\[
\delta y_{4x}(x,0) = 0 \quad \text{and} \quad \delta y_{4x}(x,0) = 0 
\]
Substituting \( v_1 = \delta y_{1x} \) for \( i = 1,2,3,4 \) in (69),(71),(73)and (75), respectively, collecting the obtained equations. Using the same way that is used to get (37), a similar equation can be obtained but with \( \ddot{y}_{1x} \) in position of \( \ddot{y}_n \), then IBS on \([0,t]\), using Lip. on \( f_i \) w.r.t. \( (y_i,u_i) \) resp. for \( i = 1,2,3,4 \) to get
\[
\int_0^T \frac{dt}{dt} \frac{d}{dt} \| \delta y_{1x}(t) \|_0^2 + \| \delta y_{2x} \|_2^2 dt \leq 
\]
\[
2 \int_0^T \| (\delta y_{2x} + \delta y_{3x} + \delta y_{4x}) \|_2^2 + \| \delta y_{1x} + \| \delta y_{2x} + \| \delta y_{3x} + \| \delta y_{4x} \|_2^2 \ dt + 
\]
\[
2 \int_0^T \| (\delta y_{1x} + \delta y_{2x} + \delta y_{3x} + \delta y_{4x}) \|_2^2 + \| \delta y_{1x} + \| \delta y_{2x} + \| \delta y_{3x} + \| \delta y_{4x} \|_2^2 \ dt + 
\]
Using the definitions of the norms and the relations between them, to get,
\[
\| \delta y_{1x} \|_0^2 + \| \delta y_{2x} \|_0^2 \leq 2 \int_0^T \| \delta y_{1x} \|_0^2 + \| \delta y_{2x} \|_0^2 dt + \int_0^T \| \delta y_{1x} \|_0^2 + \| \delta y_{2x} \|_0^2 dt 
\]
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$$+ \tilde{L}_4 \int_0^t \| \delta \tilde{u} \|_Q^2 \, dt + \tilde{L}_4 \int_0^t \| \delta \tilde{y}_{et} \|_Q^2 \, dt$$

$$\leq \tilde{L}_4 \| \delta \tilde{u} \|_Q^2 + L_5 \int_0^t \| \delta \tilde{y}_{e} \|_Q^2 + \| \delta \tilde{y}_{et} \|_Q^2 \, dt$$

Where $\tilde{L}_3 = \max(\tilde{L}_1, \tilde{L}_2, \tilde{L}_3, \tilde{L}_4), \tilde{L}_4 = \max(\tilde{L}_1, \tilde{L}_2, \tilde{L}_3, \tilde{L}_4), \tilde{L}_5 = \max(3 + \tilde{L}_3, 3 + \tilde{L}_4 + \tilde{L}_3)$. 

Applying the BGI, with $L^2 = \tilde{L}_4 e^{L_5 \int_0^t dt}$, to get

$$\| \delta \tilde{y}_{et} \|_Q^2 + \| \delta \tilde{y}_{e} \|_Q^2 \leq L^2 \| \delta \tilde{u} \|_Q^2, \forall t \in I \Rightarrow \| \delta \tilde{y}_{e} \|_Q^2 \leq L \| \delta \tilde{u} \|_Q^2 \text{ and } \| \delta \tilde{y}_{et} \|_Q \leq L \| \delta \tilde{u} \|_Q.$$ 

4. The Existence of an OCQCV

4.1. Assumptions (B): Consider $g_{i i}$ for $(l = 0 \land i = 1, 2, 3, 4)$ is of Carathéodory type on $Q \times (\mathbb{R} \times U)$, and satisfies the following sub quadratic condition w.r.t. $y_i \in \mathbb{R}$ and $u_i \in U_i$,

$$|g_{i i}(x, t, y_i, u_i)| \leq G_{i i}(x, t) + C_{i i} y_i^2 + C_{i i} u_i^2,$$ 

where $G_{i i} \in L^1(Q)$, for $(x, t) \in Q, l = 0$

4.1 Lemma: With assum. (B), the functional $\tilde{u} \to G_0(\tilde{u})$ is continuous on $(L^2(Q))^4$.

Proof: Using Assumptions (B) and proposition 1.3, the integral $\int_Q g_{i i}(x, t, y_i, u_i) \, dx \, dt$ is continuous on $L^2(Q), \forall i = 1, 2, 3, 4$, hence $G_0(\tilde{u})$ is continuous on $(L^2(Q))^4$.

4.2 Lemma: Let $g: Q \times \mathbb{R} \to \mathbb{R}$ is of Carathéodory type on $Q \times (\mathbb{R} \times \mathbb{R})$ and satisfies

$$|g(x, t, y, u)| \leq G(x, t) + c y^2 + \epsilon u^2,$$ 

where $G(x, t) \in L^1(Q), u \in U, c, \epsilon \geq 0, U \in \mathbb{R}$, is compact. Then, $\int_Q g(x, y, u) \, dx$ is continuous on $L^2(Q)$ w.r.t. $y$.

4.1 Theorem: In addition to Assumptions (A&B), if the set $\tilde{U}$ is convex and compact, $\tilde{W}_A \neq \emptyset$, the function $f_i$, for $(i = 1, 2, 3, 4)$ have the form

$$f_i(x, t, y_i, u_i) = f_{i 1}(x, t, y_i) + f_{i 2}(x, t) u_i, \text{where } |f_{i 1}(x, t, y_i)| \leq \eta_i(x, t) + c_i \land y_i|, |f_{i 2}(x, t)| \leq K_i, \eta_i \in L^2(Q), c_i \geq 0, \text{ for } i = 1, 2, 3, 4.$$ Then there exists an OCQCV.

Proof: From the Assumptions on $U_i \subset \mathbb{R}$ for $i = 1, 2, 3, 4$ and the Egorov’s theorem, once get that $\tilde{W}$ is weakly compact, since $\tilde{W}_A \neq \emptyset$, there exists a minimum sequence $\{\tilde{u}_k\} = \{(u_{1 k}, u_{2 k}, u_{3 k}, u_{4 k})\} \in \tilde{W}_A, \forall k \text{ s.t. } \lim_{k \to \infty} G_0(\tilde{u}_k) = \inf_{\tilde{u} \in \tilde{W}_A} G_0(\tilde{u}).$ Since $\{\tilde{u}_k\} \in \tilde{W}_A, \forall k \text{ and } \tilde{W}$ is weakly compact, there exists a subsequence of $\{\tilde{u}_k\}$ say again $\{\tilde{u}_k\}$ s.t. $\tilde{u}_k \to \tilde{W}$ WK in $(L^2(Q))^4$ and $\tilde{u}_k \|_Q \leq d, \forall k$. From Theorem 3.1, for each control $\{\tilde{u}_k\}$ the WF (18), (20), (22), (24) has a unique SQVS, $\{\tilde{y}_k = \tilde{y}_{u_k}\}$ s.t. the norm $\| \tilde{y}_k \|_{L^2(L(V))}, \| \tilde{y}_{k t} \|_{L^2(Q)}$ are bounded, then by ATH there exists a subsequence of $\{\tilde{y}_k\}$ and $\{\tilde{y}_{k t}\}, \text{say again } \{\tilde{y}_k\}$ and $\{\tilde{y}_{k t}\}, \text{s.t. } \tilde{y}_k \to \tilde{y} \text{ WK in } (L^2(L(V)))^4, \tilde{y}_{k t} \to \tilde{y} \text{ WK in } (L^2(Q))^4$. Now for each $k$. and by applying the ACTH [15], we get that there exists a subsequence of $\{\tilde{y}_k\}$ say a gain $\{\tilde{y}_k\}$ s.t. $\tilde{y}_k \to \tilde{y} \text{ ST in } (L^2(Q))^4$. Now since for each $k$, $\tilde{y}_k$ is a SQVS of the WF ((18), (20), (22), (24)) resp. substituting in these equations, MBs of each equation by $\phi_i(t)$, $\forall i = 1, 2, 3, 4$ (with $\phi_i \in C^2[0, T]$, s.t. $\phi_i(T) = \phi_i(0) = 0, \phi_i(0) \neq 0, \phi_i(T) \neq 0$). Rewriting the $1^{st}$ term in the LHS of each one then IBS on $[0, T]$, finally IBPs for the $1^{st}$ terms, one has

$$\int_0^T \frac{d}{dt}(y_{1 k t}, v_1) \phi_1(t) dt + \int_0^T [(\nabla y_{1 k}, \nabla v_1) + (y_{1 k}, v_1) - (y_{2 k}, v_1) + (y_{3 k}, v_1) + (y_{4 k}, v_1)] \phi_1(t) dt$$

$$= \int_0^T (f_{1 1}(x, t, y_{1 k}), v_1) \phi_1(t) dt + \int_0^T (f_{1 2}(x, t) u_{1 k}, v_1) \phi_1(t) dt \quad (77)$$

$$\int_0^T \frac{d}{dt}(y_{2 k t}, v_2) \phi_2(t) dt + \int_0^T [(\nabla y_{2 k}, \nabla v_2) + (y_{1 k}, v_2) + (y_{2 k}, v_2) - (y_{3 k}, v_2) - (y_{4 k}, v_2)] \phi_2(t) dt$$

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Now \( \sin \left( \frac{\pi}{2} \right) \) as follows. Let

\[
\text{The same steps that are utilized in the proof of Theorem 3.1, can also be used here to passage the limit in the LHS of ((77)-(80)). We persist in the passage of the limit in RHS of ((77))-(80)) as follows. Let } \forall i = 1, 2, 3, 4, \text{ } v_i \in C(\bar{Q}), \text{ } w_i \in C(\bar{Q}) \subseteq L^2(Q), \text{ set } \tilde{f}_i(y_{ik}) = f_i(y_{ik})w_i, \text{ then } \tilde{f}_i: Q \times \mathbb{R} \to \mathbb{R} \text{ is of Carathéodory type, utilizing proposition 3.1, to get the integral } \int_Q f_i(y_{ik})w_i \, dx \, dt, \text{ is continuous w.r.t. } y_{ik} \forall i = 1, 2, 3, 4.
\]

but \( y_{ik} \to y_i \) ST in \( L^2(Q) \) and \( u_{ik} \to u_i \) WK in \( L^2(Q) \), then

\[
\int_Q f_{i1}(y_{ik})w_i \, dx \, dt \to \int_Q f_{i1}(y_i)w_i \, dx \, dt, \forall \ w_i \in C(\bar{Q}), \text{ for } i = 1, 2, 3, 4
\]

\[
\int_Q f_{i2}(x, t)u_{ik}w_i \, dx \, dt \to \int_Q f_{i2}(x, t)u_iw_i \, dx \, dt, \forall \ w_i \in C(\bar{Q}), \text{ for } i = 1, 2, 3, 4
\]

Form the density of \( C(\bar{Q}) \) in \( V \), (81) & (82) are satisfies for each \( v_i \in V \) for \( i = 1, 2, 3, 4 \), hence the following WF is obtained

\[
(y_{t}, v_i) + (\nabla y_i, \nabla v_i) + (y_{1}, v_{1}) - (y_{2}, v_{1}) + (y_{3}, v_{1}) + (y_{4}, v_{1}) = (f_{i1}(x, t, y_{i}), v_{1}) + (f_{i2}(x, t)u_{i}, v_{1}) \forall v_i \in V \text{ a.e. on } I
\]

\[
(y_{2t}, v_{2}) + (\Delta y_{2}, \nabla v_{2}) + (y_{1}, v_{2}) + (y_{2}, v_{2}) - (y_{3}, v_{2}) - (y_{4}, v_{2}) = (f_{i2}(x, t, y_{2}), v_{2}) + (f_{i2}(x, t)u_{2}, v_{2}) \forall v_{2} \in V \text{ a.e. on } I
\]

\[
(y_{3t}, v_{3}) + (\nabla y_{3}, \nabla v_{3}) - (y_{1}, v_{3}) + (y_{2}, v_{3}) + (y_{3}, v_{3}) + (y_{4}, v_{3}) = (f_{i3}(x, t, y_{3}), v_{3}) + (f_{i3}(x, t)u_{3}, v_{3}) \forall v_{3} \in V \text{ a.e. on } I
\]

\[
(y_{4t}, v_{4}) + (\nabla y_{4}, \nabla v_{4}) - (y_{1}, v_{4}) + (y_{2}, v_{4}) - (y_{3}, v_{4}) + (y_{4}, v_{4}) = (f_{i4}(x, t, y_{4}), v_{4}) + (f_{i4}(x, t)u_{4}, v_{4}) \forall v_{4} \in V \text{ a.e. on } I
\]

Also, the same steps employed in Theorem 3.1 can be employed here to obtain that the ICs are held, hence \( y \) is the SQVS.

Now since \( \forall i = 1, 2, 3, 4 \), \( g_{0i}(x, t, y_i, u_i) \) is continuous w.r.t. \( (y_i, u_i) \), and \( U_i \) is compact. With \( u_i(x, t) \in U_i \) a.e. in \( Q \), then using lemma 4.2, to get

\[
\int_Q g_{0i}(x, t, y_{ik}, u_{ik}) \, dx \, dt \to \int_Q g_{0i}(x, t, y_i, u_{ik}) \, dx \, dt
\]

But \( g_{0i}(x, t, y_i, u_i) \) is continuous and convex w.r.t. \( u_i \) then \( \int_Q g_{0i}(x, t, y_i, u_i) \, dx \, dt \) is weakly

lower semi cont. (WLSC) w.r.t. \( u_i \), \( \forall i = 1, 2, 3, 4 \), i.e.

\[
\int_Q g_{0i}(x, t, y_i, u_i) \, dx \, dt \leq \liminf_{k \to \infty} \int_Q \left[ g_{0i}(x, t, y_i, u_{ik}) - g_{0i}(x, t, y_{ik}, u_{ik}) \right] \, dx \, dt
\]

\[
+ \liminf_{k \to \infty} \int_Q g_{0i}(x, t, y_{ik}, u_{ik}) \, dx \, dt
\]

\[
\leq \liminf_{k \to \infty} \int_Q g_{0i}(x, t, y_{ik}, u_{ik}) \, dx \, dt
\]

\[
\Rightarrow \sum_{i=1}^{4} \int_Q g_{0i}(x, t, y_i, u_i) \, dx \, dt \leq \sum_{i=1}^{4} \int_Q g_{0i}(x, t, y_{ik}, u_{ik}) \, dx \, dt, \text{ thus}
\]
\[ G_0(\bar{u}) \leq \lim_{k \to \infty} \inf_{\bar{u}_k \in W_A} G_0(\bar{u}_k) = \inf_{\bar{u}_k \in W_A} G_0(\bar{u}), \text{ then} \]
\[ G_0(\bar{u}) \leq \inf_{\bar{u}_k \in W_A} G_0(\bar{u}) \Rightarrow G_0(\bar{u}) \leq \min_{\bar{u}_k \in W_A} G_0(\bar{u}), \text{ then } \bar{u} \text{ is OQCCCV}. \]

5. Conclusion

The method of Galerkin with the Aubin compactness theorem is used successfully to demonstrate the existence theorem of a unique state quaternary vector solution for the weak form for the quaternary nonlinear hyperbolic boundary value problem where the classical continuous control quaternary vector is considered given under suitable hypotheses.

The continuity operator between the state quaternary vector solution of the weak form for the quaternary nonlinear hyperbolic boundary value problem and the corresponding classical continuous control quaternary vector is demonstrated. The existence theorem of an optimal classical continuous control quaternary vector under suitable hypotheses is demonstrated.

References


