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# The Optimal Classical Continuous Control Quaternary Vector of Quaternary Nonlinear Hyperbolic Boundary Value Problem

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## Abstract

This work is concerned with studying the optimal classical continuous control quaternary vector problem. It is consisted of; the quaternary nonlinear hyperbolic boundary value problem and the cost functional. At first, the weak form of the quaternary nonlinear hyperbolic boundary value problem is obtained. Then under suitable hypotheses, the existence theorem of a unique state quaternary vector solution for the weak form where the classical continuous control quaternary vector is considered known is stated and demonstrated by employing the method of Galerkin and the compactness theorem. In addition, the continuity operator between the state quaternary vector solution of the weak form and the corresponding classical continuous control quaternary vector is demonstrated in three different infinite dimensional spaces (Hilbert spaces). Furthermore, with suitable hypotheses, the existence theorem of an optimal classical continuous control quaternary vector dominated by the weak form of the quaternary nonlinear hyperbolic boundary value problem is stated and demonstrated.

**Keywords:** Optimal Classical Continuous Control Quaternary Vector, Quaternary Nonlinear Hyperbolic Boundary Value Problem, Weak form.

## 1. Introduction

Different applications in real-life are classified as optimal control problems (OCPs). For example, in medicine [1], economics [2], robotics [3], Aircraft [4], and many other fields. Usually, this importance encouraged many researchers to be interested in studying OCPs in general and optimal classical continuous control problems (OCCCP) in particular. During the last decade, great attention has been made to studying the subject of OCCCP for a system dominated by nonlinear PDEs (NLPDEs) of the three types elliptic [5], hyperbolic [6], and parabolic [7]. Latter, the study of this subject expanded to include OCCCP for systems dominated by a couple of NLPDEs of their three types [8-10]; through recent years, these studies for these three types expanded to deal with OCCCP for systems dominated by triple NLPDEs [11-13]. All these studies encouraged us to investigate the OCCCP dominated by QNLHBVP.

This article first concerns the mathematical formulation for the optimal classical continuous control quaternary vector problem. Then the existence theorem (ETH) of a unique state quaternary vector solution (SQVS) for the weak form (WF) "of the quaternary nonlinear hyperbolic boundary value problem (QNLHBVP)" is stated and demonstrated using the method of Galerkin (MGA) and the Aubin compactness theorem (ACTH) when the classical continuous control quaternary vector (CCCQV) is fixed under suitable hypotheses. Furthermore, the continuity operator between the SQVS of the WF for the QNLHBVP and the corresponding CCCQV is demonstrated. Lastly, the ETH of an optimal classical continuous control quaternary vector (OCCCQV) is stated and demonstrated with suitable hypotheses.

## 2. Problem Description

Let $I = [0, T]$, $T < \infty, \Omega \subset \mathbb{R}^2$, be an open bounded region with boundary $\Gamma = \partial\Omega$ ,$Q = \Omega \times I$ , $\Sigma = \Gamma \times I$. The OCCCQV includes the quaternary state equations (QSEs) which are considered by the following QNLHBVP:

$$y_{1tt} - \Delta y_1 + y_1 - y_2 + y_3 + y_4 = f_1(x, t, y_1, u_1), \text{ in } Q \quad (1)$$

$$y_{2tt} - \Delta y_2 + y_1 + y_2 - y_3 - y_4 = f_2(x, t, y_2, u_2), \text{ in } Q \quad (2)$$

$$y_{3tt} - \Delta y_3 - y_1 + y_2 + y_3 + y_4 = f_3(x, t, y_3, u_3), \text{ in } Q \quad (3)$$

$$y_{4tt} - \Delta y_4 - y_1 + y_2 - y_3 + y_4 = f_4(x, t, y_4, u_4), \text{ in } Q \quad (4)$$

With the following boundary conditions (BCs) and the initial conditions (ICs)

$$y_i(x, t) = 0, \text{ on } \Sigma, \text{ for } i = 1,2,3,4 \quad (5)$$

$$y_1(x, 0) = y_i^0(x), \text{and } y_{it}(x, 0) = y_i^1(x), \text{ in } \Omega \text{ for } i = 1,2,3,4 \quad (6)$$

Where $\vec{y} = (y_1, y_2, y_3, y_4)$ belongs to the Hilbert space $(H^2(\Omega))^4$ is the SQVS, corresponding to the CCCQV $\vec{u} = (u_1, u_2, u_3, u_4) \in (L^2(Q))^4$ and $(f_1, f_2, f_3, f_4) \in (L^2(Q))^4$ is a vector of a given function on $(Q \times \mathbb{R} \times U_1) \times (Q \times \mathbb{R} \times U_2) \times (Q \times \mathbb{R} \times U_3) \times (Q \times \mathbb{R} \times U_4)$, with $U_i \subset \mathbb{R}, \forall i = 1,2,3,4$ .

The quaternary controls constraints (QCCs) are

$\vec{W}_A = \{\vec{w} \in \vec{W} \subset (L^2(Q))^4 | \vec{w} \in \vec{U} \subset \mathbb{R}^4 \ a.e. in\ Q\}$,with $\vec{U} \subset \mathbb{R}^4$is a convex.

The cost function will is considered as

$$G_0(\vec{u}) = \sum_{i=1}^{4} \int_Q g_{0i}\,(x, t, y_i, u_i)dxdt \quad (7)$$

The OCCCP is to find $\vec{u} \in \vec{W}_A$, s.t. $G_0(\vec{u}) = \min_{\vec{w} \in \vec{W}_A} G_0\,(\vec{w})$

Let $\vec{\mathrm{V}} = (V)^4; V = H_0^1(\Omega)$, and $\vec{V} = \{\vec{v}: \vec{v} \in (H^1(\Omega))^4, v_1 = v_2 = v_3 = v_4 = 0\ on\ \partial\Omega\}$. $\vec{v} = (v_1, v_2, v_3, v_4)$, we denote by $(v, v)$ and $\| v \|_0$ the inner product (IP) and the norm in $(L^2(\Omega))^4$, by $(\vec{v}, \vec{v})_1 = \sum_{i=1}^{4} \| v_1 \|_1^2$ the IP and the norm in $\vec{\mathrm{V}}$, and $\vec{\mathrm{V}}^*$ is the dual of $\vec{\mathrm{V}}$.

The WF of ((1)-(6)) when $\vec{y} \in (H_0^1(\Omega))^4$ is given a.e. on I and $\forall v_i \in V_i$ ($\forall i = 1,2,3,4$ ) by :

$$(y_{1tt}, v_1) + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1) = (f_1, v_1) \quad (8)$$

$$(y_1^0, v_1) = (y_1(0), v_1), \text{ and } (y_{1t}^1, v_1) = (y_{1t}(0), v_1) \quad (9)$$

$(y_{2tt}, v_2) + (\Delta y_2, \nabla v_2) + (y_1, v_2) + (y_2, v_2) - (y_3, v_2) - (y_4, v_2) = (f_2, v_2)$ (10)

$(y_2^0, v_2) = (y_2(0), v_2)$, and $(y_{2t}^1, v_2) = (y_{2t}(0), v_2)$ (11)

$(y_{3tt}, v_3) + (\nabla y_3, \nabla v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3) = (f_3, v_3)$ (12)

$(y_3^0, v_3) = (y_3(0), v_3)$, and $(y_{3t}^1, v_3) = (y_{3t}(0), v_3)$ (13)

$(y_{4tt}, v_4) + (\nabla y_4, \nabla v_4) - (y_1, v_4) + (y_2, v_4) - (y_3, v_4) + (y_4, v_4) = (f_4, v_4)$ (14)

$(y_4^0, v_2) = (y_4(0), v_4)$, and $(y_{4t}^1, v_4) = (y_{4t}(0), v_4)$ (15)

**2.1. Assumptions (A):** Suppose that $f_i$ is of the Carathéodory type on $Q \times (\mathbb{R} \times U_i)$ and satisfies (for $i = 1,2,3,4$):

(i)$|f_i(x, t, y_i, u_i)| \leq F_i(x, t) + \gamma_i \mid u_i| + \beta_i|y_i|$, where $y_i, u_i \in \mathbb{R}, \beta_i, \gamma_i \mid > 0$ and $F_i \in L^2(\mathrm{Q})$.

(ii) $f_i$ is satisfied Lipschitz condition (LIPC) w.r.t. $y_i$, i.e.

$|f_i(x, t, y_i, u_i) - f_i(x, t, \bar{y}_i, u_i)| \leq L_i|y_i - \bar{y}_i|$ , $y_i, \bar{y}_i, u_i \in \mathbb{R}$, $L_i > 0$, for $(x, t) \in Q$.

**2.2 Lemma1: (Gronwall inequality):** Let $K$ be a nonnegative constant and let $f$ and $g$ be continuous nonnegative functions on $[\alpha, \beta]$, satisfies: $f(t) \leq K + \int_\alpha^t f(s)g(s)ds$. Then, $f(t) \leq Ke^{\int_\alpha^t g(s)ds}$, for $\alpha \leq t \leq \beta$.

## 3. The Solution for the QSEs:

**3.1 Proposition [14]:** Let $D$ is a measurable subset of $\mathbb{R}^d$ $(d = 2,3)$, $f: D \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is of Carathéodory type satisfies $\|f(v, x)\| \leq \zeta(v) + \eta(v)\|x\|^\alpha, \forall (v, x) \in D \times \mathbb{R}^n$, where $x \in L^p(D \times \mathbb{R}^n)$, $\zeta \in L^1(D \times \mathbb{R})$, $\eta \in L^{\frac{p}{p-\alpha}}(D \times \mathbb{R})$, $\alpha \in [0, p]$, if $p \neq \infty$. Then, the functional

$F(x) = \int_D f(v, x(v))dv$ is continuous.

**3.2 Theorem (ETH of a Unique SQVS):** with Assumptions (A), for each given $\vec{u} \in L^2(\mathrm{Q})$, the WF ((8)-(15)) has a unique solution $\vec{y} = (y_1, y_2, y_3, y_4) \in (L^2(\mathrm{I} \times \mathrm{V}))^4$ and $\vec{y}_t = (y_{1t}, y_{2t}, y_{3t}, y_{4t}) \in (L^2(\mathrm{Q}))^4$, $\vec{y}_{tt} = (y_{1tt}, y_{2tt}, y_{3tt}, y_{4tt}) \in (L^2(\mathrm{I} \times \mathrm{V}^*))^4$.

**Proof:** Let $\overrightarrow{V_n} = (V_n)^4 \subset \vec{V}$ (for each $n$) be the set of piecewise affine functions in $\Omega$, let $\{v_n\}_{n=1}^\infty$ be a sequence of subspaces of $\vec{V}$, s.t. $\forall\, \vec{v} = (v_1, v_2, v_3, v_4) \in \vec{V}$, there is a sequence $\{v_n\}$ with $\vec{v}_n = (v_{1n}, v_{2n}, v_{3n}, v_{4n}) \in \vec{V}_n$, $\forall n$ and $\vec{v}_n \to \vec{v}$ strongly (ST) in $\vec{V}$ then $\vec{v}_n \to \vec{v}$ ST in $(L^2(\Omega))^4$. Let $\{\overrightarrow{v_j} = (v_{1j}, v_{2j}, v_{3j}, v_{4j}): j = 1,2, \dots, M(n)\}$ be a finite basis of $\vec{V}_n$ (where $\overrightarrow{v_j}$ is a piecewise affine function in $\Omega$, with $\overrightarrow{v_j}(x) = 0$ on the boundary $\Gamma$) and let $\vec{y}_n = (y_{1n}, y_{2n}, y_{3n}, y_{4n})$ be the Galerkin approximate solution (GAS) to the exact solution $\vec{y} = (y_1, y_2, y_3, y_4)$ s.t.:

$$y_{in} = \sum_{j=1}^n c_{ij}(t) v_{ij}(x) \quad (16)$$

$$z_{in} = \sum_{j=1}^n d_{ij}(t) v_{ij}(x) \quad (17)$$

Where $c_{ij}(t), d_{ij}(t)$ are unknown functions, $\forall i = 1,2,3,4$, $j = 1,2, \dots, n$.

The MGA is utilized to approximate the WF ((8), (10), (12), (14)) w.r.t. $x$, they become after substituting $y_{int} = z_{in}$ $(\forall v_i \in V_n, \forall i = 1,2,3,4)$:

$(z_{1nt}, v_1) + (\nabla y_{1n}, \nabla v_1) + (y_{1n}, v_1) - (y_{2n}, v_1) + (y_{3n}, v_1) + (y_{4n}, v_1) = (f_1, v_1)$ (18)

$(y_{1n}^0, v_1) = (y_1^0, v_1)$, and $(z_{1n}^1, v_1) = (y_1^1, v_1)$ (19)

$(z_{2nt}, v_2) + (\nabla y_{2n}, \nabla v_2) + (y_{1n}, v_2) + (y_{2n}, v_2) - (y_{3n}, v_2) - (y_{4n}, v_2) = (f_2, v_2)$ (20)

$(y_{2n}^0, v_2) = (y_2^0, v_2)$, and $(z_{2n}^1, v_2) = (y_2^1, v_2)$ (21)

$(z_{3nt}, v_3) + (\nabla y_{3n}, \nabla v_3) - (y_{1n}, v_3) + (y_{2n}, v_3) + (y_{3n}, v_3) + (y_{4n}, v_3) = (f_3, v_3)$ (22)

$(y_{3n}^0, v_3) = (y_3^0, v_3)$, and $(z_{3n}^1, v_3) = ((y_3^1, v_3)$ (23)

$(z_{4nt}, v_4) + (\nabla y_{4n}, \nabla v_4) - (y_{1n}, v_4) + (y_{2n}, v_4) - (y_{3n}, v_4) + (y_{4n}, v_4) = (f_4, v_4)$ (24)

$(y_{4n}^0, v_2) = (y_4^0, v_4)$, and $(z_{4n}^1, v_4) = ((y_4^1, v_4)$ (25)

Where $y_{in}^0 = y_{in}^0(x) = y_{in}(x,0) \in V_n$ (respectively $z_{in}^0 = y_{in}^1 = y_{in}^1(x) = y_{int}(x,0) \in L^2(\Omega)$ be the projection of $y_i^0$ onto $V$ (be the projection of $y_i^1 = y_{it}$ on to $L^2(\Omega)$, $\forall i = 1,2,3,4$ ), i.e.

$y_{in}^0 \to y_i^0$ ST in $V$, with $\| \vec{y}_n^0 \|_1 \le b_0$ and $\| \vec{y}_n^0 \|_0 \le b_0$ (26)

$y_{in}^1 \to y_i^1$ ST in $L^2(\Omega)$ , with $\| \vec{y}_n^1 \| \le b_1$ (27)

Substituting (16) & (17) with $i = 1,2,3,4$ in ((18)-(25)) and setting $v_i = v_{il}$ ,$\forall l = 1,2,\dots,n$, then the obtained equations are equivalent to the following system of nonlinear ODEs of 1st order with ICs (which has a unique solution), i.e.

$A_1 D_1(t) + B_1 C_1(t) - E C_2(t) + F C_3(t) + K C_4(t) = b_1(\bar{V}_1^T(x) C_1(t))$

$A_1 C_1(0) = b_1^0$ and $A_1 \overline{D_1}(0) = b_1^1$

$A_2 D_2(t) + B_2 C_2(t) + H C_1(t) - G C_3(t) + D C_4(t) = b_2\ (\bar{V}_2^T(x) C_1(t))$

$A_2 C_2(0) = b_2^0$ and $A_2 \overline{D_2}(0) = b_2^1$

$A_3 D_3(t) + B_3 C_3(t) - R C_1(t) + W C_2(t) + Z C_4(t) = b_3(\bar{V}_3^T(x) C_1(t))$

$A_3 C_3(0) = b_3^0$ and $A_3 \overline{D_3}(0) = b_3^1$

$A_4 D_4(t) + B_4 C_4(t) - T C_1(t) + M C_2(t) - N C_3(t) = b_4(\bar{V}_4^T(x) C_1(t))$

$A_4 C_4(0) = b_4^0$ and $A_4 \overline{D_4}(0) = b_4^1$

where $A_i = (a_{ilj})_{n\times n}$, $a_{ilj} = (v_{ij}, v_{il})$, $B_i = (b_{ilj})_{n\times n}$, $b_{ilj} = \left(\nabla v_{ij}, \nabla v_{il}\right) + (v_{ij}, v_{il})$, $E = (e_{lj})_{n\times n}$,$e_{lj} = \left(v_{2j}, v_{1l}\right)$,$F = (f_{lj})_{n\times n}$, $f_{lj} = \left(v_{3j}, v_{1l}\right)$,$G = (g_{lj})_{n\times n}$, $g_{lj} = \left(v_{3j}, v_{2l}\right)$, $H = (h_{lj})_{n\times n}$, $h_{lj} = \left(v_{1j}, v_{2l}\right)$,$R = (r_{lj})_{n\times n}$, $r_{lj} = \left(v_{1j}, v_{3l}\right)$, $W = (w_{lj})_{n\times n}$,
$w_{lj} = \left(v_{2j}, v_{3l}\right)$, $K = (k_{lj})_{n\times n}$, $k_{lj} = \left(v_{4j}, v_{1l}\right)$, $D = (d_{lj})_{n\times n}$, $d_{lj} = \left(v_{4j}, v_{2l}\right)$, $Z = (z_{lj})_{n\times n}$, $z_{lj} = \left(v_{4j}, v_{3l}\right)$, $T = (t_{lj})_{n\times n}$, $t_{lj} = \left(v_{1j}, v_{4l}\right)$, $M = (m_{lj})_{n\times n}$, $m_{lj} = \left(v_{2j}, v_{4l}\right)$, $N = (n_{lj})_{n\times n}$, $n_{lj} = \left(v_{3j}, v_{4l}\right)$,$b_l = (b_{li}) = (b_{li})_{n\times 1}$, $b_{li} = (f_i(\bar{V}_l^T(x) C_l(t), u_i), v_{li})$, $b_l^k = \left(b_{lj}^k\right)$, $b_{lj}^0 = \left(y_l^k, v_{lj}\right)$, $C_i(t) = (C_{ij}(t))_{n\times 1}$, $D_i(0) = (d_{ij}(0))_{n\times 1}$, $C_i(0) = (C_{ij}(0))_{n\times 1}$, $D_i(t) = (d_{ij}(t))_{n\times 1}$.

Then corresponding to the sequence $\{\vec{v}_n\}$, the following approximation problems are held, i.e. for each $\vec{v}_n = (v_{1n}, v_{2n}, v_{3n}, v_{4n}) \subset \overrightarrow{V_n}$ , and $n = 1,2,\dots$

$(y_{1ntt}, v_{1n}) + (\nabla y_{1n}, \nabla v_{1n}) + (y_{1n}, v_{1n}) - (y_{2n}, v_{1n}) + (y_{3n}, v_{1n}) + (y_{4n}, v_{1n})$
$= (f_1(y_{1n}, u_1), v_{1n})$ (28)

$(y_{1n}^0, v_{1n}) = (y_1^0, v_{1n})$, and $(y_{1n}^1, v_{1n}) = (y_1^1, v_{1n})$ (29)

$(y_{2ntt}, v_{2n}) + (\nabla y_{2n}, \nabla v_{2n}) + (y_{1n}, v_{2n}) + (y_{2n}, v_2) - (y_{3n}, v_{2n}) - (y_{4n}, v_{2n}) =$
$(f_2(y_{2n}, u_2), v_{2n})$ (30)

$(y_{2n}^0, v_{2n}) = (y_1^0, v_{2n})$, and $(y_{2n}^1, v_{2n}) = (y_1^1, v_{2n})$ (31)

$(y_{3ntt}, v_{3n}) + (\nabla y_{3n}, \nabla v_{3n}) - (y_{1n}, v_{3n}) + (y_{2n}, v_{3n}) + (y_{3n}, v_{3n}) + (y_{4n}, v_{3n}) =$
$(f_3(y_{3n}, u_3), v_{3n})$ (32)

$(y_{3n}^0, v_{3n}) = (y_3^0, v_{3n})$, and $(y_{3n}^1, v_3) = (y_3^1, v_{3n})$
(33)

$(y_{4ntt}, v_{4n}) + (\nabla y_{4n}, \nabla v_{4n}) - (y_{1n}, v_{4n}) + (y_{2n}, v_{4n}) - (y_{3n}, v_{4n}) + (y_{4n}, v_{4n}) =$
$(f_4(y_{4n}, u_4), v_{4n})$ (34)

$(y_{4n}^0, v_{4n}) = (y_4^0, v_{4n})$, and $(y_{4n}^1, v_{4n}) = (y_4^1, v_{4n})$ (35)

Which have a sequence of unique solutions $\{\vec{y}_n\}$. Substituting $v_{in} = y_{int}$ , for $i = 1,2,3,4$ in (28), (30), (32) and (34) resp. , using Lemma 1.2 in [15] for the 1st terms of the LHS of each equality, then adding the resulting equation, to get

$$\frac{d}{dt}[\| \vec{y}_{nt} \|_0^2 + \frac{d}{dt} \| \vec{y}_n \|_1^2] = 2[(y_{2n}, y_{1nt}) - (y_{3n}, y_{1nt}) - (y_{4n}, y_{1nt}) - (y_{1n}, y_{2nt}) + (y_{3n}, y_{2nt}) + (y_{4n}, y_{2nt}) + (y_{1n}, y_{3nt}) - (y_{2n}, y_{3nt}) - (y_{4n}, y_{3nt}) + (y_{1n}, y_{4nt}) - (y_{2n}, y_{4nt}) + (y_{3n}, y_{4nt}) + (f_1(y_{1n}, u_1), y_{1nt}) + (f_2(y_{2n}, u_2), y_{2nt}) + (f_3(y_{3n}, u_3), y_{3nt}) + (f_4(y_{4n}, u_4), y_{4nt})] \quad (36)$$

Taking the absolute value for both sides, we get:

$$\frac{d}{dt}[\| \vec{y}_{nt} \|_0^2 + \| \vec{y}_n \|_1^2] \leq 2[| (y_{2n}, y_{1nt}) | + | (y_{3n}, y_{1nt}) | + | (y_{4n}, y_{1nt}) | + |(y_{1n}, y_{2nt})| + |(y_{3n}, y_{2nt})| + |(y_{4n}, y_{2nt})| + |(y_{1n}, y_{3nt})| + |(y_{2n}, y_{3nt})| + | (y_{4n}, y_{3nt}) | + | (y_{1n}, y_{4nt}) | + | (y_{2n}, y_{4nt}) | + | (y_{3n}, y_{4nt}) | + | (f_1(y_{1n}, u_1), y_{1nt}) | + | (f_2(y_{2n}, u_2), y_{2nt}) | + | (f_3(y_{3n}, u_3), y_{3nt}) | + | (f_4(y_{4n}, u_4), y_{4nt}) |] \quad (37)$$

Using Assumptions (A) for the R.H.L. of (37), integrating both sides (IBS) on $[0, t]$, using $\| y_{in} \|_0 \leq \| \vec{y}_n \|_1$, $\| \vec{y}_{int} \|_0 \leq \| \vec{y}_{int} \|_1$, $\| \vec{y}_{nt} \|_0 \leq \| \vec{y}_{nt} \|_1$, to get

$$\| \vec{y}_{nt} \|_0^2 + \| \vec{y}_n \|_1^2 \leq 3 \int_0^t [\| \vec{y}_{nt} \|_0^2 + \| \vec{y}_n \|_1^2] dt + \left(\| F_1 \|_Q^2 + \| F_2 \|_Q^2 + \| F_3 \|_Q^2 + \| F_4 \|_Q^2\right) + \beta_5 \int_0^t \| \vec{y}_n \|_1^2 \, dt + (1 + \beta_5 + \gamma_5) \int_0^t \| \vec{y}_{nt} \|_0^2 \, dt + (\gamma_5 + c_5) \| \vec{u} \|_Q^2 + b_0 + b_1$$

$$\leq d_6 + 3 \int_0^t [\| \vec{y}_n \|_1^2 + \| y_{nt} \|_0^2] dt + \beta_5 \int_0^t \| y_n \|_1^2 \, dt + \beta_6 \int_0^t \| y_{nt} \|_0^2 \, dt$$

$$\leq d_6 + \beta_7 \int_0^t [\| \vec{y}_n \|_1^2 + \| y_{nt} \|_0^2] dt \quad (38)$$

Where$\beta_5 = \max(\beta_1, \beta_2, \beta_3, \beta_4)$, $\gamma_5 = \max(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$,$\| u_i \|_Q^2 \leq c_i (\forall i = 1,2,3,4)$, $\| F_i \|_Q^2 \leq d_i$, $d_5 = \sum_{i=1}^4 d_i$, $c_5 = \max(c_1, c_2, c_3, c_4)$, $d_6 = \gamma_5 + c_5 + d_5 + b_0 + b_1$, $\beta_6 = 1 + \beta_5 + \gamma_5$ $\beta_7 = \max(3, \beta_6)$.

Using Lemma 2.2, $\forall t \in [0, t]$ to get

$\| \vec{y}_{nt}(t) \|_0^2 + \| \vec{y}_n(t) \|_1^2 \leq d_6 e^{\beta_7 \int_0^T dt} = b^2(c) \Longrightarrow \| \vec{y}_{nt}(t) \|_0^2 \leq b^2(c)$ and $\| \vec{y}_n(t) \|_1^2 \leq b^2(c)$.
Easily once can obtain that $\| \vec{y}_{nt}(t) \|_Q \leq b_1(c)$ and $\| \vec{y}_n(t) \|_{L^2(I,V)} \leq b(c)$.

Then, by applying the Alaoglu's theorem (ALTh), there is a subsequence of $\{\vec{y}_n\}_{n \in N}$, for simplicity say $\{\vec{y}_n\}$ s.t. $\vec{y}_{nt} \to \vec{y}$ weakly (WK) in $(L^2(Q))^4$ and $\vec{y}_n \to \vec{y}$ WK in $(L^2(I, V))^4$ . but

$$(L^2(\mathbb{R}, V))^4 \subset (L^2(\mathbb{R}, \Omega))^4 \cong ((L^2(\mathbb{R}, \Omega))^*)^4 \subset (L^2(\mathbb{R}, V^*))^4 \quad (39)$$

Then the (ACTH)[15] can be employed here to get that $\vec{y}_n \to \vec{y}$ ST in $(L^2(Q))^4$ . Now multiplying both sides (MBS) of ((28), (30),(32)&(34)) by $\phi_i(T) \in C^2[0, T]$ s.t. $\phi_i(T) = \phi_i'(T) = 0$, $\phi_i(0) \neq 0$, $\phi_i'(0) \neq 0$, $\forall i = 1,2,3,4$, IBS on $[0, T]$, finally integrating by parts twice (IBPs2) the 1st term in each equation yield to

$$-\int_0^T \frac{d}{dt}(y_{1n}, v_{1n}) \phi_1' dt + \int_0^T [(\nabla y_{1n}, \nabla v_{1n}) + (y_{1n}, v_{1n}) - (y_{2n}, v_{1n}) + (y_{3n}, v_{1n}) + (y_{4n}, v_{1n})] \phi_1(t) dt = \int_0^T (f_1(y_{1n}, u_1), v_{1n}) \phi_1(t) dt + (y_{1n}', v_{1n}) \phi_1(0) \quad (40)$$

$$\int_0^T (y_{1n}, v_{1n})\phi_1'' dt + \int_0^T [(\nabla y_{1n}, \nabla v_{1n}) + (y_{1n}, v_{1n}) - (y_{2n}, v_{1n}) + (y_{3n}, v_{1n}) + (y_{4n}, v_{1n})]\phi_1(t)dt = \int_0^T (f_1(y_{1n}, u_1), v_{1n})\phi_1(t)dt + (y'_{1n}, v_{1n})\phi_1(0) - (y^0_{1n}, v_{1n})\phi'_1(0) \quad (41)$$

$$-\int_0^T \frac{d}{dt}(y_{2n}, v_{2n})\phi'_2 dt + \int_0^T [(\nabla y_{2n}, \nabla v_{2n}) + (y_{1n}, v_{2n}) + (y_{2n}, v_{2n}) - (y_{3n}, v_{2n}) - (y_{4n}, v_{2n})]\phi_2(t)dt = \int_0^T (f_2(y_{2n}, u_2), v_{2n})\phi_2(t)dt + (y'_{2n}, v_{2n})\,\phi_1(0) \quad (42)$$

$$\int_0^T (y_{2n}, v_{2n})\phi''_2(t)dt + \int_0^T [(\nabla y_{2n}, \nabla v_{2n}) + (y_{1n}, v_{2n}) + (y_{2n}, v_{2n}) - (y_{3n}, v_{2n}) - (y_{4n}, v_{2n})]\phi_2(t)dt$$
$$= \int_0^T (f_2(y_{2n}, u_2), v_{2n})\phi_2(t)dt + (y'_{2n}, v_{2n})\phi_2(0) - (y^0_{2n}, v_{2n})\phi'_2(0) \quad (43)$$

$$-\int_0^T \frac{d}{dt}(y_{3n}, v_{3n})\phi'_3(t)dt + \int_0^T [(\nabla y_{3n}, \nabla v_{3n}) - (y_{1n}, v_{3n}) + (y_{2n}, v_{3n}) + (y_{3n}, v_{3n}) + (y_{4n}, v_{3n})]\phi_3(t)dt = \int_0^T (f_3(y_{3n}, u_3), v_{3n})\phi_3(t)dt + (y'_{3n}, v_{3n})\,\phi_3(0) \quad (44)$$

$$\int_0^T (y_{3n}, v_{3n})\phi''_3 dt + \int_0^T [(\nabla y_{3n}, \nabla v_{3n}) - (y_{1n}, v_{3n}) + (y_{2n}, v_{3n}) + (y_{3n}, v_{3n}) + (y_{4n}, v_{3n})]\phi_3(t)dt$$
$$= \int_0^T (f_3(y_{3n}, u_3), v_{3n})\phi_3(t)dt + (y'_{3n}, v_{3n})\phi_3(0) - (y^0_{3n}, v_{3n})\phi'_3(0) \quad (45)$$

$$-\int_0^T \frac{d}{dt}(y_{4n}, v_{4n})\phi'_4(t)dt + \int_0^T [(\nabla y_{4n}, \nabla v_{4n}) - (y_{1n}, v_{4n}) + (y_{2n}, v_{4n}) - (y_{3n}, v_{4n}) + (y_{4n}, v_{4n})]\phi_4(t)dt = \int_0^T (f_4(y_{4n}, u_4), v_{4n})\phi_4(t)dt + (y'_{4n}, v_{4n})\,\phi_4(0) \quad (46)$$

$$\int_0^T (y_{4n}, v_{4n})\phi''_4 dt + \int_0^T [(\nabla y_{4n}, \nabla v_{4n}) - (y_{1n}, v_{4n}) + (y_{2n}, v_{4n}) - (y_{3n}, v_{4n}) + (y_{4n}, v_{4n})]\phi_4(t)dt = \int_0^T (f_4(y_{4n}, u_4), v_{4n})\phi_4(t)dt + (y'_{4n}, v_{4n})\phi_4(0) - (y^0_{4n}, v_{4n})\phi'_4(0) \quad (47)$$

First, since $v_{in} \to v_i$ ST in $L^2(\Omega) \Longrightarrow \begin{cases} v_{in}\phi_i(t) \to v_i\phi_i(t) \\ v_{in}\phi'_i(t) \to v_i\phi'_i(t) \end{cases}$ ST in $L^2(\mathrm{I}, \mathrm{V})$,

$v_{in}\phi_i(0) \to v_i\phi_i(0)$ ST in $L^2(\Omega)$ and $v_{in}\phi'_i(0) \to v_i\phi'_i(0)$ ST in $L^2(\Omega)$ for $i = 1,2,3,4$

Also since $v_{in} \to v_i$ ST in $V \Longrightarrow \begin{cases} v_{in}\phi'_i(t) \to v_i\phi'_i(t) \\ v_{in}\phi''_i(t) \to v_i\phi''_i(t) \end{cases}$ ST in $L^2(\mathrm{Q})$.

Second, we have $y_{int} \to y_{it}$ WK in $L^2(\mathrm{Q})$ and $y_{int} \to y_{it}$ WK in $L^2(\mathrm{I}, \mathrm{V})$ and ST in $L^2(\mathrm{Q})$. Third and on the other hand, let $w_{in} = v_{in}\,\phi_i$ and $w_i = v_i\,\phi_i$ then $w_{in} \to w_i$ ST in $L^2(\mathrm{Q})$ and then $w_{in}$ is measurable w.r.t. $(x, t)$, so using Assm (A-(i)), employing Proposition 1.3, the $(f_i(x, t, y_{in}, u_i), w_{in})dxdt$ is cont. w.r.t. $(y_{in}, u_i, w_{in})$, then

$$\int_0^T (f_i(y_{in}, u_i), v_{in})\phi_i(t)dt \to \int_0^T (f_i(y_i, u_i), v_i)\phi_i(t)dt, \forall i = 1,2,3,4$$

From these convergences, (26) , and (27) we can passage the limits in ((40)-(47)), to get

$$-\int_0^T (y_{1t}, v_1)\phi_1'(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1)]\phi_1(t)dt$$

$$=\int_0^T (f_1(y_1, u_1), v_1)\phi_1(t)dt + (y_1', v_1)\,\phi_1(0) \quad (48)$$

$$\int_0^T (y_1, v_1)\phi_1''dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1)]\phi_1(t)dt$$

$$=\int_0^T (f_1(y_1, u_1), v_1)\phi_1(t)dt + (y_1', v_1)\phi_1(0) - (y_1^0, v_1)\phi_1'(0) \quad (49)$$

$$-\int_0^T (y_{2t}, v_2)\phi_2'dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_1, v_2) + (y_2, v_2) - (y_3, v_2) - (y_4, v_2)]\phi_2(t)dt$$

$$=\int_0^T (f_2(y_2, u_2), v_2)\phi_2(t)dt + (y_2', v_2)\,\phi_1(0) \quad (50)$$

$$\int_0^T (y_2, v_2)\phi_2''(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_1, v_2) + (y_2, v_2) - (y_3, v_2) - (y_4, v_2)]\phi_2(t)dt$$

$$=\int_0^T (f_2(y_2, u_2), v_2)\phi_2(t)dt + (y_2', v_2)\phi_2(0) - (y_2^0, v_2)\phi_2'(0) \quad (51)$$

$$-\int_0^T (y_{3t}, v_3)\phi_3'(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3)]\phi_3(t)dt$$

$$=\int_0^T (f_3(y_3, u_3), v_3)\phi_3(t)dt + (y_3', v_3)\,\phi_3(0) \quad (52)$$

$$\int_0^T (y_3, v_3)\phi_3''dt + \int_0^T [(\nabla y_3, \nabla v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3)]\phi_3(t)dt$$

$$=\int_0^T (f_3(y_3, u_3), v_3)\phi_3(t)dt + (y_3', v_3)\phi_3(0) - (y_3^0, v_3)\phi_3'(0) \quad (53)$$

$$-\int_0^T (y_{4t}, v_4)\phi_4'(t)dt + \int_0^T [(\nabla y_4, \nabla v_4) - (y_1, v_4) + (y_2, v_4) - (y_3, v_4) + (y_4, v_4)]\phi_4(t)dt$$

$$=\int_0^T (f_4(y_4, u_4), v_4)\phi_4(t)dt + (y_4', v_4)\,\phi_4(0) \quad (54)$$

$$\int_0^T (y_4, v_4)\phi_4''dt + \int_0^T [(\nabla y_4, \nabla v_4) - (y_1, v_4) + (y_2, v_4) - (y_3, v_4) + (y_4, v_4)]\phi_4(t)dt$$

$$= \int_0^T (f_4(y_4, u_4), v_4)\phi_4(t)dt + (y_4', v_4)\phi_4(0) - (y_4^0, v_4)\phi_4'(0) \quad (55)$$

**Case1:** Choose $\phi_i \in C^2[0,T]$ s.t. $\phi_i(0) = \phi_i'(0) = \phi_i'(T) = \phi_i(T) = 0, \quad \forall i = 1,2,3,4$ substituting in (49), (51), (53), (55) IBPs2 the 1st terms in the LHS, i.e.

$$\int_0^T (y_{1tt}, v_1)\phi_1 dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1)]\phi_1(t)dt$$

$$=\int_0^T (f_1(y_1, u_1), v_1)\phi_1(t)dt \quad (56)$$

$$\int_0^T (y_{2tt}, v_2)\phi_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_1, v_2) + (y_2, v_2) - (y_3, v_2) - (y_4, v_2)]\phi_2(t)dt$$

$$=\int_0^T (f_2(y_2, u_2), v_2)\phi_2(t)dt) \quad (57)$$

$$\int_0^T (y_{3tt}, v_3)\phi_3 dt + \int_0^T [(\nabla y_3, \nabla v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3)]\phi_3(t)dt$$

$$=\int_0^T (f_3(y_3,u_3),v_3)\phi_3(t)dt \quad (58)$$

$$\int_0^T (y_{4tt},v_4)\phi_4 dt + \int_0^T [(\nabla y_4,\nabla v_4)-(y_1,v_4)+(y_2,v_4)-(y_3,v_4)+(y_4,v_4)]\phi_4(t)dt$$
$$= \int_0^T (f_4(y_4,u_4),v_4)\phi_4(t)dt \quad (59)$$

Hence $\vec{y}$ is a solution of (16),(18), (20) &(22) a.e. on I

**Case2:** Choose $\phi_i \in C^2[0,T]$ s.t. $\phi_i(T)=0, \& \phi_i(0)=0, \forall i=1,2,3,4$. MBS of (8), (10), (12) and (14) by $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$, and $\phi_4(t)$resp., IBS on $[0,T]$, then IBPs the 1st term in the LHS of each equation, then subtracting each one of these obtained equations from those corresponding in (48),(50),(52) &(54) resp. to get

$(y_{it}(0),v_i)\phi_i(0)=(y_i'(0),v_i)\phi_i(0), \forall i=1,2,3,4.$

**Case 3:** Choose $\phi_i \in C^2[0,T]$ s.t. $\phi_i(0)= \phi_i'(T)=\phi_i(T)=0, \phi_i'(0)\neq 0,\ \forall i=1,2,3,4$. MBS of (8), (10), (12) and (14) by $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$, and $\phi_4(t)$resp., IBS on $[0,T]$ then IBPs2 the 1st term in the LHS of each equation, then subtracting each one of these obtained equations from those corresponding in (49),(51),(53), and (55) resp. to get

$(y_i(0),v_i)\phi_i'(0)=(y_i^0,v_i)\phi_i'(0), \forall i=1,2,3,4$

In the last two cases, the ICs (19), (11), (13) and (14) are held.

To prove that $\vec{y}_n \to \vec{y}$ ST in $L^2(I,V)$, we start by IBS (36) on $[0,T]$, to get

$$\int_0^T \frac{d}{dt}\| \vec{y}_n \|_0^2\, dt + 2\int_0^T \| \vec{y}_n \|_1^2\, dt = 2\int_0^T [(y_{2n},y_{1nt})-(y_{3n},y_{1nt})-(y_{4n},y_{1nt})-$$
$$(y_{1n},y_{2nt})+(y_{3n},y_{2nt})+(y_{4n},y_{2nt})+(y_{1n},y_{3nt})-(y_{2n},y_{3nt})-(y_{4n},y_{4nt})+$$
$$(y_{1n},y_{4nt})-(y_{2n},y_{4nt})+(y_{3n},y_{4nt})]dt + 2\int_0^T [(f_1(y_1,u_1),y_{1nt})+(f_2(y_2,u_2),y_{2nt})+$$
$$(f_3(y_3,u_3),y_{3nt})+(f_4(y_4,u_4),y_{4nt})]dt \quad (60)$$

The same way applied to acquire (36)&(60), can be utilize here to obtain, i.e.

$$\| y_t(T) \|_0^2 - \| y_t(0) \|_0^2 + 2\int_0^T \| \vec{y}(t) \|_1^2\, dt = 2\int_0^T [(y_2,y_{1t})-(y_3,y_{1t})-(y_4,y_{1t})-(y_1,y_{2t})$$
$$+(y_3,y_{2t})+ (y_4,y_{2t})+(y_1,y_{3t})-(y_2,y_{3t})-(y_4,y_{4t})+(y_1,y_{4t})-(y_2,y_{4t})+$$
$$(y_3,y_{4t})]dt + 2\int_0^T [(f_1(y_1,u_1),y_{1t})+(f_2(y_2,u_2),y_{2t})+(f_3(y_3,u_3),y_{3t})+$$
$$(f_4(y_4,u_4),y_{4t})]dt \quad (61)$$

Since

$$\| \vec{y}_{nt}(T)-\vec{y}_t(T) \|_0^2 - \| \vec{y}_{nt}(0)-\vec{y}_t(0) \|_0^2 + 2\int_0^T \| \vec{y}_n(t)-\overline{y}(t) \|_1^2\, dt = \text{(a)-(b)-(c)} \quad (62)$$

$$\text{(a)}= \| \vec{y}_{nt}(T) \|_0^2 - \| \vec{y}_{nt}(0) \|_0^2 + 2\int_0^T \| \vec{y}_n(t) \|_1^2\, dt\ .$$

$$\text{(b)}=\Big(\vec{y}_{nt}(T),\vec{y}_t(T)) - \big(\vec{y}_{nt}(0),\vec{y}_t(0)\big)\Big) + 2\int_0^T \big(\vec{y}_n(t),\overline{y}(t)\big)_1 dt$$

$$\text{(c)}=(\ \vec{y}_t(T),\vec{y}_{nt}(T)-\vec{y}_t(T)) - (\ \vec{y}_t(0),\vec{y}_{nt}(0)-\vec{y}_t(0)) + 2\int_0^T \big(\overline{y}(t),\vec{y}_n(t)-\overline{y}(t)\big)_1 dt$$

Since $\vec{y}_n \to \vec{y}$ ST in $L^2(Q)$, and $\vec{y}_{nt} \to \vec{y}$ WK in $L^2(Q)$, then from (60) and the Assm on $f_i$, for $i=1,2,3,4$, we get

$(a) = 2\int_0^T [(y_{2n}, y_{1nt}) - (y_{3n}, y_{1nt}) - (y_{4n}, y_{1nt}) - (y_{1n}, y_{2nt}) + (y_{3n}, y_{2nt}) + (y_{4n}, y_{2nt})$
$+(y_{1n}, y_{3nt}) - (y_{2n}, y_{3nt}) - (y_{4n}, y_{4nt}) + (y_{1n}, y_{4nt}) - (y_{2n}, y_{4nt}) + (y_{3n}, y_{4nt})]dt +$
$2\int_0^T [(f_1(y_1, u_1), y_{1nt}) + (f_2(y_2, u_2), y_{2nt}) + (f_3(y_3, u_3), y_{3nt}) + (f_4(y_4, u_4), y_{4nt})]dt \longrightarrow$
$2\int_0^T [(y_2, y_{1t}) - (y_3, y_{1t}) - (y_4, y_{1t}) - (y_1, y_{2t}) + (y_3, y_{2t}) + (y_4, y_{2t}) + (y_1, y_{3t}) -$
$(y_2, y_{3t}) - (y_4, y_{4t}) + (y_1, y_{4t}) - (y_2, y_{4t}) + (y_3, y_{4t})]dt + 2\int_0^T [(f_1(y_1, u_1), y_{1t}) +$
$(f_2(y_2, u_2), y_{2t}) + (f_3(y_3, u_3), y_{3t}) + (f_4(y_4, u_4), y_{4t})]dt$

By the same way that was employed to acquire (27), it used here to acquire
$\vec{y}_{nt}(T) \to \vec{y}(T)$ ST in $L^2(\Omega)$ (63)

On the other hand, since $\vec{y}_n \to \vec{y}$ in $L^2(\mathrm{I}, \mathrm{V})$, then using (27) & (63), yield to

(b)⟶ RHS of (61)$= 2\int_0^T [(y_2, y_{1t}) - (y_3, y_{1t}) - (y_4, y_{1t}) - (y_1, y_{2t}) + (y_3, y_{2t}) +$
$(y_4, y_{2t}) + (y_1, y_{3t}) - (y_2, y_{3t}) - (y_4, y_{4t}) + (y_1, y_{4t}) - (y_2, y_{4t}) + (y_3, y_{4t})]dt +$
$2\int_0^T [(f_1(y_1, u_1), y_{1t}) + (f_2(y_2, u_2), y_{2t}) + (f_3(y_3, u_3), y_{3t}) + (f_4(y_4, u_4), y_{4t})]dt$

All the term in (c) approach to zero, so as the 1st two terms in the LHS of (62), hence (62) gives
$\int_0^T \| \vec{y}_n(t) - \vec{y}(t) \|_1^2 \, dt \to 0$ as $n \to \infty$, therefore $\vec{y}_n \to \vec{y}$ ST in $L^2(\mathrm{I}, \mathrm{V})$

**Uniqueness of the Solution:**

Let $\vec{y} = (y_1, y_2, y_3, y_4)$ and $\vec{\bar{y}} = (\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4)$ be two solutions of the SQVS of the WF ((8), (10), (12), and (14)), subtracting each equation from the other and replace $v_i = y_i - \bar{y}_i$ for each $i = 1,2,3,4$, i.e.

$((y_1 - \bar{y}_1)_{tt}, y_1 - \bar{y}_1) + \nabla(y_1 - \bar{y}_1, y_1 - \bar{y}_1) + (y_1 - \bar{y}_1, y_1 - \bar{y}_1) - (y_2 - \bar{y}_2, y_1 - \bar{y}_1) +$
$(y_3 - \bar{y}_3, y_1 - \bar{y}_1) + (y_4 - \bar{y}_4, y_1 - \bar{y}_1) = (f_1(y_1, u_1) - f_1(\bar{y}_1, u_1), y_1 - \bar{y}_1)$ (64)

$((y_2 - \bar{y}_2)_{tt}, y_2 - \bar{y}_2) + \nabla(y_2 - \bar{y}_2, y_2 - \bar{y}_2) + (y_1 - \bar{y}_1, y_2 - \bar{y}_2) + (y_2 - \bar{y}_2, y_2 - \bar{y}_2) -$
$(y_3 - \bar{y}_3, y_2 - \bar{y}_2) - (y_4 - \bar{y}_4, y_2 - \bar{y}_2) = (f_2(y_2, u_2) - f_2(\bar{y}_2, u_2), y_2 - \bar{y}_2)$ (65)

$((y_3 - \bar{y}_3)_{tt}, y_3 - \bar{y}_3) + \nabla(y_3 - \bar{y}_3, y_3 - \bar{y}_3) - (y_1 - \bar{y}_1, y_3 - \bar{y}_3) + (y_2 - \bar{y}_2, y_3 - \bar{y}_3) +$
$(y_3 - \bar{y}_3, y_3 - \bar{y}_3) + (y_4 - \bar{y}_4, y_3 - \bar{y}_3) = (f_3(y_3, u_3) - f_3(\bar{y}_3, u_3), y_3 - \bar{y}_3)$ (66)

$((y_4 - \bar{y}_4)_{tt}, y_4 - \bar{y}_4) + \nabla(y_4 - \bar{y}_4, y_4 - \bar{y}_4) + (y_1 - \bar{y}_1, y_4 - \bar{y}_4) + (y_2 - \bar{y}_2, y_4 - \bar{y}_4) -$
$(y_3 - \bar{y}_3, y_4 - \bar{y}_4) + (y_4 - \bar{y}_4, y_4 - \bar{y}_4) = (f_4(y_4, u_4) - f_4(\bar{y}_4, u_4), y_4 - \bar{y}_4)$ (67)

Collecting the above equalities for $i = 1,2,3,4$ using Lemma 1.2 in ref. [15] for the 1st in LHS of above equations, to get

$\frac{1}{2}\frac{d}{dt} \| (\vec{y} - \vec{\bar{y}})_t(t) \|_0^2 + 2 \| \vec{y} - \vec{\bar{y}} \|_1^2 = (f_1(y_1, u_1) - f_1(\bar{y}_1, u_1), y_1 - \bar{y}_1) +$
$(f_2(y_2, u_2) - f_2(\bar{y}_2, u_2), y_2 - \bar{y}_2) + (f_3(y_3, u_3) - f_3(\bar{y}_3, u_3), y_3 - \bar{y}_3) +$
$(f_4(y_4, u_4) - f_4(\bar{y}_4, u_4), y_4 - \bar{y}_4)$ (68)

The LHS of (68) is positive, IBS of it w.r.t. $t$ from 0 to $t$, and using Assumptions (A-ii) of the RHS of it, yields to

$\int_0^t \frac{d}{dt} \| \vec{y} - \vec{\bar{y}} \|_0^2 \le 2\int_0^t L \| \vec{y} - \vec{\bar{y}} \|_0^2 \, dt$, where $L = \max(L_1, L_2, L_3, L_4)$

Then, $\parallel \vec{y}-\vec{\bar{y}} \parallel_0^2 \leq 2\int_0^T L \parallel \vec{y}-\vec{\bar{y}} \parallel_0^2,$.

By using Lemma (1.2), to acquire

$\parallel \vec{y}-\vec{\bar{y}} \parallel_0^2 \leq 0e^{\int_0^T 2Ldt} = 0, \forall t \in I$

Again IBS of (68) w.r.t. $t$ from 0 to $T$, using the ICs and the above result for the RHS of the equations, to acquire

$\int_0^T \frac{d}{dt} \parallel \vec{y}-\vec{\bar{y}} \parallel_0^2 + 2 \parallel \vec{y}-\vec{\bar{y}} \parallel_1^2 dt \leq L\int_0^T \parallel (\vec{y}-\vec{\bar{y}}) \parallel_0^2 dt \Rightarrow \int_0^T \parallel (\vec{y}-\vec{\bar{y}})(t) \parallel_1^2 \leq 0 \quad \Rightarrow \parallel (\vec{y}-\vec{\bar{y}})(t) \parallel_{L^2(I,V)} = 0 \Rightarrow \vec{y}=\vec{\bar{y}}$ i.e. The solution is unique

**3.1 Lemma:** In addition to Assumptions (A), if the functions $f_i$ (for each $i = 1,2,3,4$) is Lipschitz w.r.t. $y_i$ & $u_i$, and if CCCQV is bounded, then the operator $\vec{u} \to \vec{y}_{\vec{u}}$ form $(L^2(Q))^4$ to $(L^\infty(I, L^2(\Omega)))^4$ or to $(L^2(Q))^4$ or to $(L^2(I,V))^4$ is continuous.

**Proof:** Let $\vec{u} = (u_1,u_2,u_3,u_4)$, $\vec{\bar{u}} = (\bar{u}_1,\bar{u}_2,\bar{u}_3,\bar{u}_4) \in (L^2(Q))^4$, $\delta\vec{u} = \vec{\bar{u}} - \vec{u}$, $\vec{u}_\varepsilon = \vec{u} + \varepsilon\delta\vec{u} \in (L^2(Q))^4$, for $\varepsilon > 0$, then by Theorem 3.1, $\vec{y} = \vec{y}_{\vec{u}} = (y_1,y_2,y_3,y_4)$ and $\vec{y}_\varepsilon = \vec{y}_{\vec{u}_\varepsilon} = (y_{1\varepsilon},y_{2\varepsilon},y_{3\varepsilon},y_{4\varepsilon})$ are their corresponding SQVS which satisfy the WF ((8)-(15)). Setting $\delta\vec{y}_\varepsilon = (\delta y_{1\varepsilon},\delta y_{2\varepsilon},\delta y_{3\varepsilon},\delta y_{4\varepsilon}) = \vec{y}_\varepsilon - \vec{y}$, to obtain

$(\delta y_{1\varepsilon tt}, v_1) + (\nabla\delta y_{1\varepsilon}, \nabla v_1) + (\delta y_{1\varepsilon}, v_1) - (\delta y_{2\varepsilon}, v_1) + (\delta y_{3\varepsilon}, v_1) + (\delta y_{4\varepsilon}, v_1)$
$= (f_1(y_1 + \delta y_{1\varepsilon}, u_1 + \varepsilon\delta u_1) - f_1(y_1,u_1), v_1)$ (69)

$\delta y_{1\varepsilon}(x,0) = 0$ and $\delta y_{1\varepsilon t}(x,0) = 0$ (70)

$(\delta y_{2\varepsilon tt}, v_2) + (\Delta\delta y_{2\varepsilon}, \nabla v_2) + (\delta y_{1\varepsilon}, v_2) + (\delta y_{2\varepsilon}, v_2) - (\delta y_{3\varepsilon}, v_2) - (\delta y_{4\varepsilon}, v_2)$
$= (f_2(y_2 + \delta y_{2\varepsilon}, u_2 + \varepsilon\delta u_2) - f_2(y_2,u_2), v_2)$ (71)

$\delta y_{2\varepsilon}(x,0) = 0$ and $\delta y_{2\varepsilon t}(x,0) = 0$ (72)

$(\delta y_{3\varepsilon tt}, v_3) + (\nabla\delta y_{3\varepsilon}, \nabla v_3) - (\delta y_{1\varepsilon}, v_3) + (\delta y_{2\varepsilon}, v_3) + (\delta y_{3\varepsilon}, v_3) + (\delta y_{4\varepsilon}, v_3)$
$= (f_3(y_3 + \delta y_{3\varepsilon}, u_3 + \varepsilon\delta u_3) - f_3(y_3,u_3), v_3)$ (73)

$\delta y_{3\varepsilon}(x,0) = 0$ and $\delta y_{3\varepsilon t}(x,0) = 0$ (74)

$(\delta y_{4\varepsilon tt}, v_4) + (\nabla\delta y_{4\varepsilon}, \nabla v_4) - (\delta y_{1\varepsilon}, v_4) + (\delta y_{2\varepsilon}, v_4) - (\delta y_{3\varepsilon}, v_4) + (\delta y_{4\varepsilon}, v_4)$
$= (f_4(y_4 + \delta y_{4\varepsilon}, u_4 + \varepsilon\delta u_4) - f_4(y_4,u_4), v_4)$ (75)

$\delta y_{4\varepsilon}(x,0) = 0$ and $\delta y_{4\varepsilon t}(x,0) = 0$ (76)

Substituting $v_i = \delta y_{i\varepsilon t}$ for $i = 1,2,3,4$ in (69),(71),(73)and (75) resp., collecting the obtained equations. Using the same way that is used to get (37), a similar equation can be obtained but with $\delta\vec{y}_\varepsilon$ in position of $\vec{y}_n$, then IBS on $[0,t]$, using Lip. on $f_i$ w.r.t. $(y_i,u_i)$ resp. for ($i = 1,2,3,4$) to get

$\int_0^t \frac{d}{dt}[\parallel \delta\vec{y}_{\varepsilon t}(t) \parallel_0^2 + \parallel \delta\vec{y}_\varepsilon \parallel_1^2]dt \leq$

$2\int_0^t [(|\delta y_{2\varepsilon}| + |\delta y_{3\varepsilon}| + |\delta y_{4\varepsilon}|)|\delta y_{1\varepsilon t}| + \bar{L}_1 |\delta y_{1\varepsilon t}|^2 + \varepsilon\bar{\bar{L}}_1 |\delta u_4||\delta y_{1\varepsilon t}|]dt +$

$+ 2\int_0^t [(|\delta y_{1\varepsilon}| + |\delta y_{3\varepsilon}| + |\delta y_{4\varepsilon}|)|\delta y_{2\varepsilon t}| + \bar{L}_2 |\delta y_{2\varepsilon t}|^2 + \varepsilon\bar{\bar{L}}_2 |\delta u_2||\delta y_{2\varepsilon t}|]dt$

$2\int_0^t [(|\delta y_{1\varepsilon}| + |\delta y_{2\varepsilon}| + |\delta y_{4\varepsilon}|)|\delta y_{3\varepsilon t}| + \bar{L}_3 |\delta y_{3\varepsilon t}|^2 + \varepsilon\bar{\bar{L}}_3 |\delta u_3||\delta y_{3\varepsilon t}|]dt$

$+2\int_0^t [(|\delta y_{1\varepsilon} + |\delta y_{2\varepsilon}| + |\delta y_{3\varepsilon}|)|\delta y_{4\varepsilon t}| + \bar{L}_4 |\delta y_{4\varepsilon t}|^2 + \varepsilon\bar{\bar{L}}_4 |\delta u_4||\delta y_{4\varepsilon t}|]dt$

Using the definitions of the norms and the relations between them, to get.

$\parallel \delta\vec{y}_{\varepsilon t} \parallel_0^2 + \parallel \delta\vec{y}_\varepsilon \parallel_1^2 \leq 3\int_0^t [\parallel \delta\vec{y}_\varepsilon \parallel_0^2 + \parallel \delta\vec{y}_{\varepsilon t} \parallel_1^2]dt + \tilde{L}_3\int_0^t [\parallel \delta\vec{y}_\varepsilon \parallel_0^2 + \parallel \delta\vec{y}_{\varepsilon t} \parallel_1^2]dt$

$$+\tilde{L}_4 \int_0^t \| \delta\vec{u} \|_0^2 \, dt + \tilde{L}_4 \int_0^t \| \delta\vec{y}_{\varepsilon t} \|_1^2 \, dt$$

$$\leq \tilde{L}_4 \| \delta\vec{u} \|_Q^2 + L_5 \int_0^t [\| \delta\vec{y}_\varepsilon \|_0^2 + \| \delta\vec{y}_{\varepsilon t} \|_1^2] dt$$

Where $\tilde{L}_3 = \max(\bar{L}_1, \bar{L}_2, \bar{L}_3, \bar{L}_4)$, $\tilde{L}_4 = \max(\bar{\bar{L}}_1, \bar{\bar{L}}_2, \bar{\bar{L}}_3, \bar{\bar{L}}_4)$, $L_5 = \max(3 + \tilde{L}_3, 3 + \tilde{L}_4 + \tilde{L}_3)$.

Applying the BGI, with$L^2 = \tilde{L}_4 e^{L_5 \int_0^t dt}$, to get

$\| \delta\vec{y}_{\varepsilon t} \|_0^2 + \| \delta\vec{y}_\varepsilon \|_1^2 \leq L^2 \| \delta\vec{u} \|_Q^2, \forall t \in I \Longrightarrow \| \delta\vec{y}_\varepsilon \|_1^2 \leq L^2 \| \delta\vec{u}(t) \|_Q^2, \forall t \in I$

$\| \delta\vec{y}_\varepsilon \|_{L^\infty(I,L^2(\Omega))} \leq L \| \delta\vec{u} \|_Q$, $\| \delta\vec{y}_\in \|_{L^2(I,V)} \leq L \| \delta\vec{u} \|_Q$ and $\| \delta\vec{y}_\varepsilon \|_Q \leq L \| \delta\vec{u} \|_Q$ .

## 4. The Existence of an OCCCQV

**4.1. Assumptions (B):** Consider $g_{li}$ for ($l = 0$ &$i = 1,2,3,4$) is of Carathéodory type on $Q \times (\mathbb{R} \times U)$, and satisfies the following sub quadratic condition w.r.t. $y_i \in \mathbb{R}$ and $u_i \in U_i$ ,

$|g_{li}(x,t,y_i,u_i)| \leq G_{li}(x,t) + C_{li1}y_i^2 + C_{li2}u_i^2$, where $G_{li} \in L^1(\mathrm{Q})$, for $(x,t) \in Q, l = 0$

**4.1 Lemma**: With assum. (B), the functional $\vec{u} \to G_0(\vec{u})$ is continuous on $(L^2(Q))^4$.

**Proof:** Using Assumptions (B) and proposition 1.3, the integral$\int_Q g_{0i}\,(x,t,y_i,u_i)dxdt$ is continuous on $L^2(Q)$, $\forall i = 1,2,3,4$, hence $G_0(\vec{u})$ is continuous on $(L^2(Q))^4$.

**4.2 Lemma :** let $g: Q \times \mathbb{R} \to \mathbb{R}$ is of Carathéodory type on $Q \times (\mathbb{R} \times \mathbb{R})$ and satisfies $|g\,(x,t,y,u)| \leq G\,(x,t) + c\,y^2 + \acute{c}u^2$, where $G(x,t) \in L^1(\mathrm{Q}), u \in U, c,\acute{c} \geq 0, U \subset \mathbb{R}$, is compact.. Then, $\int_Q g\,(x,y,u\,)dx$ is continuous on $L^2(Q)$ w.r.t. $y$.

**4.1 Theorem:** In addition to Assumptions (A&B), if the set $\vec{U}$ is convex and compact., $\vec{W}_A \neq \phi$, the function $f_i$, for ($i = 1,2,3,4$) have the form

$f_i(x,t,y_i,u_i) = f_{i1}(x,t,y_i) + f_{i2}(x,t)u_i$,where $|f_{i1}(x,t,y_i)| \leq \eta_i(x,t) + c_i \mid y_i \mid$,

$|f_{i2}(x,t)| \leq K_i, \eta_i \in L^2(\mathrm{Q}), c_i \geq 0$, for $i = 1,2,3,4$. Then there exists an OCCQV.

**Proof:** From the Assumptions on $U_i \subset \mathbb{R}$ for $i = 1,2,3,4$ and the Egorov's theorem, once get that $\vec{W}$ is weakly compact, since $\vec{W}_A \neq \phi$, there exists a minimum sequence $\{\vec{u}_k\} = \{(u_{1k}, u_{2k}, u_{3k}, u_{4k})\} \in \vec{W}_A$, $\forall k$ s.t. $\lim_{k\to\infty} G_0(\vec{u}_k) = \inf_{\vec{u}_k \in \vec{W}_A} G_0(\vec{\bar{u}})$. Since $\{\vec{u}_k\} \in \vec{W}_A$, $\forall k$ and $\vec{W}$ is weakly compact, there exists a subsequence of $\{\vec{u}_k\}$ say again $\{\vec{u}_k\}$ s.t. . $\vec{u}_k \to \vec{u}$ WK in $(L^2(Q))^4$ and $\| \vec{u}_k \|_Q \leq d$,$\forall k$. From Theorem 3.1, for each control $\{\vec{u}_k\}$ the WF (18), (20), (22), (24) has a unique SQVS , $\{\vec{y}_k = \vec{y}_{u_k}\}$ s.t. the norm $\| \vec{y}_k \|_{L^2(I,V)}$, $\| \vec{y}_{kt} \|_{L^2(Q)}$ are bounded, then by ATH there exists a subsequence of $\{\vec{y}_k\}$ and $\{\vec{y}_{kt}\}$, say again $\{\vec{y}_k\}$ and $\{\vec{y}_{kt}\}$, s.t. $\vec{y}_k \to \vec{y}$ WK in $(L^2(I,V))^4$, $\vec{y}_{kt} \to \vec{y}$ WK in $(L^2(Q))^4$. Now for each $k$. and by applying the ACTH [15], we get that there exists a subsequence of$\{\vec{y}_k\}$ say a gain $\{\vec{y}_k\}$ s.t. $\vec{y}_k \to \vec{y}$ ST in $(L^2(Q))^4$. Now since for each $k$, $\vec{y}_k$ is a SQVS of the WF ((18), (20), (22), (24)) resp. , substituting in these equations, MBSs of each equation by $\phi_i(t)$, $\forall i = 1,2,3,4$ (with $\phi_i \in C^2[0,T]$, s.t. $\phi_i(T) = \phi_i'(T) = 0, \phi_i(0) \neq 0$, $\phi_i'(0) \neq 0$). Rewriting the 1st term in the LHS of each one then IBS on $[0,T]$, finally IBPs for the 1st terms, one has

$$\int_0^T \frac{d}{dt}(y_{1kt}, v_1)\phi_1 dt + \int_0^T [(\nabla y_{1k}, \nabla v_1) + (y_{1k}, v_1) - (y_{2k}, v_1) + (y_{3k}, v_1) + (y_{4k}, v_1)]\phi_1 dt$$

$$= \int_0^T (f_{11}(x,t,y_{1k}), v_1)\phi_1(t)dt + \int_0^T (f_{12}(x,t)u_{1k}, v_1)\phi_1(t)dt \qquad (77)$$

$$\int_0^T \frac{d}{dt}(y_{2kt}, v_2)\phi_2 dt + \int_0^T [(\nabla y_{2k}, \nabla v_2) + (y_{1k}, v_2) + (y_{2k}, v_2) - (y_{3k}, v_2) - (y_{4k}, v_2)]\phi_2 dt$$

$= \int_0^T (f_{21}(x,t,y_{2k}),v_2)\phi_2(t)dt + \int_0^T (f_{22}(x,t)u_{2k},v_2)\phi_2(t)dt$ (78)

$\int_0^T \frac{d}{dt}(y_{3k},v_3)\phi_3 dt + \int_0^T [(\nabla y_{3k},\nabla v_3) - (y_{1k},v_3) + (y_{2k},v_3) + (y_{3k},v_3) + (y_{4k},v_3)]\phi_3 dt$

$= \int_0^T (f_{31}(x,t,y_{3k}),v_3)\phi_3(t)dt + \int_0^T (f_{32}(x,t)u_{3k},v_3)\phi_3(t)dt$ (79)

$\int_0^T \frac{d}{dt}(y_{4k},v_4)\phi_4 dt + \int_0^T [(\nabla y_{4k},\nabla v_4) - (y_{1k},v_4) + (y_{2k},v_4) - (y_{3k},v_4) + (y_{4k},v_4)]\phi_4 dt$

$= \int_0^T (f_{41}(x,t,y_{4k}),v_4)\phi_4(t)dt + \int_0^T (f_{42}(x,t)u_{4k},v_4)\phi_4(t)dt$ (80)

The same steps that are utilized in the proof of Theorem 3.1, can also be used here to passage the limit in the LHS of ((77)-(80)). We persist in the passage of the limit in RHS of ((77))-(80)) as follows. Let $\forall i = 1,2,3,4$, $v_i \in C[\bar{\Omega}]$, $w_i = v_i\phi_i(t)$, then $w_i \in C[\bar{Q}] \in L^\infty(Q) \subset L^2(Q)$, set $\bar{f}_{i1}(y_{ik}) = f_{i1}(y_{ik})w_i$, then $\bar{f}_{i1}: Q \times \mathbb{R} \to \mathbb{R}$ is of Carathéodory type , utilizing proposition 3.1, to get the integral $\int_Q f_{i1}(y_{ik})w_i\, dxdt$, is continuous w.r.t. $y_{ik}$ $\forall i = 1,2,3,4$. but $y_{ik} \to y_i$ ST in $L^2(Q)$ and $u_{ik} \to u_i$ WK in $L^2(Q)$, then

$\int_Q f_{i1}(y_{ik})w_i\, dxdt \to \int_Q f_{i1}(y_i)w_i\, dxdt, \forall\ \ w_i \in C[\bar{Q}]$ , for $i = 1,2,3,4$ (81)

$\int_Q f_{i2}(x,t)u_{ik}w_i\, dxdt \to \int_Q f_{i2}(x,t)u_i w_i\, dxdt, \forall\ \ w_i \in C[\bar{Q}]$ , for $i = 1,2,3,4$ (82)

Form the density of $C(\bar{\Omega})$ in $V$, (81) &(82) are satisfies for each $v_i \in V$ for $i = 1,2,3,4$, hence the following WF is obtained

$(y_{1t},v_1) + (\nabla y_1,\nabla v_1) + (y_1,v_1) - (y_2,v_1) + (y_3,v_1) + (y_4,v_1)$
$= (f_{11}(x,t,y_1),v_1) + (f_{12}(x,t)u_1,v_1), \forall v_1 \in V$ a.e. on I (83)

$(y_{2t},v_2) + (\Delta y_2,\nabla v_2) + (y_1,v_2) + (y_2,v_2) - (y_3,v_2) - (y_4,v_2)$
$= (f_{21}(x,t,y_2),v_2) + (f_{22}(x,t)u_2,v_2), \forall v_2 \in V$ a.e. on I (84)

$(y_{3t},v_3) + (\nabla y_3,\nabla v_3) - (y_1,v_3) + (y_2,v_3) + (y_3,v_3) + (y_4,v_3)$
$= (f_{31}(x,t,y_3),v_3) + (f_{32}(x,t)u_3,v_3)\forall v_3 \in V$ a.e. on I (85)

$(y_{4t},v_4) + (\nabla y_4,\nabla v_4) - (y_1,v_4) + (y_2,v_4) - (y_3,v_4) + (y_4,v_4)$
$= (f_{41}(x,t,y_4),v_4) + (f_{42}(x,t)u_4,v_4)\ \forall v_4 \in V$ a.e. on I (86)

Also, the same steps employed in Theorem 3.1 can be employed here to obtain that the ICs are held, hence $\vec{y}$ is the SQVS.

Now since $\forall i = 1,2,3,4$, $g_{0i}(x,t,y_i,u_i)$ is continuous w.r.t. $(y_i,u_i)$, and $U_i$ is compact. With $u_i(x,t) \in U_i$ a.e. in $Q$, then using lemma 4.2, to get

$\int_Q g_{0i}(x,t,y_{ik},u_{ik})\, dxdt \to \int_Q g_{0i}\,(x,t,y_i,u_{ik})dxdt$ (87)

But $g_{0i}(x,t,y_i,u_i)$ is continuous and convex w.r.t. $u_i$ then $\int_Q g_{0i}\,(x,t,y_i,u_i)dxdt$ is weakly lowe semi cont. (WLSC) w.r.t. $u_i$, $\forall i = 1,2,3,4$, i.e.

$$\int_Q g_{0i}\,(x,t,y_i,u_i)dxdt \le \liminf_{k\to\infty} \int_Q [g_{0i}\,(x,t,y_i,u_{ik}) - g_{0i}\,(x,t,y_{ik},u_{ik})]dxdt$$

$$+\liminf_{k\to\infty} \int_Q g_{0i}\,(x,t,y_{ik},u_{ik})dxdt,$$

$$\le \liminf_{k\to\infty} \int_Q g_{0i}\,(x,t,y_{ik},u_{ik})dxd$$

$\Rightarrow \sum_{i=1}^{4} \int_Q g_{0i}\,(x,t,y_i,u_i)dxdt \le \sum_{i=1}^{4} \int_Q g_{0i}\,(x,t,y_{ik},u_{ik})dxdt$, thus

$G_0(\vec{u}) \leq \lim_{k\to\infty} \inf_{\vec{u}_k \in \vec{W}_A} G_0(\vec{u}_k) = \lim_{k\to\infty} G_0(\vec{u}_k) = \inf_{\vec{u}_k \in \vec{W}_A} G_0(\vec{\vec{u}})$, then

$G_0(\vec{u}) \leq \inf_{\vec{u}_k \in \vec{W}_A} G_0(\vec{\vec{u}}) \Rightarrow G_0(\vec{u}) \leq \min_{\vec{u}_k \in \vec{W}_A} G_0(\vec{\vec{u}})$, then $\vec{u}$ is OQCCCV.

## 5. Conclusion

The method of Galerkin with the Aubin compactness theorem is used successfully to demonstrate the existence theorem of a unique state quaternary vector solution for the weak form for the quaternary nonlinear hyperbolic boundary value problem where the classical continuous control quaternary vector is considered given under suitable hypotheses.

The continuity operator between the state quaternary vector solution of the weak form for the quaternary nonlinear hyperbolic boundary value problem and the corresponding classical continuous control quaternary vector is demonstrated. The existence theorem of an optimal classical continuous control quaternary vector under suitable hypotheses is demonstrated.

## References

1. Grigorenko, N.; Grigorieva, È. V.; Roi, P. K.; Khailov, E.N. Optimal Control Problems for a Mathematical Model of the Treatment of Psoriasis, *Computational Mathematics and Modeling*. **2019**, *304*, 352-363.

2. Syahrini, I.; Masabar, R.; Aliasuddin, A.; Munzir, S.; Hazim, Y. The Application of Optimal Control through Fiscal Policy on Indonesian Economy. *The Journal of Asian Finance, Economics and Business.* ***2021**, 8, 3, 0741-0750.*

3. Rigatos, G.; Abbaszadeh, M. Nonlinear Optimal Control for Multi-DOF Robotic Manipulators with Flexible Joints. *Optimal Control Applications and Methods*. **2021**,*6*,1708-1733.

4. Kahina, L.; Spiteri, P.; Demim, F.; Mohamed, A. ; Nemra, A.; Messine, F. Application Optimal Control for a Problem Aircraft Flight, *Journal of Engineering Science and Technology Review*. **2018**, *11*, 156-164.

5. Bors, D. ; Walczak, S. Optimal Control Elliptic Systems with Distributed and Boundary Controls . *Nonlinear Analysis***. 2005**, *63*,5-7,1367-1376.

6. Al-Hawasy, J. "The Continuous Classical Optimal Control Problem of a Nonlinear Hyperbolic Partial Differential Equations (CCOCP)", Al-Mustansiriyah Journal of Science. **2008,** *19, 3*, 96-110,.

7. Chryssoverghi I., Al-Hawasy J. The Continuous Classical Optimal Control Problem of Semi Linear Parabolic Equations (CCOCP), *Journal of Karbala University*. **2010,**8,3.

8. Al-Hawasy J., and Al-Rawdhanee E. H, The Continuous Classical Optimal Control of a Couple of Non-linear Elliptic Equation", *Mathematical Theory and Modeling*. 2014,*4*,14.

9. Al-Hawasy J. The Continuous Classical Optimal Control of a Coupled Nonlinear Hyperbolic Partial Differential Equations with Equality and Inequality Constraints, *Iraqi Journal of Science*. **2016**,*57*, 2C, 1528-1538.

10. Al-Hawasy, J.; Kadhem, G.M. The Continuous Classical Optimal Control for a Coupled Nonlinear Parabolic Partial Differential Equations with Equality and Inequality Constraints. *Journal of Al-Nahrain University*, **2016,** *19*, *1*, 173-186.

11. Al-Hawasy J., Jasim D. K.The Continuous Classical Optimal Control Problems for Triple Nonlinear Elliptic Boundary Value Problem. *Ibn Al-Haitham Jour. for Pure and Applied Sciences*. **2020**, *33, 3*, 101-112**.**

12. Al-Hawasy, J.; Rasheed, Th. Continuous Classical Optimal Control of Triple Nonlinear Parabolic Partial Differential Equation , Iraqi J. Sci. **2022**,*63*,*2*,690-701.

13. Al-Hawasy, J.A.; Ali, L.H. Constraints Optimal Control Governing by Triple Nonlinear Hyperbolic Boundary Value Problem . Hindawi. *Journal of Applied Mathematics*, **2020**.
14. Chryssoverghi, I. Optimization . National Technical university of Athens-Greece, **2003.**

15 Temam, R. Navier-Stokes Equations. North- Holand Publishing Company,**1977**.