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Exponentially Fitted Diagonally Implicit EDITRK Method for Solving ODEs

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Abstract

This paper derives the EDITRK4 technique, which is an exponentially fitted diagonally implicit RK method for solving ODEs $y'''(x) = f(x, y)$. This approach is intended to integrate exactly initial value problems (IVPs), their solutions consist of linear combinations of the group functions e^{wx} and e^{-wx} for exponentially fitting problems, with $w \in R$ being the problem's major frequency utilized to improve the precision of the method. The modified method EDITRK4 is a new three-stage fourth-order exponentially-fitted diagonally implicit approach for solving IVPs with functions that are exponential as solutions. Different forms of 3^{rd}-order ODEs must be derived using the modified system, and when the same issue is reduced to a 1^{st} framework of equations that can be solved using conventional RK approaches, numerical comparisons must be done. The findings show that the novel approach is more efficacious than previously published methods.

Keyword: Numerical Methods, Exponentially Fitted, Ordinary Differential Equations, Diagonal Implicit Type Runge Kutta Methods**,** Initial Value Problems.

1.Introduction

This paper is about RK type methods directly for solving 3^{rd}-order ODEs that are exponentially-fitted diagonally implicit in the form:

$$v'''(t) = f\big(x, v(t)\big), \qquad v(t_0) = v_0, \;\; v'(t_0) = v_0', \;\; v''(t_0) = v_0'', \qquad t \geq t_0 \tag{1}$$

Where $v \in R^d, f : R \times R^d \rightarrow R^d$ is a continuous vector-valued operation that does not rely on the second derivatives directly. Numerous physical issues, such as thin-film flow and gravity-driven fluxes, need to be addressed, and so on, have this type of problem [1-3]. Then, some researchers have constructed explicit RK methods for solving 1^{st}-order and 2^{nd}-order ODEs that are exponentially and trigonometrically fitted in recent years. Because implicit techniques can attain higher levels of accuracy for the same stage number as explicit methods, they are essential. It becomes easier to resolve challenging problems as a result of this. Implicit (RK) approaches, on the other hand, play a crucial role in other types of problems, such as differential-algebraic neutralization. Furthermore, because they have a lower triangular A-matrix with at least one non-zero diagonal element, diagonally implicit RK methods are sometimes known as semi-(implicit or explicit) RK approaches. For trigonometric polynomial periodic solutions to ODEs, [4] developed Runge-Kutta-Nyström techniques. Exponentially adapted RK algorithms were developed by [5]. While [6] presents an extended version of the RK method that tackles the problems of the Schrödinger equation. The majority of scientists, engineers, and researchers utilized to resolve (1) by reducing third-order differential equations to a three-dimensional system of 1^{st}-order equations. However, if the problem can be solved directly using numerical methods, it is more efficient. In [7-9] are examples of this type of work. Two explicit two-derivative RKN techniques, one exponentially fitted and the other trigonometrically fitted, are constructed in [10]. Then, using the Simos technique, Demba et al. [11] implemented an explicit trigonometrically fitted RKN method. Furthermore, [12] and [13] are to demonstrate how the direct method is used to solve specific third and fourth-order ODEs. The major purpose of this research is to show how to solve special 3^{rd}-order ODEs using an exponentially-fitted diagonally implicit RK method. In addition, (1) is solved numerically using the approach's algebraic order must be considered, as it is the most essential aspect in achieving high accuracy.

The required criteria and derivation for exponentially fitted 3^{rd}-order ODEs are solved using RK techniques and presented in part 3. Section 4 compares the effectiveness of the new method to that of previous methods. For addressing the IVPs problem (1), the general structure of the EDITRK4 approach with an m-stage:

$$v_{n+1} = v_n + hv'_n + \frac{h^2}{2} v''_n + h^3 \sum_{i=1}^{m} b_i\, k_i \,, \tag{2}$$

$$v'_{n+1} = v'_n + hv''_n + h^2 \sum_{i=1}^{m} b'_i\, k_i \,, \tag{3}$$

$$v''_{n+1} = v''_n + h \sum_{i=1}^{m} b''_i\, k_i \,. \tag{4}$$

Where

$$k_i = f\left(x_n + c_i h, v_n + c_i h v'_n + \frac{h^2}{2}\, c_i^2\, v''_n + h^3 \sum_{j=1}^{i-1} a_{ij}\, k_j\right), \tag{5}$$

for $i = 2,3, \dots, m$.

The diagonal implicit RK type parameters (EDITRK4) techniques are $b_i, b'_i, b''_i, a_{i,j}$ and c_i of where $i = 2, 3, \dots, m$, are genuine integers as well as m is the approach's digit of the stage. When $a_{i,j} \neq 0$ for $i \leq j$, this method is known as diagonal implicit. The final designation comprises the single EDITRK4 techniques, It means that the values of A in lower triangle diagonal matrices are the same as $a_{ij} \neq 0$ where $i = j$ in the diagonal.

Table 1. Butcher form EDITRK4 method.

$$
\begin{array}{c|ccc}
c_1 & a_{11} & & \\
c_2 & a_{21} & a_{22} & \\
c_3 & a_{31} & a_{32} & a_{33} \\
\hline
 & b_1 & b_2 & b_3 \\
 & b'_1 & b'_2 & b'_3 \\
 & b''_1 & b''_2 & b''_3
\end{array}
$$

The parameters of the new method supplied by (2)-(5) are obtained by expanding the EDITRK4 method statement using Taylor's series enlargement. In accordance with a few algebraic adjustments, this enlargement equals the correct solution found through Taylor's series expansion. The general order criteria for the new technique were derived using the direct truncation error on the local level. This idea is founded on the derivation of order criteria to the RK technique see [14] and [15].

The new EDITRK4 technique is formed as follows:

$$
\begin{aligned}
y_{n+1} &= y_n + h\Phi(x_n\,, y_n),\\
y'_{n+1} &= y'_n + h\Phi'(x_n\,, y_n),\\
y''_{n+1} &= y''_n + h\Phi''(x_n\,, y_n),
\end{aligned} \tag{6}
$$

where the increment functions are:

$$
\begin{aligned}
\Phi(x_n\,, y_n) &= y'_n + \frac{h}{2}\, y''_n + h^2 \sum_{i=1}^{m} b_i\, k_i,\\
\Phi'(x_n\,, y_n) &= y''_n + h \sum_{i=1}^{m} b'_i\, k_i,\\
\Phi''(x_n\,, y_n) &= \sum_{i=1}^{m} b''_i\, k_i.
\end{aligned} \tag{7}
$$

In which k_i is shown in (5). If we suppose the Taylor increment function is Δ, Δ' and Δ'' · Thus, by inserting the exact solution of (1) into (7), the local truncation error *(LTE)* of $y(x)$, $y'(x)$ and $y''(x)$ can be obtained:

$$
\begin{aligned}
\tau_{n+1} &= h[\Phi - \Delta],\\
\tau'_{n+1} &= h[\Phi' - \Delta'],\\
\tau''_{n+1} &= h[\Phi'' - \Delta''].
\end{aligned} \tag{8}
$$

In the terms of elementary differentials, these expressions are best given and the Taylor series can be expressed as follows:

$$\Delta = y' + \frac{1}{2}hy'' + \frac{1}{6}\, h^2 F_1^{(3)} + \frac{1}{24}\, h^3 F_1^{(4)} + O(h^4),$$

$$\Delta' = y'' + \frac{1}{2}\, hF_1^{(3)} + \frac{1}{6}\, h^2 F_1^{(4)} + \frac{1}{24}\, h^3 F_1^{(5)} + O(h^4),$$

$$\Delta'' = F_1^{(3)} + \frac{1}{2}\, hF_1^{(4)} + \frac{1}{6}\, h^2 F_1^{(5)} + O(h^3). \qquad (9)$$

The first few basic differentials in the scalar case are as follows:

$$F_1^{(3)} = f,$$

$$F_1^{(4)} = f_x + f_y y' \ ,$$

$$F_1^{(5)} = f_{xx} + 2f_{xy}y' + f_{xy'}y_{xx} + f_y y'' + f_{yy}(y')^2. \qquad (10)$$

Substituting (10) into (7), for new method, the increment functions Φ, Φ' and Φ'' will become

$$\sum_{i=1}^{m} b_i k_i = \sum_{i=1}^{m} b_i\, f + \sum_{i=1}^{m} b_i c_i \left(f_x + f_y y'\right)h + \frac{1}{2}\sum_{i=1}^{m} b_i\, c_i^2 \left(f_{xx} + 2f_{xy}y' + f_{xy'}y_{xx} + f_y y'' + f_{yy}(y')^2\right)h^2 + O(h^3),$$

$$\sum_{i=1}^{m} b_i' k_i = \sum_{i=1}^{m} b_i'\, f + \sum_{i=1}^{m} b_i' c_i \left(f_x + f_y y'\right)h + \frac{1}{2}\sum_{i=1}^{m} b_i'\, c_i^2 \left(f_{xx} + 2f_{xy}y' + f_{xy'}y_{xx} + f_y y'' + f_{yy}(y')^2\right)h^2 + O(h^3),$$

$$\sum_{i=1}^{m} b_i'' k_i = \sum_{i=1}^{m} b_i''\, f + \sum_{i=1}^{m} b_i'' c_i \left(f_x + f_y y'\right)h + \frac{1}{2}\sum_{i=1}^{m} b_i''\, c_i^2 \left(f_{xx} + 2f_{xy}y' + f_{xy'}y_{xx} + f_y y'' + f_{yy}(y')^2\right)h^2 + O(h^3)\ ,$$

From (9) and (11), The following is how LTE in (8) is expressed:

$$\tau_{n+1} = h^3 \left[\sum_{i=1}^{m} b_i k_i - \left(\frac{1}{6}F_1^{(3)} + \frac{1}{24}hF_1^{(4)} + \cdots\right)\right],$$

$$\tau_{n+1}' = h^2 \left[\sum_{i=1}^{m} b_i' k_i - \left(\frac{1}{2}F_1^{(3)} + \frac{1}{6}hF_1^{(4)} + \cdots\right)\right],$$

$$\tau_{n+1}'' = h \left[\sum_{i=1}^{m} b_i'' k_i - \left(F_1^{(3)} + \frac{1}{2}hF_1^{(4)} + \frac{1}{6}h^2 F_1^{(5)} \ldots\right)\right]. \qquad (12)$$

The *LTE* or the order conditions up to order six for m-stage for the new technique can be solved by substituting (11) into (12) and expanding as a Taylor expansion using the Maple package see [14] and [12].

2.Exponentially Fitted EDITRK4 Method

Definition 1: *To create the exponentially fitted RK kind 3-stage 4^{th}-order technique, the functions e^{w} and e^{-w} must integrate perfectly at each stage; consequently, the following equations are derived for y, y' and y''*

$$e^{\pm v} = 1 \pm v + \frac{1}{2} v^2 \pm v^3 \sum_{i=1}^{m} b_i \, e^{\pm c_i v}, \quad (13)$$

$$e^{\pm v} = 1 \pm v + v^2 \sum_{i=1}^{m} b_i' \, e^{\pm c_i v}, \quad (14)$$

$$e^{\pm v} = 1 \pm v \sum_{i=1}^{m} b_i'' \, e^{\pm c_i v}. \quad (15)$$

Where $v = wh$, $w \in R$. The relations $\cosh(v) = \frac{e^{v}+e^{-v}}{2}$ and $\sinh(v) = \frac{e^{v}-e^{-v}}{2}$ will be used in the derivation process. The following equations corresponding y, y' and y'' are:

$$\cos(v) = 1 + \frac{1}{2} v^2 + v^3 \sum_{i=1}^{m} b_i \sin h(vc_i), \quad (16)$$

$$\sin h(v) = v + v^3 \sum_{i=1}^{m} b_i \cosh(vc_i), \quad (17)$$

$$\cosh(v) = 1 + v^2 \sum_{i=1}^{m} b_i' \cosh(vc_i), \quad (18)$$

$$\sinh(v) = v + v^2 \sum_{i=1}^{m} b_i' \sinh(vc_i), \quad (19)$$

$$\cosh(v) = 1 + v \sum_{i=1}^{m} b_i'' \sinh(vc_i), \quad (20)$$

$$\sinh(v) = v \sum_{i=1}^{m} b_i'' \cosh(vc_i). \quad (21)$$

In [12], devised the following *3-stage* 4^{th}*-order* diagonally implicit approach.

$$c_1 = \frac{4}{5}, c_2 = \frac{2}{5}, c_3 = \frac{7}{10}, a_{11} = \frac{1}{600}, a_{21} = 0, a_{22} = \frac{1}{600}, a_{31} = 0, a_{32} = \frac{2}{25}, a_{33} = \frac{1}{600}, b_1 = 0, b_2 = \frac{1}{10}, b_3 = \frac{1}{100}, b_1' = 0, b_2' = \frac{3}{10}, b_3' = \frac{1}{10}, b_1'' = 0, b_2'' = \frac{1}{2}, b_3'' = \frac{1}{2}.$$

Next, we solve (16) – (21) and use of the coefficients listed above to find $b_1, b_2, b_1', b_2', b_1''$ and b_2''.

$$b_1 = \frac{(\cosh(\frac{v}{3})\sinh\left(\frac{7v}{10}\right) - \cosh(\frac{7v}{10})\sinh(\frac{v}{3})}{100\cosh(\frac{4}{5}v)\sinh\left(\frac{v}{5}\right) - 100\cosh(\frac{v}{5})\sinh(\frac{4}{5}v)} + {}^{1}/_{2}\ \frac{\cosh\left(\frac{v}{5}\right)v^2 + 2\sinh(v)\sinh\left(\frac{v}{5}\right) - 2\cosh(\frac{v}{5})\cosh(v) - 2\sinh\left(\frac{v}{5}\right)v + 2\cosh(\frac{v}{5})}{v^3(\cosh(\frac{4}{5}v)\sinh\left(\frac{v}{5}\right) - \cosh(\frac{v}{5})\sinh(\frac{4}{5}v))}$$

$$b_2 = -\frac{(\cosh\left(\frac{4}{5}v\right)\sinh\left(\frac{7v}{10}\right) - \cosh\left(\frac{7v}{10}\right)\sinh\left(\frac{4}{5}v\right)}{100\cosh\left(\frac{4}{5}v\right)\sinh\left(\frac{v}{5}\right) - 100\cosh\left(\frac{v}{5}\right)\sinh\left(\frac{4}{5}v\right)} - {}^{1}/_{2}\ \frac{\cosh\left(\frac{4}{5}v\right)v^2 + 2\sinh(\frac{4}{5}v)\sinh(v) - 2\cosh(v)\cosh\left(\frac{4}{5}v\right) - 2\sinh\left(\frac{4}{5}v\right)v + 2\cosh(\frac{4}{5}v)}{v^3(\cosh(\frac{4}{5}v)\sinh\left(\frac{v}{5}\right) - \cosh(\frac{v}{5})\sinh(\frac{4}{5}v))}$$

$$b_1' = \frac{1}{10}\frac{\left(cosh\left(\frac{v}{5}\right)sinh\left(\frac{7v}{10}\right) - cosh\left(\frac{7v}{10}\right)sinh\left(\frac{v}{5}\right)\right)}{\cosh\left(\frac{4}{5}v\right)\sinh\left(\frac{v}{5}\right) - \cosh\left(\frac{v}{5}\right)\sinh\left(\frac{4}{5}v\right)} - \frac{\cosh\left(\frac{v}{5}\right)\sinh(v) - \cosh(\frac{v}{5})v - \sinh\left(\frac{v}{5}\right) + \sinh(\frac{v}{5})}{v^2(\cosh(\frac{4}{5}v)\sinh\left(\frac{v}{5}\right) - \cosh(\frac{v}{5})\sinh(\frac{4}{5}v))}$$

$$b_1'' = \frac{\left(\frac{1}{2}cosh\left(\frac{v}{5}\right)sinh\left(\frac{7v}{10}\right)v - \frac{1}{2}cosh\left(\frac{7v}{10}\right)sinh\left(\frac{v}{5}\right)v - \cosh\left(\frac{v}{5}\right)\cosh(v) + \sinh(v)\sinh\left(\frac{v}{5}\right) + \cosh(\frac{v}{5})\right)}{v\left(cosh\left(\frac{4}{5}v\right)sinh\left(\frac{v}{5}\right) - cosh\left(\frac{v}{5}\right)sinh\left(\frac{4}{5}v\right)\right)}$$

$$b_2'' = \frac{\left(\frac{1}{2}cosh\left(\frac{4}{5}v\right)sinh\left(\frac{7v}{10}\right)v - \frac{1}{2}cosh\left(\frac{7v}{10}\right)sinh\left(\frac{4}{5}v\right)v - \cosh(v)\cosh\left(\frac{4}{5}v\right) + \sinh\left(\frac{4}{5}v\right)\sinh(v) + \cosh(\frac{4}{5}v)\right)}{v\left(cosh\left(\frac{4}{5}v\right)sinh\left(\frac{v}{5}\right) - cosh\left(\frac{v}{5}\right)sinh\left(\frac{4}{5}v\right)\right)}$$

As a result, we developed EDITRK4, a 3-stage 4^{th}-order diagonally implicit exponentially fitted RK type approach. The solution's matching Taylor series expansion is given by:

$$b_1 = \frac{1}{180} - \frac{v^2}{8000} + \frac{4057v^4}{1008000000} - \frac{732037v^6}{5443200000000} + \frac{380948723v^8}{79833600000000000} - \frac{150590645851v^{10}}{871782912000000000000}$$

$$b_2 = \frac{34}{225} + \frac{29v^2}{24000} + \frac{5333v^4}{1008000000} - \frac{658169v^6}{5443200000000} + \frac{41437183v^8}{8870400000000000} - \frac{149781459959v^{10}}{871782912000000000000}$$

$$b_1' = \frac{1}{36} - \frac{v^2}{1440} + \frac{1259v^4}{33600000} - \frac{108659v^6}{77760000000} + \frac{1226395609v^8}{23950080000000000} - \frac{90543546703v^{10}}{48432384000000000000}$$

$$b_2' = \frac{67}{180} + \frac{11v^2}{7200} + \frac{1471v^4}{33600000} - \frac{152837v^6}{108864000000} + \frac{1228905701v^8}{23950080000000000} - \frac{10065377507v^{10}}{5381376000000000000}$$

$$b_1'' = \frac{1}{12} + \frac{43v^2}{7200} - \frac{169v^4}{2880000} + \frac{117487v^6}{60480000000} - \frac{29389781v^8}{435456000000000} + \frac{1665975073v^{10}}{684288000000000000} - \frac{6617676448637v^{12}}{74724249600000000000000}$$

$$b_2'' = \frac{5}{12} + \frac{23v^2}{7200} - \frac{101v^4}{2880000} + \frac{104747v^6}{60480000000} - \frac{28558129v^8}{435456000000000} - \frac{11578525811v^{10}}{4790016000000000000} + \frac{6605830989193v^{12}}{74724249600000000000000}$$

3.Numerical Experiments

Problem 1: (Non-homogeneous Linear Problem).

$v'''(t) = v(t) + cos(v), \quad v(0) = 0, \quad v'(0) = 0, \; v''(0) = 1.$

Theoretical solution:

$$v(t) = (e^t - \cos(t) - \sin(t)) \,.$$

Problem 2: (Non-homogeneous Nonlinear Problem).

$v'''(t) = (v(t))2 + cos2(v) - cos(t) - 1,$

$v(0) = 0, v'(0) = 1, v''(0) = 1.$

Theoretical solution:

$$v(t) = sin(t).$$

Problem 3: **(Non-homogeneous Nonlinear Problem)**.

$$v'''(\mathrm{t}) = 8\left(\frac{v^2(t)}{e^{2t}}\right),$$

$$v(0) = 1, \qquad v'(0) = 2, \qquad v''(0) = 4\,.$$

Theoretical solution:
$v(t) = e^{2t}.$

Problem 4: (Non-linear System).

$y_1'''(\mathrm{t}) = y_2(\mathrm{t}), \qquad y_1(0) = 1, y_1'(0) = 0, y_1''(0) = 1,$

$y_2'''(\mathrm{t}) = -y_1(t) - 2y_2(t) + 2y_3(t), \qquad y_2(0) = 0, y_2'(0) = 1, y_2''(0) = 0,$

$y_3'''(\mathrm{t}) = y_1(t) + y_2(t) \qquad y_3(0) = 1, y_3'(0) = 1, y_3''(0) = 1.$

The precise solution is provided by

$$y_1(t) = cosh(t),$$
$$y_2(t) = sinh(t),$$

$$y_3(t) \; = e^t.$$

Problem 5: (Non-linear System).

$$y_1'''(\mathrm{t}) = y_2(\mathrm{t}), \qquad y_1(0) = 1\,, y_1'(0) = 0\,, y_1''(0) = 1\,,$$

$$y_2'''(\mathrm{t}) = y_1(\mathrm{t}), \qquad y_2(0) = 0\,, y_2'(0) = 1\,, y_2''(0) = 0\,,$$

$$y_3'''(\mathrm{t}) = y_1(t) + y_2(t) - \sinh(t) \qquad y_3(0) = 1\,, y_3'(0) = 0\,, y_3''(0) = 1.$$

The precise solution is provided by

$$y_1(t) = \cosh(t),$$

$$y_2(t) = \sinh(t),$$

$$y_3(t) = e^t + 1 - \cosh(t) + \frac{t^2}{2} - t.$$

Figures 1–5 show the decimal logarithm of the largest global error and the logarithm function valuations, which illustrate how effective the EDITRK4 techniques. The EDITRK4 approach requires fewer function evaluations than other implicit RK approaches of the same order. This is because the number of equations tripled when the problems were converted into a system of 1^{st}-order ODEs. Furthermore, as shown in **Figures 1–5**, the EDITRK4 approaches have the smallest maximum global error and the fewest number of function evaluations for each step. The EDITRK4 produces more accurate findings than the other results in the literature, as seen in Figures 1-5 (DITRKM4, RKLIIIB4, and DIRKN4). The logarithm of function evaluations with different step sizes $h = 0.1, 0.05, 0.025, 0.00125$, and 0.00625 in this study is a function of the decimal logarithm of the greatest global fault for 5 test problems.

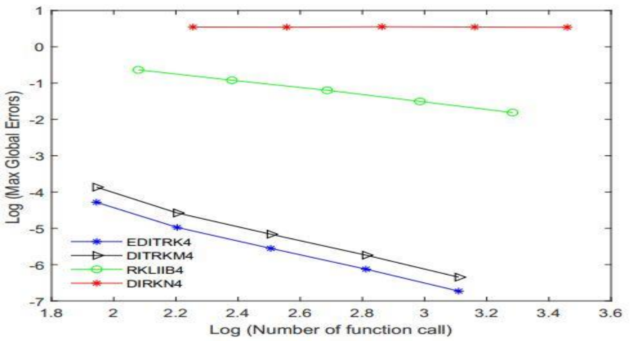

Figure 1. Accuracy curve for EDITRK4, DITRKM4, RKLIIIB4, and DIRKN4 with h = 0.1, 0.05 0.025, 0.00125, and 0.00625 for the problem 1.

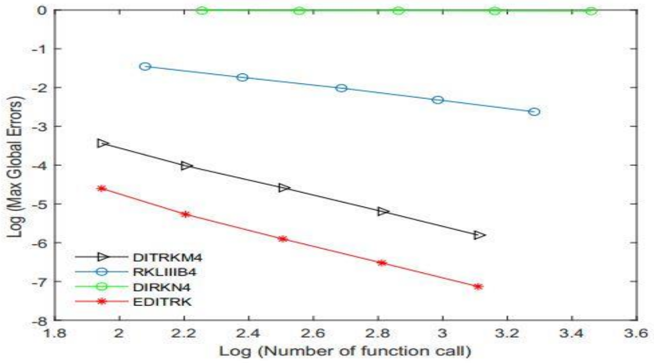

Figure 2. Accuracy curve for EDITRK4, DITRKM4, RKLIIIB4, and DIRKN4 with h = 0.1, 0.05 0.025, 0.00125, and 0.00625 for the problem 2.

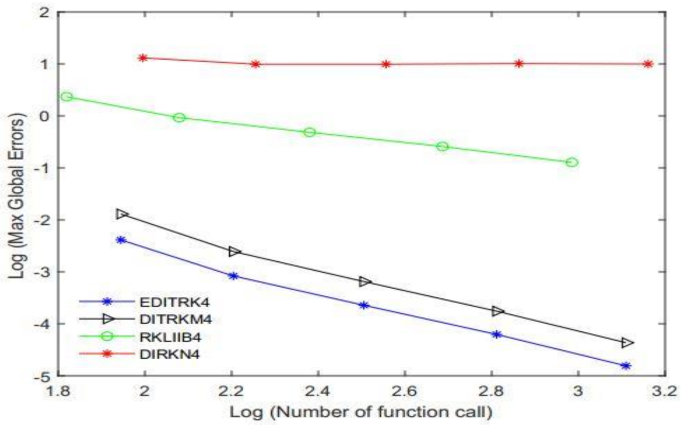

Figure 3. Accuracy curve for EDITRK4, DITRKM4, RKLIIIB4, and DIRKN4 with h = 0.1, 0.05 0.025, 0.00125, and 0.00625 for the problem 3.

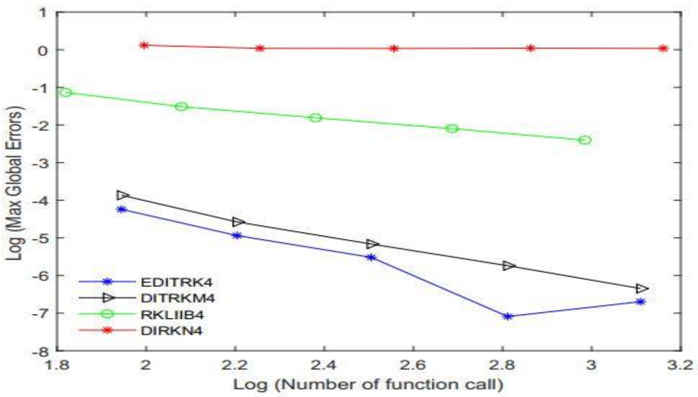

Figure 4. Accuracy curve for EDITRK4, DITRKM4, RKLIIIB4, and DIRKN4 with h = 0.1, 0.05 0.025, 0.00125, and 0.00625 for the problem 4.

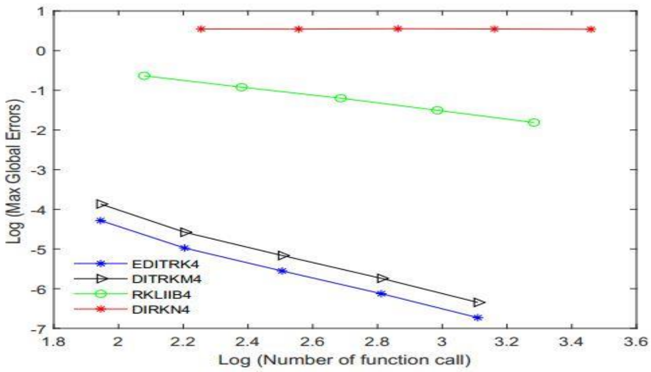

Figure 5. Accuracy curve for EDITRK4, DITRKM4, RKLIIIB4, and DIRKN4 with h = 0.1, 0.05 0.025, 0.00125, and 0.00625 for the problem 5.

4.Conclusion

In this paper, we devise an exponentially fitted diagonally implicit RK type technique to address the $y'''(x) = f(x, y)$ problem. As a result, the EDITRK4 method, a diagonally implicit three-stage fourth-order exponentially-fitted technique based on computing the solution's greatest error $(max(|\, y(t_n) - y_n|\,))$. This is the greatest difference between actual and computed solution absolute errors, was developed and used in the numerical comparison of criteria. The numerical results are shown in Figures 1–5. Then, the EDITRK4 method requires fewer capacity assessments than the DITRK4, RKLIIIB4, and DIRKN4 procedures. The common logarithm of the greatest global error during integration and computing cost was calculated using the number of function evaluations, as indicated in the figures. The numerical results showed that over a brief duration of integration, the unique exponentially fitted methodology RK type approach has a lower global error than the other current approaches. The innovative EDITRK4 methodology is significantly more effective than the competition present approaches when solving third-order ODEs of the kind $y''' = f(x, y)$ directly.

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