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Solving of the Quadratic Fractional Programming Problems by a Modified Symmetric Fuzzy Approach

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Abstract

The aims of the paper are to present a modified symmetric fuzzy approach to find the best workable compromise solution for quadratic fractional programming problems (QFPP) with fuzzy crisp in both the objective functions and the constraints. We introduced a modified symmetric fuzzy by proposing a procedure, that starts first by converting the quadratic fractional programming problems that exist in the objective functions to crisp numbers and then converts the linear function that exists in the constraints to crisp numbers. After that, we applied the fuzzy approach to determine the optimal solution for our quadratic fractional programming problem which is supported theoretically and practically. The computer application for the algorithm was tested, and finally compared modified symmetric fuzzy approach with the modified simplex approach which is shown in the table 1. Finally, the procedures of numeric results in the paper indicate that modified symmetric fuzzy approach is reliable and saves valuable time.

Keywords: fuzzy system, modified simplex approach, modified symmetric fuzzy, quadratic fractional programming problems.

1. Introduction

Mathematical programming finds many applications in the field of management. Optimization of resources in any organization is mostly addressed with the use of mathematical programming. QFPP, which deals with situations where a ratio between two mathematical functions is either maximized or minimized, is an important class of mathematical programming problems that has attracted a lot of research and interest because it is useful in production planning, economic and financial planning, medical services, and healthcare planning.

There are many managerial decision-making situations where the uncertainties in working situations are best explained by use of fuzzy set theory. The nation of fuzzy set theory is developed by [1], since then a considerable number of scholars have expressed their interest in the application of fuzzy set theory. [2] established the notion of decision making in a fuzzy environment and their concept of fuzzy decision making is used in mathematical programming [3]. Many authors have discussed the use of fuzzy set theory in QFPP e.g., [4, 5 ,6 ,7].

The fuzzy system plays a very significant role in the theory of linear programming (LP). Thus, researchers have shown their interest in the concept of the script for a linear program under a fuzzy environment as well e.g., [8], their works formulates a quadratic fractional bi-level (QFBL) programming problem with probabilistic constraints in both the first (ruler) and second (supporter) levels, with two-parameter exponential random variables with recognized probabilistic, and fuzziness as a triangular and trapezoidal fuzzy number. Using a chance-constrained programming approach, the issue is first turned into an a like deterministic quadratic fractional fuzzy bi-level programming model in the suggested model. Second, each objective function of the bi-level QFP issue has its own nonlinear membership function in the proposed model. In [9], the fuzzy environment with parabolic concave membership functions is used to investigate a certain form of convex fractional programming issue and its dual. An ambition level technique is used to determine appropriate duality outcomes. It differs from other research in that it uses parabolic concave membership functions to reflect the decision-level maker's of pleasure. The Fully Fuzzy Quadratic Fractional Programming problem (FFQFPP) is solved using a solution process in which all variables and parameters are triangular fuzzy integers. In this case, FFQFPP was converted into a Multi-Objective Quadratic Fractional Programming issue (MOQFPP). Using a mathematical programming method, MOQFPP is then turned into an analogous multi-objective quadratic programming problem. A fuzzy goal programming paradigm for handling bilevel programming issues with decision-makers' objectives in quadratic fractional form. The membership functions for the defined fuzzy goals of the decision-makers' objectives at both levels are established first in the suggested approach. Then, in order to find the most satisfactory answer in the decision-making environment, a fuzzy goal programming model is constructed to minimize the group regret of the degree of satisfaction of both decision-makers. In [10], the linear constraints of quadratic fractional objective programming problem (QFOPP) have been established and developed. The Wolfe's technique and a modified simplex approach were used to solve the specific case for this problem.

To extend this work, we study and introduce a modified symmetric fuzzy approach to solve QFPP. An algorithm and practical technique with theoretical support are suggested, also solving such a problem by a modified simplex approach, and test the validity will be compared against both outcomes.

2. Fundamentals of Fuzzy set, and Fuzzy Algebra Operation

Some fundamentals and algebra operations of the fuzzy environment that are utilized in this research are listed [11, 12]:

2.1. Fundamentals of Fuzzy set

Fuzzy set: A fuzzy set $\widetilde{\mathrm{A}}$ in X is a set of ordered pairs: $\widetilde{\mathrm{A}} = \{(x, \mu_{\widetilde{\mathrm{A}}}(x)) | x \in \mathrm{X}\}$, $\mu_{\widetilde{\mathrm{A}}}(x)$ is titled by the membership function of x in $\widetilde{\mathrm{A}}$. If X is a assemble of contraption defined generally by x. The support: $S(\widetilde{\mathrm{A}})$ is the crisp set of all $x \in \mathrm{X}$ if $\mu_{\widetilde{\mathrm{A}}}(x) > 0$ is called by the support of fuzzy set $\widetilde{\mathrm{A}}$. α-level set :The fragile set of elements that pertinence to the fuzzy set $\widetilde{\mathrm{A}}$ at the lower to the grade α is named by α-level set: $\mathrm{A}_\alpha = \{x \in \mathrm{X} | \mu_{\widetilde{\mathrm{A}}}(x) \geq \alpha\}$, $\mathrm{A}'_\alpha = \{x \in \mathrm{X} | \mu_{\widetilde{\mathrm{A}}}(x) > \alpha\}$ is defined by "strong α-level set" or "strong α-cut".

2.2. Fuzzy Algebra Operation

Algebraic sum: The algebraic sum (likelihood sum) $\tilde{\mathrm{C}} = \widetilde{\mathrm{A}} + \widetilde{\mathrm{B}}$ is illustrated as $\tilde{\mathrm{C}} = \{(x, \mu_{\widetilde{\mathrm{A}}+\widetilde{\mathrm{B}}}(x)) | x \in \mathrm{X}\}$, where $\mu_{\widetilde{\mathrm{A}}+\widetilde{\mathrm{B}}}(x) = \mu_{\widetilde{\mathrm{A}}}(x) + \mu_{\widetilde{\mathrm{B}}}(x) - \mu_{\widetilde{\mathrm{A}}}(x).\mu_{\widetilde{\mathrm{B}}}(x)$.

Bounded sum: The bounded sum $\tilde{\mathrm{C}} = \widetilde{\mathrm{A}} \oplus \widetilde{\mathrm{B}}$ is defined as $\tilde{\mathrm{C}} = \{(x, \mu_{\widetilde{\mathrm{A}} \oplus \widetilde{\mathrm{B}}}(x)) | x \in \mathrm{X}\}$, where $\mu_{\widetilde{\mathrm{A}} \oplus \widetilde{\mathrm{B}}}(x) = \min\{1, \mu_{\widetilde{\mathrm{A}}}(x) + \mu_{\widetilde{\mathrm{B}}}(x)\}$.

Bounded difference: The bounded difference $\tilde{\mathrm{C}} = \widetilde{\mathrm{A}} \ominus \widetilde{\mathrm{B}}$ is defined as $\tilde{\mathrm{C}} = \{(x, \mu_{\widetilde{\mathrm{A}} \ominus \widetilde{\mathrm{B}}}(x)) | x \in \mathrm{X}\}$, where $\mu_{\widetilde{\mathrm{A}} \ominus \widetilde{\mathrm{B}}}(x) = \min\{1, \mu_{\widetilde{\mathrm{A}}}(x) + \mu_{\widetilde{\mathrm{B}}}(x)\}$.

3. Mathematical Model Formula

3.1. Linear Programming models

A special kind of decision model introduced by [13], is considered as Linear programming models, the restrictions define the decision space; the objective function defines the "target" (utility function), and the kind of choice is decision making under conditions. The standard linear programming model is as follows:

$$\left.\begin{array}{l} maximize\ f(x) = c^T x \\ \text{Such that:}\quad Ax \leq b \\ \quad x \geq 0 \\ \text{With } c, x \in R^n, b \in R^n, A \in R^{m \times n} \end{array}\right\} \qquad (1)$$

3.2. Linear Fractional Programming Formula

The linear fractional programming problem (LFPP) introduced by [14] can be stated in the following manner:

$$\left.\begin{array}{l} Max. W = \dfrac{c'_1 x + \beta}{c'_2 x + \delta} \\ \text{subject to:}\quad Ax = b, x \geq 0\ . \end{array}\right\} \qquad (2)$$

Where (i) $x, c_1,$ and c_2 are $n \times 1$ column vectors. (ii) A is $n \times m$ matrix. (iii) b is an $m \times 1$ vectors. (iv) The prime { $'$ } on the vectors c_1, and c_2 indicate as a transpose of vectors and (v) β, δ are comparatively numeric. The procedure solution of such problems solved by [15].

3.3. Quadratic Programming Problem (QPP)

Quadratic Programming is a particular kind of mathematical optimization problem It is a problem of reducing or maximizing a quadratic function of numerous variables under linear constraints. The QPP can be written as follows::

$$\left.\begin{array}{l} \text{Max.} z \text{ (or Min. z)} = x^T G x + c^T x + \alpha \\ \qquad \text{Subject to:} \\ \qquad Ax \begin{pmatrix} \le \\ \ge \\ = \end{pmatrix} \text{b} \\ \qquad x \ge 0 \end{array}\right\} \tag{3}$$

where $A = (a_{ij})_{m \times n}$ matrix of coefficients, for $i = 1,2, \dots, m$ and $j = 1,2, \dots, n$, $b = (b_1, b_2, \dots, b_m)^T$, $x = (x_1, x_2, \dots, x_n)^T$, $c^T = (c_1, c_2, \dots, c_n)$, α is scalar and $G = (g_{ij})_{n \times n}$ is assumed negative definite of the problem in maximization, and positive definite of problem in minimization, The objective function is quadratic, whereas the restrictions are linear. [16].

3.4. Quadratic Fractional Programming Problem (QFPP)

This section discusses a specific situation of QFPP's mathematical form, which contains a quadratic function in the numerator and a denominator that can be represented as the product of two linear components, as the following:

$$\left.\begin{array}{l} Max. Z = \frac{(c_1{}^T x + \alpha)(c_2{}^T x + \beta)}{(e_1{}^T x + \gamma)(e_2{}^T + \delta)} = \frac{(w_1)(w_2)}{(w_3)(w_4)} \\ \text{Subject to:} \\ \qquad Ax \begin{bmatrix} \ge \\ \le \\ = \end{bmatrix} b \\ \qquad x \ge 0 \end{array}\right\} \tag{4}$$

A is $m \times n$ matrix, where x is $n \times 1$ column vectors of decision variables, c_1, c_2, e are $n \times 1$ column vectors of constants, b is $m \times 1$ column vectors of constants. α, β, γ and δ are scalars, the prime (T) over the vectors c_1, c_2, e_1 and e_2 denoted the transpose of the vectors. $w_1 w_2$ and $w_3 w_4$ are the values of numerator and denominator of objective function respectively

3.5. Symmetric Fuzzy linear programming

We will suppose that the decision-maker can select an ambition level, z, for the value of the objective function he or she wishes to reach, and that each restriction is modelled as a fuzzy set in model (1). Our fuzzy album then transforms into:

Find x such that

$$\left.\begin{aligned} &c^T x \geqq z \\ &Ax \leqq b \\ &x \geq 0 \end{aligned}\right\} \qquad (5)$$

Here ≦ signifies the fuzzified form of ≤ and has the linguistic interpretation "essentially smaller than or equal to." ≧ signifies the fuzzified form of ≥ and has the linguistic interpretation "essentially greater than or equal to." The objective function of a model (1) might have to be written as a minimizing goal to consider z as an upper bound founded by [13].

3.6. Zimmermann's Approach: A Symmetric Model

Principle for determining the fuzzy choice to solve the problem's fuzzy system of inequalities (FSI) (1). As a result, the following neat linear programming issue arises.

$$Max\ \alpha$$

subject to,

$$\mu_0(c^T x) = \left(1 - \frac{z - c^T}{p_0}\right) \geq \alpha$$

$$\mu_i(A_i x) = \left(1 - \frac{A_i x - b_i}{p_i}\right) \geq \alpha \ , i = 1, \dots, m$$

$$\alpha \in [0,1]$$

$$x \geq 0$$

or

$$Max\ \alpha$$

subject to

$$\left.\begin{aligned} &c_j^T x \geq z_0 - (1-\alpha)p_0 \\ &A_i x \geq b_i + (1-\alpha)p_i. \quad , i = 1, \dots, m \\ &x \geq 0 \quad , \quad \lambda \in [0,1] \end{aligned}\right\} \qquad (6)$$

4. Modified Simplex Approach Development

Dantzig invented the simplex approach in 1947. The Simplex technique is a systematic approach that involves moving from one basic variable solution (on the vertex) to another in a predefined order in order to elicit the value of the objective function. This leaping from vertex to vertex operation is repeated. If the goal function improves with each leap, no basis will ever be duplicated, and there will be no need to return to the vertex because it has already been covered. The algorithm must lead to the optimum vertex in a finite number of steps since the number of vertices is finite.

For addressing linear programming problems, the Simplex method is an iterative (step-by-step) approach. It entails the following: Having a fundamental workable solution to constraint equations.

ii) Determine whether it is the best option.

iii) Using a set of criteria to improve the initial trial solution and continuing the procedure until an ideal solution is found. For more details.

4.1. Modified Simplex Approach to Solve Quadratic Fractional Programming Problem

This section deals with the solution of the quadratic fractional programming problem developed by [17], which the solution procedure has some similarity. This technology can be used to high-speed computing with success. This strategy can be used if the problem's constraints are linear functions. i.e., the issue is of the following existence:

$$Max. z\ (or\ Min. z) = \frac{({c_1}^T x + \alpha)({c_2}^T x + \beta)}{({e_1}^T x + \gamma)({e_2}^T + \delta)}$$

Subject to:

$$Ax \begin{pmatrix} \leq \\ \geq \\ = \end{pmatrix} b$$

$$x \geq 0$$

A is $m \times n$ matrix;

x, c_1, c_1, e_1 and e_2 are $n \times 1$ column vectors;

b is $m \times 1$ column vectors;

α, β, δ are γ scalars and prime (T) denoted the transpose of the vector.

Where $z_1 = ({c_1}^T x + \alpha)({c_2}^T x + \beta)$

$z_2 = ({e_1}^T x + \gamma)({e_2}^T + \delta)$

And supposed $z_1^1 = ({c_1}^T x + \alpha)$

$z_1^2 = ({c_2}^T x + \beta)$

$z_2^1 = ({e_1}^T x + \gamma)$

$z_2^2 = ({e_2}^T + \delta)$

To apply simplex process, first Δ_{j1} and Δ_{j2}need to be found from the coefficients of the first linear vector and second linear vector of objective function respectively, by using the following formula:

$\Delta_{ij} = c_{ij} - c_{Bi} x_{ij}\ , i\ = 1,2. j\ =\ 1,2, \dots, m\ +\ n$

$z_1 = {z_1}^{(1)} {z_1}^{(2)}$

$z_2 = {z_2}^{(1)}{z_2}^{(2)}$

$z = z_1/z_2$ the values of objective functions

in this approach the formula is defined to find Δ_j from $\Delta\xi_{1j}, \Delta\xi_{2j}, z_1, z_2$ following:

$\Delta_j = z_1\Delta\xi_{1j} + z_2\Delta\xi_{2j}$.

5. Modify Symmetric Fuzzy Approach (MSFA) to Solve Quadratic Fractional Programming Problems

Our modified approach depends on the adopted "fuzzy" version of formula (4) is:
find x such that

$$\left.\begin{array}{c} \frac{({c_1}^T x+\alpha)({c_2}^T x+\beta)}{({e_1}^T x+\gamma)({e_2}^T+\delta)} \gtrsim \frac{(w_1)(w_2)}{(w_3)(w_4)} \\ Ax \lesssim b \\ x \geqq 0 \end{array}\right\} \tag{7}$$

here $c_1, c_2, e_1$and e_2 are the vector of coefficients of numerator and denominator respectively of the ratio of the goal function, b denoted as a vector of constraints, and A is the factor of matrix. The sign " $\lesssim$" indicate the fuzzified type of " $\leqq$ " and read out "basically smaller than or commensurate to". Note that (7) is wholly symmetric with observance to goal function and constraints,
To solve the above problem, First choose an appropriate membership function for each of the fuzzy inequality of (7). In particular, let f_0 denote the membership function for objective function and $f_i, i = 1, \dots, m$ denote the membership function for constraint, let p_0 and $p_i, i = 1, \dots, m$ be the permissible tolerances for objective function and constraint and let f_0 and f_i , $i = 1, \dots, m$ be a continuous and nondecreasing linear membership which is explained below:
We have two cases that include only for f_0:
<u>First case</u> if $(w_3)(w_4) > (w_1)(w_2)$

$$f_0(c^T x) = \begin{cases} 1 \\ \frac{1-(((w_1)(w_2)-(w_3)(w_4)) - (({c_1}^T x+\alpha)({c_2}^T x+\beta)-({e_1}^T x+\gamma)({e_2}^T+\delta)))}{p_0} \\ 0 \end{cases}$$
$$\left.\begin{array}{c} for\ (({c_1}^T x + \alpha)({c_2}^T x + \beta) - ({e_1}^T x + \gamma)({e_2}^T + \delta)) > ((w_1)(w_2) - (w_3)(w_4)) \\ for\ ((w_1)(w_2) - (w_3)(w_4)) - p_0 \le (({c_1}^T x + \alpha)({c_2}^T x + \beta) - ({e_1}^T x + \gamma)({e_2}^T + \delta)) \le ((w_1)(w_2) - (w_3)(w_4)) \\ for\ (({c_1}^T x + \alpha)({c_2}^T x + \beta) - ({e_1}^T x + \gamma)({e_2}^T + \delta)) < ((w_1)(w_2) - (w_3)(w_4)) - p_0 \end{array}\right\} \tag{8}$$

With

$$\left. f_i(A_i x) = \begin{cases} 1 & for\ A_i x < b_i \\ \frac{1-(A_i x - b_i)}{p_i} & for\ b_i \le A_i x \le b_i + p_i, \quad \mathrm{i} = 1, \dots, \mathrm{m} \\ 0 & for\ A_i x > b_i + p_i \end{cases} \right\} \tag{9}$$

Now, in order to solve the problem we dentify the fuzzy decision as following approach:
The crisp linear programming denoted by $(LP)_{(f,\lambda)}$ is:
In case $(w_3)(w_4) > (w_1)(w_2)$ we have:

$$Max\ \lambda$$

subject to,

$$f_0(c^T x) = \left(\frac{1-(((w_1)(w_2)-(w_3)(w_4)) -((c_1{}^T x+\alpha)(c_2{}^T x+\beta)-(e_1{}^T x+\gamma)(e_2{}^T+\delta)))}{p_0}\right) \geq \lambda$$

$$f_i(A_i x) = \left(\frac{1-(A_i x - b_i)}{p_i}\right) \geq \lambda \ , i = 1, \dots, m$$

$$\lambda \in [0,1]$$

$$x \geq 0$$

Or

$$Max\, \lambda$$

$$\left.\begin{array}{l} \text{subject to,} \\ ((w_1)(w_2) - (w_3)(w_4)) \leq 1 - \lambda p_0 - ((c_1{}^T x + \alpha)(c_2{}^T x + \beta) - (e_1{}^T x + \gamma)(e_2{}^T + \delta)) \\ \sum_{i=1}^{n} a_{ik} x_k \leq b_k + 1 - \lambda(p_k) \, , k = 1, \dots, m \\ \lambda, x_k \geq 0 \, , k = 1, \dots, m. \end{array}\right\} \quad (10)$$

<u>Second case</u> if $(w_3)(w_4) < (w_1)(w_2)$

$$f_0(c^T x) = \begin{cases} 1 & for\ ((c_1{}^T x + \alpha)(c_2{}^T x + \beta) - (e_1{}^T x + \gamma)(e_2{}^T + \delta)) > ((w_3)(w_4) - (w_1)(w_2)) \\ \frac{1-(((w_3)(w_4)-(w_1)(w_2)) -((c_1{}^T x+\alpha)(c_2{}^T x+\beta)-(e_1{}^T x+\gamma)(e_2{}^T+\delta)))}{p_0} & for\ ((w_3)(w_4) - (w_1)(w_2)) - p_0 \leq ((c_1{}^T x + \alpha)(c_2{}^T x + \beta) - (e_1{}^T x + \gamma)(e_2{}^T + \delta)) \leq ((w_3)(w_4) - (w_1)(w_2)) \\ 0 & for\ ((c_1{}^T x + \alpha)(c_2{}^T x + \beta) - (e_1{}^T x + \gamma)(e_2{}^T + \delta)) < ((w_3)(w_4) - (w_1)(w_2)) - p_0 \end{cases} \quad (11)$$

With

$$f_i(A_i x) = \begin{cases} 1 & for\ A_i x < b_i \\ \frac{1-(A_i x - b_i)}{p_i} & for\ b_i \leq A_i x \leq b_i + p_i \, , \quad \mathrm{i} = 1, \dots, \mathrm{m} \\ 0 & for\ A_i x > b_i + p_i \end{cases} \quad (12)$$

Now, in order to solve the problem we identify the fuzzy decision as following approach:
The crisp linear programming denoted by $(LP)_{(f,\lambda)}$ is:
In case $(w_3)(w_4) < (w_1)(w_2)$ we have:

$$Max\, \lambda$$

subject to,

$$f_0(c^T x) = \left(\frac{1-(((w_3)(w_4)-(w_1)(w_2))-((c_1^T x+\beta)-(c_2^T x+\delta)))}{p_0}\right) \geq \lambda$$

$$f_i(A_i x) = \left(\frac{1-(A_i x - b_i)}{p_i}\right) \geq \lambda \ , i = 1, \dots, m$$

$$\lambda \in [0,1] \quad , \ x \geq 0$$

Or

$$\left.\begin{aligned}
&Max\ \lambda\\
&\text{subject to,}\\
&((w_3)(w_4)-(w_1)(w_2)) \le 1-\lambda p_0-((c_1^T x+\beta)-(c_2^T x+\delta))\\
&\textstyle\sum_{i=1}^{n} a_{ik}x_k \le b_k+1-\lambda(p_k)\ ,k=1,\dots,m\\
&\lambda, x_k \ge 0\ ,k=1,\dots,m.
\end{aligned}\right\} \qquad (13)$$

$w_1 w_2$ and $w_3 w_4$ are the values of numerator and denominator of objective function respectively, and $c_j^T x$ are $n \times 1$ column vectors for $j = 1,2$

By using Simplex algorithm, we can solve problem (10 or 13), the optimum result to (10 or 13) is the optimal solution as well to (7) and as well to (4).

Theorem 5.1: In problem (7), let the membership functions $f_i: \mathbb{R} \to [0,1], (i = 0, \dots, m)$, be continuous and nondecreasing. Then the fuzzy solution of (7) is assumed by the parametric solution of the parametric fractional linear programming problem.

Proof. We have to solve the following problem to obtain the solution of (7).

$(LP)_{(f,\lambda)}$

$$\begin{aligned}
&Max\ \lambda\\
&\text{subject to}\\
&f_0(c^T x) \ge \lambda\\
&f_i(A_i x) \ge \lambda\ ,i=1,\dots,m\\
&x \ge 0 \quad ,\ \lambda \in [0,1]
\end{aligned}$$

Where $f_0(c^T x) = \inf(f_{0j}(c_j^T x))$ and $f_{\text{i}}(A_i x) = \inf\left(f_{\text{ij}}(A_{ij}x)\right), i = 1, \dots, m$. But f_{ij} is nondecreasing and continuous function, given that f_{ij}^{-1} exist, and $f_{0j}(c_j^T x) \ge \lambda \Rightarrow c_j^T x \ge f_{0j}^{-1}(\lambda)$ with $f_{\text{ij}}(A_{ij}x) \ge \lambda \Rightarrow A_{ij}x \ge f_{ij}^{-1}(\lambda)$.

Therefore, the problem $(LP)_{(f,\lambda)}$ can be written as

$$\begin{aligned}
&Max\ \lambda\\
&\text{subject to}\\
&c_j^T x \ge f_{0j}^{-1}(\lambda)\\
&A_i x \ge f_{ij}^{-1}(\lambda). \quad ,i=1,\dots,m\\
&x \ge 0 \quad ,\ \lambda \in [0,1]
\end{aligned}$$

Which is equivalent to (6)

$$\begin{aligned}
&Max\ \alpha\\
&\text{subject to}\\
&c_j^T x \ge z_0-(1-\alpha)p_0\\
&A_i x \ge b_i+(1-\alpha)p_i. \quad ,i=1,\dots,m\\
&x \ge 0 \quad ,\ \alpha \in [0,1]
\end{aligned}$$

Thus, the fuzzy linear programming (10 or 13) can be solved by the (crisp) linear programming

$$\begin{aligned}
&Max\ \alpha\\
&\text{subject to}
\end{aligned}$$

$$c_j^T x \geq z_0 - (1-\alpha)p_0$$
$$A_i x \geq b_i + (1-\alpha)p_i. \quad , i = 1, \dots, m$$
$$x \geq 0 \quad , \qquad \alpha \in [0,1]$$

Which is the same as problem (6) with $\lambda = \alpha$.

Remark 5.1 If the problem (7) has fuzzy as well as crisp constraints, then in the alike (crisp) linear fractional programming problem, the original crisp constraints will not have any modification as for them the tolerances are nil.

6. Algorithm to Solve QFPP by Using MSFA

To find the result of optimal solution for the QFPP of formulation (4), a procedure is given as below:

Step One: Convert the main problem to the fuzzy version as formula 7

Step Two: Find the value of p_0 and $p_i, i = 1, \dots, m$ which are the permissible tolerances for objective function and constraint by using formula (8, 11) and (9, 12), respectively.

Step Three: Construct the crisp linear programming $(LP)_{(f,\lambda)}$ by using formula (10, 13)

Step Four: Optimize step 3 by using simplex algorithm**.**

7. An Illustrative Numerical Example and Results:

In this section, we solve an example through MSFA and compared with MODIFIED SIMPLEX APPROACH.

Example: Maximize Z = $\frac{(4x_1+6x_2-2)(2x_1+3x_2+1)}{(6x_1+9x_2+3)^2}$

Subject to:

$$x_1 + 3x_2 \leq 5$$
$$2x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

Step one:

The fuzzy version of problem is:

find x such that:

$$\frac{(4x_1+6x_2-2)(2x_1+3x_2+1)}{(6x_1+9x_2+3)(6x_1+9x_2+3)} \gtrsim \frac{(w_1)(w_2)}{(w_3)(w_4)}$$
$$x_1 + 3x_2 \lesssim 5$$
$$2x_1 + x_2 \lesssim 5$$
$$x_1, x_2 \geqq 0$$

Step Two: Find the value of p_0 and $p_i, i = 1, \dots, m$ which are the permissible tolerances for objective function and constraint.

Firstly, obtain the value of $(w_1), (w_2), (w_3)$ and (w_4) by simplex algorithm as follows:

$Max. w_1 = (4x_1 + 6x_2 - 2)$ Subject to: $x_1 + 3x_2 \leq 5$ $2x_1 + x_2 \leq 2$ $x_1, x_2 \geq 0$ Optimal solution: $x_1 = 0.2, x_2 = 1.6$ $Max. w_1 = 8.4$	$Max. w_1 = (2x_1 + 3x_2 + 1)$ Subject to: $x_1 + 3x_2 \leq 5$ $2x_1 + x_2 \leq 2$ $x_1, x_2 \geq 0$ Optimal solution: $x_1 = 0.2, x_2 = 1.6$ $Max. w_1 = 6,2$

$Max. w_1 = (6x_1 + 9x_2 + 3)$ Subject to: $x_1 + 3x_2 \leq 5$ $2x_1 + x_2 \leq 2$ $x_1, x_2 \geq 0$ Optimal solution: $x_1 = 0.2, x_2 = 1.6$ $Max. w_1 = 18.6$	$Max. w_1 = (6x_1 + 9x_2 + 3)$ Subject to: $x_1 + 3x_2 \leq 5$ $2x_1 + x_2 \leq 2$ $x_1, x_2 \geq 0$ Optimal solution: $x_1 = 0.2, x_2 = 1.6$ $Max. w_1 = 18.6$

Now, find p_0 by using **case 1** at decision variable $x_1 = 0.2, x_2 = 1.6$ since $(w_3)(w_4) > (w_1)(w_2)$, so can get it as follow:

$$\left(\frac{1 - ((18.6)(18.6)) - (8.6)(6.2)) - ((4x_1 + 6x_2 - 2)(2x_1 + 3x_2 + 1) - (6x_1 + 9x_2 + 3)(6x_1 + 9x_2 + 3))}{p_0}\right) \geq \lambda$$

$\rightarrow p_0 \geq 2$

Now, find p_1 by using formula (9) at decision variable $x_1 = 0.2, x_2 = 1.6$ as follow:

$$\left(\frac{1 - ((x_1 + 3x_2) - 5)}{p_1}\right) \geq \lambda$$

$\rightarrow p_1 \geq 2$

For p_2 by using formula (9) at decision variable $x_1 = 0.2, x_2 = 1.6$ as follow:

$$\left(\frac{1 - (2x_1 + x_2) - 5)}{p_2}\right) \geq \lambda$$

$\rightarrow p_2 \geq 2$

Step Three: Construct the crisp linear programming $(LP)_{(f,\lambda)}$ by using formula (10,13)

$$Max\, \lambda$$

subject to,

$$2\lambda - 6x_1 + 3x_2 \leq 30.6$$
$$2\lambda + x_1 + 3x_2 \leq 6$$
$$2\lambda + 2x_1 + x_2 \leq 3$$
$$\lambda, x_1, x_2 \geq 0$$

Step four: Optimize crisp linear programming $(LP)_{(f,\lambda)}$ by using simplex algorithm**,** found that the decision variable is $\boldsymbol{x_1 = 0.2, x_2 = 1.6}$ **and** $\boldsymbol{Max. Z = 0.15}$

In additional, will solve the such numerical example by computational procedure of quadratic fractional algorithm introduced by [10], section 4.1 to compare between two methods (MSFA) and (Modified Simplex Approach).

Now,

Maximize Z = $\frac{(4x_1+6x_2-2)(2x_1+3x_2+1)}{(6x_1+9x_2+3)^2}$

Subject to:

$$x_1 + 3x_2 \leq 5$$
$$2x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

introduce the slack variables $S_1 \geq 0$ and $S_2 \geq 0$ the problem in the standard form becomes:

$$Max. W = \frac{(4x_1+6x_2-2)(2x_1+3x_2+1)}{(6x_1+9x_2+3)(6x_1+9x_2+3)} = \frac{z^{(1)}z^{(2)}}{z^{(3)}z^{(4)}}$$

subject to:

$$x_1 + 3x_2 + S_1 = 5$$
$$2x_1 + x_2 + S_2 = 2$$
$$x_1, x_2, S_1, S_2 \geq 0$$

for starting table, we will find $\Delta_1 = -72$, $\Delta_2 = -108$, $\Delta_3 = 0, \Delta_4 = 0$. We choose $\min \Delta_j$ (Δ_2 in this case). Thus, z can be increased by taking X_2 in the basis. The method to determine the leaving variables and also the new value of $X_{ij}, X_B, \Delta_1, \Delta_2, \Delta_3$ and Δ_4 corresponding to improved solution will be the same as for ordinary simplex method. Thus, S_1 will be the departing variable.

Starting table

					c_{1B}	4	6	0	0	
					c_{2B}	2	3	0	0	
					C_{1B}	6	9	0	0	
					C_{2B}	6	9	0	0	
B	C_{1B}	C_{2B}	c_{1B}	c_{2B}	X_B	X_1	X_2	S_1	S_2	$\text{Min}\left(\frac{X_B}{X_1}\right)$
S_1	0	0	0	0	5	1	3	1	0	1.6
S_2	0	0	0	0	2	2	1	0	1	2
	$z_1^{(1)} = c_{1B}X_B = -2$				Δk_1^{1j}	-4	-6	0	0	
	$z_1^{(2)} = c_{2B}X_B = 1$				Δk_1^{2j}	-2	-3	0	0	
	$z_2^{(1)} = C_{1B}X_B = 3$				Δk_2^{1j}	-6	-9	0	0	
	$z_2^{(2)} = C_{2B}X_B = 3$				Δk_2^{2j}	-6	-9	0	0	
	$z_1 = z_1^{(1)} z_1^{(2)} = -2$				$\Delta\xi_{1j}$	-8	-12	0	0	
	$z_2 = z_2^{(1)} z_2^{(2)} = 9$				$\Delta\xi_{2j}$	0	0	0	0	
	$z = \frac{z_1}{z_2} = -0.222$				Δ_j	-72	-108	0	0	

First iteration table

Introducing X_2 and dropping S_1, we get the following table:

					c_{1B}	4	6	0	0	
					c_{2B}	2	3	0	0	
					C_{1B}	6	9	0	0	
					C_{2B}	6	9	0	0	
B	C_{1B}	C_{2B}	c_{1B}	c_{2B}	X_B	X_1	X_2	S_1	S_2	$\text{Min}\left(\frac{X_B}{X_1}\right)$
X_2	9	9	6	3	5/3	1/3	1	1/3	0	5
S_2	0	0	0	0	0.33	$\boxed{1.66}$	0	-0.33	1	1.19
	$z_1^{(1)} = c_{1B}X_B = 8$				Δk_1^{1j}	-2	0	2	0	
	$z_1^{(2)} = c_{2B}X_B = 6$				Δk_1^{2j}	-1	0	1	0	
	$z_2^{(1)} = C_{1B}X_B = 18$				Δk_2^{1j}	-3	0	3	0	
	$z_2^{(2)} = C_{2B}X_B = 18$				Δk_2^{2j}	-3	0	3	0	
	$z_1 = z_1^{(1)} z_1^{(2)} = 48$				$\Delta\xi_{1j}$	-4	0	4	0	
	$z_2 = z_2^{(1)} z_2^{(2)} = 324$				$\Delta\xi_{2j}$	0	0	0	0	
					Δ_j	-1296	0	1296	0	

Second iteration table

Introducing X_1 and dropping S_2, we get the following table:

c_{1B}	4	6	0	0

B	C_{1B}	C_{2B}	c_{1B}	c_{2B}	X_B	X_1	X_2	S_1	S_2	Min $\left(\frac{X_B}{X_1}\right)$
					c_{2B}	2	3	0	0	
					C_{1B}	6	9	0	0	
					C_{2B}	6	9	0	0	
X_2	9	9	6	3	1.6	0	1	0.39	-0.2	
X_1	6	6	4	2	0.2	1	0	-1.19	0.602	
	$z_1^{(1)} = c_{1B}X_B = 8.4$				Δk_1^{1j}	0	0	1.34	2	
	$z_1^{(2)} = c_{2B}X_B = 6.2$				Δk_1^{2j}	0	0	0.67	0.6	
	$z_2^{(1)} = C_{1B}X_B = 18.6$				Δk_2^{1j}	0	0	2.01	1.8	
	$z_2^{(2)} = C_{2B}X_B = 18.6$				Δk_2^{2j}	0	0	2.01	1.8	
	$z_1 = z_1^{(1)} z_1^{(2)} = 52.08$				$\Delta\xi_{1j}$	0	0	2.68	2.4	
	$z_2 = z_2^{(1)} z_2^{(2)} = 345.95$				$\Delta\xi_{2j}$	0	0	0	0	
	$z = \frac{z_1}{z_2} = 0.15$				Δ_j	0	0	926342	83020	

Since all $\Delta_j \geq 0$ we have reached the optimal solution:

$\mathbf{X_1 = 0.2},\ \mathbf{X_2 = 1.6},\ \mathbf{S_1 = 0},\ \mathbf{S_2 = 0},\ \boldsymbol{Max.W = 0.15}$.

The fuzzy and unfuzzy results are shown in the following:

Table 1- The result of numerical example solved in two methods Unfuzzy [Modified Simplex Approach] and Fuzzy [MSFA]

Unfuzzy (Modified Simplex Approach)		Fuzzy (MSFA)	
$x_1 = 0.2$ $x_2 = 1.6$ $W = 0.15$		$x_1 = 0.2$ $x_2 = 1.6$ $W = 0.15$	
Constraints:			
1	5	1	6
2	2	2	3

We conclude from the solution we have same profit under regard all constraints.

8. Conclusion

This paper used modify symmetric fuzzy approach. After converting to crisp linear programming, the maximum value of QFPP was discovered and compared to the modified simplex technique. The values of the objective function are used to compare these techniques, the study found that $Max. z$ resulted of those two methods are same but modify symmetric fuzzy solve the problem directly without iteration while modified simplex approach take two iterations. Therefore, we can solve special case of QFPP by these two methods under our approach and algorithm. consequently, reliable results have been found.

References

1. Zadeh, L.A., Fuzzy sets, in Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh. *World Scientific*, **1996**, 394-432.
2. Bellman, R.E.; L.A. Zadeh, Decision-making in a fuzzy environment. *Management science*, **1970**, *17, 4*, 141-164.
3. Tanaka, H., T. Okuda,; K. Asai, Fuzzy mathematical programming. *Transactions of the society of instrument and control engineers*, **1973**, *9, 5,* 607-613.
4. Biswas, A.; K. Bose. Fuzzy goal programming approach for quadratic fractional bilevel programming. *in Proceedings of the International Conference on Scientific Computing (CSC)*. **2011**. Citeseer.
5. Kassa, S.M.; T.H. Tsegay, An iterative method for tri-level quadratic fractional programming problems using fuzzy goal programming approach. *Journal of Industrial Engineering International*, **2018**, *14, 2*, 255-264.
6. Rakshit, M.; M.S. Kumar, A Bilevel Quadratic–Quadratic Fractional Programming through Fuzzy Goal Programming approach. *International Journal of Mathematics Trends and Technology (IJMTT)*, **2016**, *38, 3*, 130-137.
7. Abdulrahim, B.K.; S.O. Abdulla, A MOQFPP Method for Solving Quadratic Fractional Programming under Fuzzy Environment, *JUG* ,**2019**, *6*, *1*, 114-122.
8. Kausar, H., A.Y. Adhami,; A. Rahman, Quadratic Fractional Bi-level Fuzzy Probabilistic Programming Problem When bi Follows Exponential Distribution. *Reliability: Theory & Applications,* **2021**, *16*, *2*, 289-300
9. Saxena, P.; R. Jain, Duality for a nonlinear fractional programming under fuzzy environment with parabolic concave membership functions. *International Journal of Advanced and Applied Sciences*, **2017**, *4, 1*, 131-136.
10. Nejmaddin A. Suleiman; Maher A. Nawkhass, Solving quadratic fractional programming problem. *International Journal of Applied Mathematics Research*, **2013**, *2, 2,* 303-309.
11. Hamadameen, A., A novel technique for solving multiobjective fuzzy linear programming problems. *ARO-the scientific journal of Koya university*, **2017**, *5, 1*, 1-8.
12. Zimmermann, H.-J., Description and optimization of fuzzy systems. *International journal of general System*, **1975**, *2, 1*, 209-215.
13. Zimmermann, H.-J., Fuzzy set theory—and its applications. **2011**: *Springer Science & Business Media.*
14. Sharma, S., Nonlinear and Dynamic Programming. *Kedar Nath Ram Nath and CO., Meerut, India,* **1980,** 547.
15. Nejmaddin A. Suleiman; Maher A. Nawkhass, Symmetric fuzzy approach to solve complementary multi-objective linear fractional programming problem. *International Journal of Nonlinear Analysis and Applications*, **2022**: 1-6.
16. Antoniou, A. and W.-S. Lu, *Practical optimization*. **2007**: Springer.
17. Maher A. Nawkhass; Nejmaddin A. Suleiman; A new modified simplex method to solve quadratic fractional programming problem and compared it to a traditional simplex method by using pseudoaffinity of quadratic fractional functions. *Applied Mathematical Sciences,* **2013**, *7, 76*, 3749-3764.