IHJPAS. 36(1)2023

Ibn Al-Haitham Journal for Pure and Applied Sciences

Journal homepage: jih.uobaghdad.edu.iq

Approximation Solution of Fuzzy Singular Volterra Integral Equation by Non-Polynomial Spline

Zahraa A. Ibrahim
Department of Mathematics, College of Sciences,Mustansiriyah University, Baghdad, Iraq.
zara.ali1431@gmail.com

Nabaa N. Hasan
Department of Mathematics, College of Sciences,Mustansiriyah University, Baghdad, Iraq.
alzaer1972@uomustansiriyah.edu.iq

Article history: Received 22 July 2022, Accepted 21 Augest 2022, Published in January 2023.

doi.org/10.30526/36.1.2860

Abstract

A non-polynomial spline (NPS) is an approximation method that relies on the triangular and polynomial parts, so the method has infinite derivatives of the triangular part of the NPS to compensate for the loss of smoothness inherited by the polynomial. In this paper, we propose polynomial-free linear and quadratic spline types to solve fuzzy Volterra integral equations (FVIE) of the 2nd kind with the weakly singular kernel (FVIEWSK) and Abel's type kernel. The linear type algorithm gives four parameters to form a linear spline. In comparison, the quadratic type algorithm gives five parameters to create a quadratic spline, which is more of a credit for the exact solution. These algorithms process kernel singularities with a simple technique. Illustrative examples use MathCad software to deal with upper and lower-bound solutions to fuzzy problems. The method provides a reliable way to ensure that an exact solution is approximated. Also, figures and tables show the potential of the method.

Keywords: Fuzzy Volterra integral equation, weakly singular kernel, non-polynomial spline.

1. Introduction

The fuzzy set theory is one of the essential theories introduced by' [1-2] Solving linear and nonlinear Abel fuzzy integral equations using Laplace transforms. [3] found the solution of the fuzzy Volterra integral equation of 2^{nd} kind with Abel's type kernel by applying Laplace Adomain decomposition method. [4] proposed a piecewise spline collocations method with gradient meshes for FVIEWSK, showing that the gradient meshes are superior to uniform ones for this problem. [5] solved linear Volterra integral equations of the second kind with a weakly singular kernel by using the sixth order of non-polynomial spline functions. In this work, we drive linear and quadratic NPS to solve FVIEWSK and numerically drive linear NPS of FVIE with Abel's types kernel. This paper is organized as follows: in section 2, some preliminaries are given about basic definitions. Section 3 is about NPS' functions in linear and quadratic. Section 4 considers the linear and quadratic methods of FVIEWSK and section 5 NPS of FVIE with Abel's type kernel. Furthermore, in section 6, some illustrated examples are given, showing the method's accuracy. Finally, conclusions are given in section 7.

2. Preliminaries concepts:

In this section some concepts and definitions of fuzzy set theory and kinds of singular kernels are given.

Definition 2.1[4]: A fuzzy number is a fuzzy set $A: R \to [0,1]$ such that:
(i) A is upper semi continuous,
(ii) $A(x) = 0$ outside some interval $[a, \mathrm{d}]$,
(iii) There are real numbers $b, : a \leq b \leq c \leq d$
for which

(1) $A(x)$ is monotonically increasing on $[a, b]$,
(2) $A(x)$ is monotonically decreasing on $[c, d]$,
(3) A $(x) = 1, b \leq x \leq c$

Definition 2.2 [6]: Let X be a Cartesian product of universes $X_1, X_2, \dots, X_r$ and $\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_r$ be r-fuzzy subsets of $X_1, X_2, \dots, X_r$, respectively, $f = \mathrm{X} \to Y$,
$y = f(x_1, x_2, \dots, x_r)$, then the extension principles allow us to define a fuzzy set $\tilde{B}$ in Y by:
$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \mid y = f(x_1, x_2, \dots, x_r), x_1, x_2, \dots, x_r \in \mathrm{X}\}$
where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup Min\{\mu_{\widetilde{A_1}}(x_1), \dots, \mu_{\widetilde{A_r}}(x_r)\}, & f^{-1}(y) \neq \emptyset \\ (x_1, x_2, \dots, x_r) \in f^{-1}(y) & \\ 0\,, other\ wise & \end{cases}$$

and f^{-1} is the inverse image of f
for $r = 1$, the fuzzy extension principles, of course reduces to
$\tilde{B} = f(\tilde{A}) = f(\{(y, \mu_{\tilde{B}}(y)) \mid y = f(x)\,, x \in \mathrm{X}\}$
where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup \mu_{\tilde{A}}(x) & , f^{-1}(y) \neq \emptyset \\ x \in f^{-1}(y) & \\ 0\,, & other\ wise \end{cases}$$

which is the definition of the fuzzy mapping.

Definition 2.3 Crisp number [6]:
A crisp number a is represented by:

$$A(x) = \begin{cases} 1\ if\ x = a \\ 0\ if x \neq a \end{cases}$$

A crisp interval $[\mathrm{c}, \mathrm{d}]$ is represented by a fuzzy set

$$B(x) = \begin{cases} 1 \text{ if } \mathrm{x} \in [\mathrm{c}, \mathrm{d}] \\ 0 \text{ if } \mathrm{x} \notin [\mathrm{c}, \mathrm{d}] \end{cases}$$

Definition 2.4 some types of kernels [7]:

i. **Cauchy kernel**
If the kernel $k(x,t)$ is of the form
$k(x,t) = \frac{H(x,t)}{x-t}$
where $k(x,t)$ is differentiable function of (x,t) with $H(x,t) \neq 0$, then the integral, equation is said to be a singular equation with Cauchy kernels.

ii. **Weakly singular kernel**
If the kernel $k(x,t)$ is of the form
$k(x,t) = \frac{H(x,t)}{|x-t|^{\alpha}}$
where $0 < \alpha < 1$ and $H(x,t)$ is a differentiable and continuous function with $H(x,t) \neq 0$,

then the integral equation is said to be weakly singular kernel.

iii. **Strongly singular kernel**
If the kernel $k(x,t)$ is of the form
$k(x,t) = \frac{H(x,t)}{|x-t|^{\alpha}} \qquad \alpha \geq 2$
where $H(x,t)$ is a differentiable function of (x,t) with $H(x,t) \neq 0$, then the integral equation is said to be strongly singular kernel.

3. NPS Functions

The standard form of the 2nd FVIE is defined below [8]: $u(x,r) = f(x,r) + \lambda \int_0^x k(x,t,u(t,r),r)dt$ (1)

Consider the partition $\Delta = \{r_0, r_1, r_2, \ldots, r_n\}$ of $[a,b] \subset R$, let $S(\Delta)$ and indicate the arrangement of piecewise polynomial on subinterval [9]
$I_i = [r_i, r_{i+1}]$ of segment Δ. Let $u(r)$ be the exact solution. The NPS of n order $S_i(r)$ at the form is:

$$S_i(r) = a_i \sin\ k(r-r_i) + b_i \cos\ k(r-r_i) + \cdots + y_i(r-r_i)^{n-1} + z_i. \qquad (2)$$

Where a_i , b_i , ... , y_i and z_i constants.

In this section, we introduce different types of NPS functions, linear NPS functions and quadratic NPS functions.

3.1 Linear Non-Polynomial Spline (LNPS):

We consider the LNPS method for finding an approximate solution of FVIE of the second kind [9]

Consider the grid point r_i on the interval $[a,b]$, as follows:

$$a = r_0 < r_1 < r_2 < \cdots < r_n = b \qquad (3)$$

$$r_i = r_0 + ih \ , i = 0,1,2, \ldots, n \qquad (4)$$

$$h = \frac{b-a}{n} \qquad (5)$$

Where n is an appositive integer. Let $u(r)$ be the exact solution of equation (1) and $S_i(r)$ be an approximation to $u_i = u(r_i)$ obtained by the segment $s_i(r)$. Each NPS segment $s_i(r)$ has the form:

$$S_i(r) = a_i \sin\ k(r-r_i) + b_i \cos\ k(r-r_i) + c_i(r-r_i) + d_i. \qquad (6)$$

Where a_i, b_i, c_i *and* d_i are constant and k is the frequency of the trigonometric function which will be used to raise the accuracy of the method. We consider the following relations:

$S_i(r_i) = u(r_i)$
$S'_i(r_i) = kb_i + c_i \approx u'_i(r_i)$
$S''_i(r_i) = -k^2 b_i \approx u''_i(r_i)$
$S'''_i(r_i) = -k^3 a_i \approx u'''_i(r_i)$

Now, we can obtain the values of a_i, b_i , c_i and d_i as follows:

$$a_i = \frac{-1}{k^3} u'''(r_i) \qquad (7)$$

$$b_i = \frac{-1}{k^2} u''(r_i) \qquad (8)$$

$$c_i = u'(r_i) + \frac{1}{k} u''(r_i) \qquad (9)$$

$$d_i = u(r_i) + \frac{1}{k} u''(r_i) \qquad (10)$$

For i=0, 1, 2,…,n

We differentiate equation (1) three times with respect to r, then put r = a to get:

$$u_0 = u(a) = f(a) \qquad (11)$$

$u_0' = u'(a) = f'(a) + k(a,a)u(a)$ (12)

Let $E\ (x,t,u(t,r);r) = \frac{\partial k(x,t\ u(t,r))}{\partial r}$

$u''(r) = f_0''(r) + \int_0^x \frac{\partial E\ (x,t,u(t,r);r)}{\partial r}\ dt + 2\mathrm{E}(\mathrm{x},\mathrm{t},\mathrm{u}(\mathrm{t},\mathrm{r}):\mathrm{r})$

$u_0''(a) = u''(a) = f_0''(a) + 2E\left(a,a,u(a)\right)$ (13)

Let $\mathrm{F}\ (\mathrm{x},\mathrm{t},\mathrm{u}(\mathrm{t}\,,\mathrm{r});\mathrm{r}) = \frac{\partial E(x,t,u(t,r);r)}{\partial r}$

$\mathrm{u}'''(r) = f'''(\mathrm{r}) + \int_0^x \frac{\partial \mathrm{F}\ (\mathrm{x},\mathrm{t},\mathrm{u}(\mathrm{t}\,,\mathrm{r});\mathrm{r})}{\partial r} dt + 3\mathrm{F}\ (\mathrm{x},\mathrm{t},\mathrm{u}(\mathrm{t}\,,\mathrm{r});\mathrm{r})$ $u_0'''(\mathrm{a}) = f'''(a) + 3\mathrm{F}\ \left(\mathrm{a},\mathrm{a},\mathrm{a},\mathrm{u}(\mathrm{a})\right)$ (14)

3.2 Quadratic Non-polynomial Splines (QNPS):

We consider the QNPS method for finding an approximate solution of FVIE of the second kind on the interval $[a,b]$ as the above equations (3),(4) and (5)

The form of QNPS function [10] is

$Q_i(x,r) = a_i \sin\ k(r-r_i) + b_i \cos\ k(r-r_i) + c_i(r-r_i) + d_i(r-r_i)^2 + e_i$ (15)

where a_i, b_i, c_i, d_i and e_i are constants. We differentiate equation (15) four times with respect to r and put r = a then replace r by r_i in the relation equation (15) to yield:

$Q_i(x,r) = a_i + e_i$

$Q'_i(x,r) = kb_i + c_i$

$Q''_i(x,r) = -k^2 a_i + 2d_i$

$Q'''_i(x,r) = -k^3 b_i$

$Q_i^{(4)}(x,r) = k^{(4)} a_i$

We obtain the values of a_i, b_i, c_i, d_i and e_i from the above relation, as follows

$a_i = \frac{Q_i^{4}(x,r_i)}{k^4}$ (16)

$b_i = -\frac{Q_i'''(x,r_i)}{k^3}$ (17)

$c_i = Q'_i(x,r_i) - kb_i$ (18)

$d_i = \frac{Q''_i(x,r_i) + k^2 a_i}{2}$ (19)

$e_i = Q_i(x,r_i) - a_i$ (20)

4. LNPS and QNPS Methods of FVIEWSK:

In this section, we use LNPS and QNPS methods to compute numerical solution of FVIEWSK which is

$u(x,r) - \int_0^x \frac{t^{\mu-1}}{x^{\mu}} u(x,t)dt = f(x,r), \qquad x \in [0,T]$ (21)

Where $0 < \mu < 1$ and f is a known function [11]. There is singularity at $x = 0$ and $t = 0$ for any positive value of t .

To solve equation (21), we multiply both sides by x^{μ} to get

$x^{\mu}\, u(x,r) - \int_0^x t^{\mu-1}\, u(x,t)dt = f(x,r)x^{\mu}$ (22)

Hence, we differentiate equation (22) with respect to x, then we get

$x^{\mu}\, u'(x,r) + \mu x^{\mu-1} u(x,r) - \frac{1}{x^{1-\mu}} u(x,r) = \mu x^{\mu-1} f(x,r) + xf'(x,r)x^{\mu}$ (23)

And multiplying both sides by $x^{1-\mu}$ to get

$xu'(x,r) + (1-\mu)u(x,r) = \mu f(x,r) + xf'(x,r)$ (24)

Hence, we differentiate equation (24) four times with respect to x and replace x with the relation to yield:

$u_0 = \frac{\mu}{\mu-1} f(a,a)$

$u'_0 = \frac{\mu+1}{\mu} f'(a,a)$

$u''_0 = \frac{\mu+2}{\mu+1} f''(a,a)$ (25)

$u_0''' = \frac{\mu+3}{\mu+2} f'''(a,a)$

$u_0^4 = \frac{\mu+4}{\mu+3} f^4(a,a)$

To approximate the solution of linear FVIEWSK in (25) using linear NPS function, we present a method of solution following the algorithm.

Algorithm of (LNPS)

Step1:- input $h = \frac{b-a}{n}$, $r_i = r_0 + ih$, $i = 0,1,\dots,n$ and $u_0 = \frac{\mu}{\mu-1} f(a,a)$

Step2:- Compute a_0, b_0, c_0 and d_0 by substituting the equations (11 – 14) into equations (7 – 10)

Step3:- Evaluate $S_0(r)$ using step2 and equation (6) for i = 0

Step4:- Approximate $u_1 = u(r_1) \approx S_0(r_1)$

Step5:- Do the following steps for $i = 1$ to $n - 1$

Step6:- Compute a_i, b_i, c_i, and d_i by using equations (7 – 10) and replacing $u_0(r_i),\ u'_0(r_i), u_0''(r_i)\ and\ u_0'''(r_i)\ in\ S(r_i), S'(r_i), S''(r_i) and\ S'''(r_i)$

Step7:- calculate $S_i(r)$ using step 6 and equations (6)

Step8:- Approximate $u_{i+1} = S_i(r_{i+1})$

Remark: To approximate solution of QNPS by the above algorithm and replace step 6 by (Compute a_i, b_i, c_i, d_i and e_i using equations (16 – 20) and replacing $Q_i(r_i)$, $Q'_i(r_i), Q''_i(r_i)\ , Q_0'''(r_i)\ and\ Q_i^{(\mathbf{4})}(r_i)\ \ in\ \ S(r_i), S'^{(r_i)}, S''(r_i), S'''(r_i) and\ S_i^{(\mathbf{4})}(r_i)$ and replace step7 by (calculate $S_i(r)$ using step 6 and equation (15)).

5. NPS of FVIE with Abel's type kernel:

Singular FVIE with Abel's type kernel [3] can be written in a general form as

$$\tilde{u}(x,r) = \tilde{f}(x,r) + \int_0^x \frac{\tilde{u}(t,r)}{\sqrt{x-t}} dt. \qquad (26)$$

Where $r, x \in [0,1]$

To solve equation (26), applying Laplace transformation to both sides, we have

$$\mathcal{L}\tilde{u}(x,r) = \mathcal{L}\tilde{f}(x,r) + \mathcal{L}\left[\int_0^x \frac{\tilde{u}(t,r)}{\sqrt{x-t}} dt\right] \qquad (27)$$

By convolution theorem, equation (27) yields

$$\mathcal{L}\tilde{u}(x,r) = \mathcal{L}\,\tilde{f}(x,r) + \mathcal{L}\left[x^{-1/2}\right] \mathcal{L}\,[\tilde{u}(x,r) \qquad (28)$$

Then, we have

$$\mathcal{L}\tilde{u}(x,r) = \mathcal{L}\,\tilde{f}(x,r) + \sqrt{\frac{\pi}{s}}\,\mathcal{L}\,[\tilde{u}(x,r) \qquad (29)$$

Upon using the inverse of the Laplace transformation to both sides of equation (29), we obtain

$$\tilde{u}(x,r) = \tilde{f}(x,r) + \mathcal{L}^{-1}\left[\sqrt{\frac{\pi}{s}}\,\mathcal{L}\,[\tilde{u}(x,r)]\right] \qquad (30)$$

Let the solution of equation (30) be in the form of the spline

where

$\tilde{u}(x,r) = [\underline{u}(x,r), \overline{u}(x,r)] = a\cos(x) + b\sin(x) + c(x) + d$

$\tilde{f}(x,r) = [\underline{f}(x,r), \overline{f}(x,r)]$

by substituting the above two equations into equation (30) to get:

$$a\cos(x) + b\sin(x) + c(x) + d = \tilde{f}(x,r) + \mathcal{L}^{-1}\left[\sqrt{\frac{\pi}{s}}\,\mathcal{L}[a\cos(x) + b\sin(x) + c(x) + d]\right] \quad (31)$$

Now, by simplifying equation (31) to obtain:

$$\left[a\left(\cos(x) - \sqrt{\pi}\cos(x)\left(\sqrt{2}-1\right) - \sqrt{\pi}\sin(x)\left(2-\sqrt{2}\right) + b\left(\sin(x) - \sqrt{\pi}\cos(x)\left(-1 + \frac{1}{\sqrt{2}}\right) - \sqrt{\pi}\sin(x)\left(2 - \frac{1}{\sqrt{2}}\right) + c\left(x - \frac{4x^{3/2}}{3}\right) + d\left(1-2\sqrt{x}\right)\right]\right. = \tilde{f}(x,r) \quad (32)$$

Let Δ be a partition for the x, s.t $\Delta: 0 = x_0 < x_1 < x_2 < x_3 = 1$.
Where $h = 1/3$ then $x_0 = 0, x_1 = 1/3, x_2 = 1/3$ and $x_3 = 1$.
The approximate equation $\tilde{s}(x,r) = \tilde{f}(x,r)[a\cos(x) + b\sin(x) + c(x) + d]$ (33)
Now, we need to solve equation (33) to find the constants (a, b, c and d) using the system: $MC = F$ to find C calculate $C = M^{-1}F$ and substitute this solution in equation (33) to find approximation solution.

6. Illustrative Examples

In this section, two test examples are illustrated below to solve (FVIEWSK) and FVIE with Abel's type kernel in upper and lower solution, $s(x,r)$ the approximate solution by the proposed method and error $= |u(x,r) - s(x,r)|$ where $u(x,r)$ the exact solution.

Example (1):

Consider LFVIEs with weakly singular

$u(x,r) - \int_0^x \frac{t^{\beta-1}}{x^\beta} u(x,t)dt = f(x,r)$, where
$f(x,r) = [(0.71428571x^3 - 0.6x^2)(r-1)\,;\,(0.71428571x^3 - 0.6x^2)(1-r)]$
The exact solution is $u(x,r) = [(x^3 - x^2)(r-1)\,;(x^3 - x^2)(1-r)]$ [11]

Table 1. shows the results of LFVIEs with weakly singular in lower solution

x	$\underline{u}(x,r)$	$\underline{s}(x,r)$ in		Error in	
		Linear	Quadratic	Linear	Quadratic
0	0.000000000	0.000000000	-0.000000000	0.000000000	0.000000000
0.1	-8.1×10^{-3}	-0.008092924	-8.1×10^{-3}	7.004×10^{-6}	4.499×10^{-7}
0.2	-0.02900000	-0.028694546	-0.028814338	1.055×10^{-4}	1.439×10^{-5}
0.3	-0.05700000	-0.056203437	-0.056809116	4.966×10^{-4}	1.091×10^{-4}
0.4	-0.08600000	-0.849492630	-0.086859048	1.451×10^{-3}	4.590×10^{-4}
0.5	-0.11300000	-0.109249304	-0.113897909	3.251×10^{-3}	1.398×10^{-3}
0.6	-0.13200000	-0.123465262	-0.133069357	6.135×10^{-3}	3.469×10^{-3}
0.7	-0.13200000	-0.122059594	-0.139775512	0.010000000	7.476×10^{-3}
0.8	-0.11500000	-0.099650843	-0.129722893	0.016000000	0.015000000
0.9	-0.07300000	-0.051067411	-0.098965315	0.022000000	0.026000000
1	0.000000000	0.02860077	-0.043943323	0.029000000	0.044000000

Table 2. Shows the results of LFVIEs with weakly singular in upper solution

x	$\overline{u}(x,r)$	$\overline{s}(x,r)$ in		Error in	
		Linear	Quadratic	Linear	Quadratic
0	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000
0.1	8.1×10^{-3}	0.008092924	8.10×10^{-3}	7.004×10^{-6}	4.499×10^{-7}
0.2	0.029000000	0.028694546	0.028814338	1.055×10^{-4}	1.439×10^{-5}

0.3	0.057000000	0.056203437	0.056809116	4.966×10^{-4}	1.091×10^{-4}
0.4	0.086000000	0.084949263	0.086859048	1.451×10^{-3}	4.590×10^{-4}
0.5	0.113000000	0.109249304	0.113897909	3.251×10^{-3}	1.398×10^{-3}
0.6	0.132000000	0.123465262	0.133069357	6.135×10^{-3}	3.469×10^{-3}
0.7	0.132000000	0.122059594	0.139775512	0.010000000	7.476×10^{-3}
0.8	0.115000000	0.099650843	0.129722893	0.016000000	0.015000000
0.9	0.073000000	0.051067411	0.098965315	0.022000000	0.026000000
1	0.000000000	0.02860077	0.043943323	0.029000000	0.044000000

Example (2):

Consider the FVIE with Abel's type kernel

$$\begin{cases} \underline{u}(x,r) = \left(x + \frac{4}{3}x^{3/2}\right)(4+r) - \int_0^x \frac{\underline{u}(t,r)}{\sqrt{x-t}}\,dt \\ \overline{u}(x,r) = \left(x + \frac{4}{3}x^{3/2}\right)(r-6) - \int_0^x \frac{\overline{u}(t,r)}{\sqrt{x-t}}\,dt \end{cases}$$

where the exact solution is $u(x,r) = [(4+r)x\,,(6-r)x][3]$

Table 3. FVIE with Abel's type kernel in the parametric form in upper and lower solution.

x	$\underline{u}(x,r)$	$\underline{s}(x,r)$	Error	$\overline{u}(x,r)$	$\underline{s}(x,r)$	Error
0	0.00000000	0.000000000	0.000000000	0.0000000000	0.0000000000	0.000000000
0.1	-0.59000000	-0.590000000	0.000000000	0.4100000000	0.4100000000	0.000000000
0.2	-1.18000000	-1.180000000	0.000000000	0.8200000000	0.8200000000	0.000000000
0.3	-1.77000000	-1.770000000	0.000000000	1.2300000000	1.2300000000	0.000000000
0.4	-2.36000000	-2.360000000	0.000000000	1.6400000000	1.6400000000	0.000000000
0.5	-2.95000000	-2.950000000	0.000000000	2.0500000000	2.0500000000	0.000000000
0.6	-3.54000000	-3.540000000	0.000000000	2.4600000000	2.4600000000	0.000000000
0.7	-4.13000000	-4.130000000	0.000000000	2.8700000000	2.8700000000	0.000000000
0.8	-4.72000000	-4.720000000	0.000000000	3.2800000000	3.2800000000	0.000000000
0.9	-5.31000000	-5.310000000	0.000000000	3.6900000000	3.6900000000	0.000000000
1	-5.90000000	-5.900000000	0.000000000	4.1000000000	4.1000000000	0.000000000

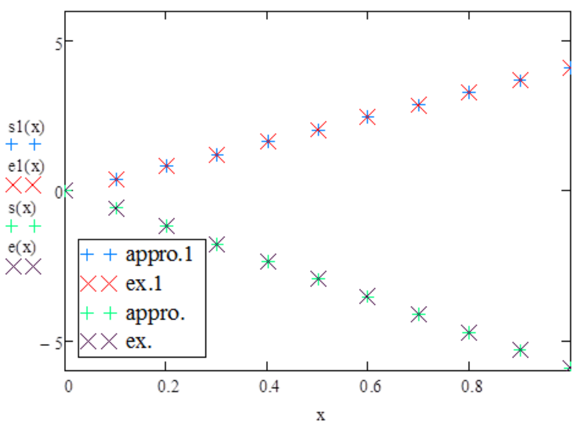

Figure 1. Results of example (1)

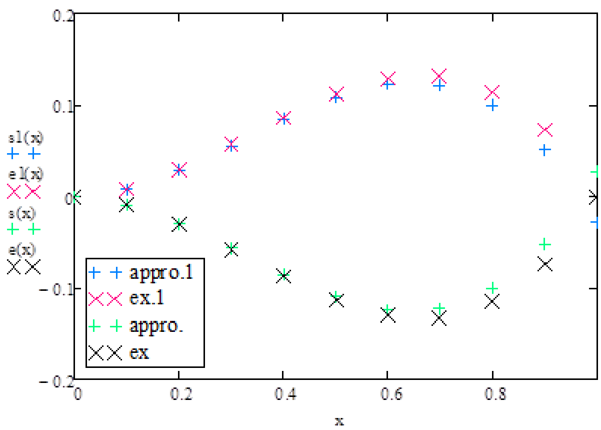

Figure 2. Results of example (2)

In **Figures 1, 2**, we plot the graphs of the approximate (**(+)** for upper and **(+)** for lower) solution and exact (**(×)** for upper and **(×)** for lower) solution among different values of x with fuzzy parameter $r = 0.1$, $0 \leq x \leq 1$, where s(x) is approximate, and e(x) is exact for the upper parametric form and s1(x) is approximate, and e1(x) is exact for the lower parametric form.

7. Conclusions

The examples show that the results of the method are convergent to the exact solution. The non-polynomial spline has been successfully used to obtain the .approximate solutions; the trigonometric term of this spline has infinite derivatives, which agree with the exact solution. Given the results, tables and figures show that the proposed technique is a powerful mathematical tool for solving FVIEWSK with MathCad programming implementation. We will consider FVIEWSK, algorithm, and applied fractional order problems for future work.

References

1. Zadeh, L. Fuzzy Sets Information and Control, **1965**, 8, 338-353.
2. Mahmoud P.; Farshid M.and Mohammad K. Solving linear and Nonlinear Abel Fuzzy Integral Equation by fuzzy Laplace Transforms. Mathematical Inverse Problems, **2014**, *1(2).*
3. Bushnaq, S.; Ullah, Z.; Ullah, A. and Shah, K. Solution of Integral Equation with Abel's Type Kernel using a Novel Hybrid Method. Advances in Difference Equation, *Springer Open Journal*, **2020,** *165*.
4. Zahra, A. Collocation Method for Fuzzy Volterra Integral Equation of the Second Kind with Weakly Singular Kernels, Research Square, **2021**.
5. Elgaili, A.I.; Abdel radi, A.A.; Zakieldeen A.M. ; Reem, H.I. Analytical and Numerical Solution of Linear Volterra Integral Equations of the Second Kind with Weakly Singular Kernel by using the Sixth Order of Non-Polynomial Spline Functions by matlab.*Turkish Journal of Computer and Mathematics Education*, **2021**, *12(14).*
6. Biswas, S.; Roy, T.K. Fuzzy Linear Integral Equation and its Application Inbiomathematical Model. Advances in Fuzzy Mathematics, **2017,** *12(5)*, 1137–1157.
7. Waffa, A.I. Numerical Method for Solving Integral Equations, **2016**, M.Sc. Thesis Mustansiriayah University.
8. Ali, F.J; Anakira, R. N.; Alomari, A.K. ; Noraziah, M. Solution and Analysis of the Fuzzy Volterra Integral Equation via Homotopy Analysis Method. Computer Modeling in Engineering and Sciences, **2021**,*127(3).*
9. Muna, M. ; Sarah, H. Solution of Second Kind Volterra Integrals Equations using Non-Polynomial Spline Functions. *Baghdad Science Journal*, **2014**, *11(2).*
10. Rawaa, I.E.; Atefa, J.S. Numerical Treatment of First Order Volterra Integro-Differential Equation using Non-Polynomial spline function. *Iraqi Journal of Science* ,**2020**, 114-121.
11. Sara, H.H. Volterra Integral Equation using Non-Polynomial Spline Functions, **2013**, M.Sc. Thesis College of Science Baghdad University.