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Fibrewise Multi-Perfect Topological Spaces

Majed .H. Jaber *
Department of Mathematics, College of Education for Pure Sciences, Ibn Al-Haitham University of Baghdad, Iraq

Yousif. Y. Yousif
Department of Mathematics, College of Education for Pure Sciences, Ibn Al-Haitham University of Baghdad, Iraq

M. El Sayed
Department of Mathematics, College of Science and Arts, Najran University, Kingdom of Saudi Arabia

***Corresponding Author:** maged.hameed1203a@ihcoedu.uobaghdad.edu.iq

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Abstract

The essential objective of this paper is to introduce new notions of fibrewise topological spaces on D that are named to be upper perfect topological spaces, lower perfect topological spaces, multi-perfect topological spaces, fibrewise upper perfect topological spaces, and fibrewise lower perfect topological spaces. fibrewise multi-perfect topological spaces, filter base, contact point, rigid, multi-rigid, multi-rigid, fibrewise upper weakly closed, fibrewise lower weakly closed, fibrewise multi-weakly closed, set, almost upper perfect, almost lower perfect, almost multi-perfect, fibrewise almost upper perfect, fibrewise almost lower perfect, fibrewise almost multi-perfect, upper* continuous fibrewise upper* topological spaces respectively, lower* continuous fibrewise lower* topological spaces respectively, multi*-continuous fibrewise multi*-topological spaces respectively multi-T_e, locally In addition, we find and prove several propositions linked to these notions.

Keywords: Fibrewise topological spaces, filter base, fibrewise upper perfect topological spaces, fibrewise lower perfect topological spaces, and fibrewise multi-perfect topological spaces.

1. Introduction

We begin our work with the concept of category of Fibrewise (briefly, 𝔽.𝕎.) set on a known set, named the base set. If the base set is stated with D, then a F.W. set on D applied to a set E with a function X is X: E → D, named the projection (briefly, project). For every point d of D, the fiber on d is the subset $E_d = X^{-1}(d)$ of E; fibers will be empty, so we do not require X to be a surjection. Also, for every subset D* of D, we regard $E_{D^*} = X^{-1}(D^*)$ as a 𝔽.𝕎. set on D* with the project determined by X. A multi-function [2] Ω of a set E into F is a correspondence such that Ω (e) is a nonempty subset of F for every e ∈ E. We will denote such a multi- function by Ω: E → F . For a multi- function Ω, the upper and lower inverse set of a set K of F, will be denoted by $\Omega^{+}(K)$ and $\Omega^{-}(K)$, respectively, that is $\Omega^{+}(K) = \{e \in E : \Omega(e) \subseteq K\}$ and $\Omega^{-}(K) = \{e \in E: \Omega(e) \cap K \neq \emptyset\}$.

Definition 1.1. [7] Suppose that *E and F are* 𝔽.𝕎. sets on *D*, with project. $X_E: E \to D$ and $X_F: F \to D$, respectively, a function Ω*: E → F* is named to be 𝔽.𝕎. *if* $X_F O \Omega = X_E$, that is to say if $\Omega(X_d) \subset F_d$ for every point *d of D.*

For other concepts or information that are undefined here, we follow nearly [3] and[4]

Recall that [7] Let D be a topological space, the 𝔽.𝕎. Topology space (briefly, 𝔽.𝕎.T.S.) on a 𝔽.𝕎. set E on D, which means any topology on E for that the project X is continuous.

Remark 1.1. [7]

i. The smaller topology is the topology trace with X, where in the open sets of E are the pre image of the open sets of *D*, this is named the 𝔽.𝕎. indiscrete topology.
ii. The 𝔽.𝕎.T.S. on *D* is stated to be a 𝔽.𝕎. set on *D* with a 𝔽.𝕎.T.S.

We regard the topology product *D × T, for any topological space T, as a* 𝔽.𝕎.*T.S. on D using the category of the first projection. The equivalences in the category of* 𝔽.𝕎.*T.S. are named* 𝔽.𝕎.*T. equivalences. If E is* 𝔽.𝕎.*T. equivalent to D × T, for some topological space T, we say that E is trivial, as a* 𝔽.𝕎.*T.S. on D. In* 𝔽.𝕎.*T. the form neighbourhood (briefly, η*ℙ𝕕*) is used in the same sense as it is in normally topology, but the forms* 𝔽.𝕎. *basic may need some illustration, so let E be* 𝔽.𝕎.*T.S. on D, if e is a point of* E_d *where in d ∈ D, appear a family N(e) of η*ℙ𝕕 *of e in E as* 𝔽.𝕎. *basic if as every η*ℙ𝕕 *H of e we have* $E_w \cap K \subset H$*, for some element K of N(e) and η*ℙ𝕕 *W of d in D. As example, in the case of the topological product D × T, where in T is a topological spaces, the family of Cartesian products D × N(t), where in N(t) runs through the η*ℙ𝕕*s of t, is* 𝔽.𝕎. *basic for (d, t).*

Definition 1.2. [7] *The* 𝔽.𝕎. *functions Ω: E→ F; E and F are* 𝔽.𝕎. *spaces on D is named:*

(a) Continuous (briefly, cont.) if every e ∈ E_d*; d ∈ D, the inverse image of every open set of Ω(e) is an open set of e.*

(b) Open if for every e∈E_d, d ∈D, the direct image of every open set of e is an open set of Ω(e).

Definition 1.3. [7] *The F.W.T.S. E on D is named* 𝔽.𝕎. *closed (resp., open) if the project. X is closed (resp., open) functions.*

Definition 1.4. [1] *Let Ω: E → F be a multi-function. Then Ω is upper cont. (briefly, U. cont.) if* $\Omega^{+}(K)$ *open in E for all K open in F. That is,* $\Omega^{+}(K) = \{x \in E: \Omega(x) \subseteq K\}$*. K ⊆ F.*

Definition 1.5. [1] *Let Ω: E → F be a multi-function. Then Ω is lower cont. (briefly, L. cont.) if* $\Omega^{-}(K)$ *open in E for all K open in F. That is,* $\Omega^{-}(K) = \{e \in E: \Omega(e) \cap K \neq \emptyset\}$*. K ⊆ F*

Let Ω: E → F be a multi-function. Then Ω is multi cont. (briefly, M. cont.) if it is U. cont. and L. cont.

Definition 1.6.[5] *Let D be topological space, the* 𝔽.𝕎. *upper topology space (briefly,* 𝔽.𝕎.*U.T.S.) on a* 𝔽.𝕎. *set E on D mean any topology on E for which the project. X is U. cont.*

Definition 1.7.[5] *Let D be topological space the* $\mathbb{F}.\mathbb{W}$. *lower topology space (briefly,* $\mathbb{F}.\mathbb{W}$.*L.T.S.) on a* $\mathbb{F}.\mathbb{W}$. *set E on D mean any topology on E for which the project. X is L. cont.*
Let D be topological space the $\mathbb{F}.\mathbb{W}$. *multi-topology space (briefly,* $\mathbb{F}.\mathbb{W}$.*M.T.S.) if it is* $\mathbb{F}.\mathbb{W}$.*U.T.S. and* $\mathbb{F}.\mathbb{W}$.*L.T.S.*

Definition 1.8. [3] A filter $\mathfrak{I}$ on topological space (E,τ) a non-empty collection of non-empty subsets of E such that

i. $\forall\ \mathbb{F}_1, \mathbb{F}_2 \in \mathfrak{I}, \mathbb{F}_1 \cap \mathbb{F}_2 \in \mathfrak{I}$

ii. If $\mathbb{F}_1 \subseteq \mathbb{F}_2 \subseteq$E and $\mathbb{F}_1 \in \mathfrak{I}$ then $\mathbb{F}_2 \in \mathfrak{I}$.

Definition 1.9. [3] If $\mathfrak{I}$,$\mathfrak{Q}$ filter bases on (E,τ), we namely $\mathfrak{Q}$ is finer than $\mathfrak{I}$ (written as $\mathfrak{I} < \mathfrak{Q}$) if for all $\mathbb{F} \in \mathfrak{I}$, there is G $\subseteq \mathbb{F}$ meets$\mathfrak{Q}$ if $\mathbb{F} \cap G \neq \emptyset$ for every $\mathbb{F} \in \mathfrak{I}$ and G $\in \mathfrak{Q}$.

Definition 1.10. [10] If E is topological space and e ∈ E a $\eta\mathbb{P}\mathbb{d}$ of e is a set $\mathfrak{U}$ which contain an open set V containing e. If $\mathcal{A}$ is open set and contains e we namely $\mathcal{A}$ is open $a\ \eta\mathbb{P}\mathbb{d}$ for a point e.

Definition 1.11. [9] A point e in (E,τ) is named to be a contact point of a subset $\mathcal{A} \subseteq$ E ff $\forall\ \mathfrak{U}$ open $\eta\mathbb{P}\mathbb{d}$ of e, cl ($\mathfrak{U}$) $\cap\ \mathcal{A} \neq \emptyset$. So, set of all contact points of $\mathcal{A}$ is named to be the closure of $\mathcal{A}$ and is symbolized by cl ($\mathcal{A}$).

Definition 1.12. [10] A subset $\mathcal{A}$ in topological spacee (E,τ). So, $\mathcal{A}$ is named to be $\mathbb{E}$.set in E (briefly, E -set) if $\forall\tau$ an open cover of $\mathcal{A}$ there is a finite sub collection H of δ; $\mathcal{A} \subset \cup\{$cl(H) : H $\in \delta$ }. If $\mathcal{A}$ = E; then, E is named to be a OHC space.

Definition 1.13. [2] Let e a point in a $\mathbb{F}.\mathbb{W}$.T.S. (E,τ) on (D,ρ) is named to be adherent point of a F*.B*. $\mathfrak{I}$. on E (briefly, ad(e)) iff all number of $\mathfrak{I}$ is contract a point. A set of all adherent point of $\mathfrak{I}$ is named to be the adherence of $\mathfrak{I}$ and is symbolizes by ad($\mathfrak{I}$).

Definition 1.14.[11] The filter base $\mathfrak{I}$ (briefly F*.B*. $\mathfrak{I}$) on topological space (E,τ) is named to be convergent (briefly, conv.) (Written, $\mathfrak{I} \xrightarrow{\text{conv.}}$ e iff every τ.open. $\eta\mathbb{P}\mathbb{d}\ \mathfrak{U}$ of e, contains some elements of $\mathfrak{I}$.

Definition 1.15.[11] The F*.B*. $\mathfrak{I}$ on topological space (E,τ) is named directed toward a set $\mathcal{A} \subset$ E,(briefly, $\mathfrak{I} \xrightarrow{\text{d.t}} \mathcal{A}$) iff all F*.B*.$\mathfrak{Q}$. larger than $\mathfrak{I}$ has an adherent point in $\mathcal{A}$, i.e. ad($\mathfrak{Q}$) $\cap\mathcal{A} \neq \emptyset$, and in another writing $\mathfrak{I} \xrightarrow{\text{ad}}$ e to imply that $\mathfrak{I} \xrightarrow{\text{d.t}} \{e\}$, in which e ∈E.
Currently, we review a characterization of a point e of a F*.B*. $\mathfrak{I}$.

2. Fibrewise Multi-Perfect Topological Spaces

In this segment we establish F.W. multi-perfect topological spaces (briefly, $\mathbb{F}.\mathbb{W}$.M.P.T.S.), and confirmation of few of its basic characteristics.

Definition 2.1. Let Ω : (E,τ) → (F,σ) be a function where E and F are $\mathbb{F}.\mathbb{W}$.T.S. on D is named to be upper perfect (briefly, U.P.) if for every F∗.B∗. $\mathfrak{I}$ on Ω(E), such that $\mathfrak{I}$ .d.t., some subset $\mathcal{A}$ of Ω(E), the F∗.B∗ $\Omega^{+}(\mathfrak{I})$ is $d.t.\ \Omega^{-1}(\mathcal{A})$ in E.

Definition 2.2. Let Ω : (E,τ) → (F,σ) be a function where E and F are $\mathbb{F}.\mathbb{W}$.T.S. on D is named to be lower perfect (briefly, L.P.) if for every F∗.B∗. $\mathfrak{I}$ on Ω(E), such that $\mathfrak{I}$ .d.t., some subset $\mathcal{A}$ of Ω(E), the F∗.B∗ $\Omega^{-}(\mathfrak{I})$ is $.d.t.\Omega^{-1}(\mathcal{A})$ in E.

Let Ω : (E,τ) → (F,σ) be a function where E and F are $\mathbb{F}.\mathbb{W}$.T.S. on D is named to be multi-perfect (briefly, M.P.) if it is U.P. and L.P.

Lemma 2.1. A function Ω : (E,τ) → (F,σ) is closed if cl($\Omega(\mathcal{A})$) $\subset \Omega$(cl($\mathcal{A}$)) for every $\mathcal{A} \subset$ E.

Proof. (⇒) Let Ω be closed and $\mathcal{A} \subset$ H. Since Ω is closed then Ω (cl($\mathcal{A}$)) is closed set in F, because cl($\mathcal{A}$) is closed set in E. so, cl(Ω($\mathcal{A}$)) ⊂ Ω (cl($\mathcal{A}$)).
(⇒) Let A be closed set in E, so $\mathcal{A}$ = cl($\mathcal{A}$), however cl(Ω ($\mathcal{A}$)) ⊂ Ω (cl($\mathcal{A}$)), so cl(Ω ($\mathcal{A}$)) ⊂ Ω ($\mathcal{A}$). Then, Ω ($\mathcal{A}$) is closed in F. Therefore Ω is closed.

Lemma 2.2. The point e in topological space (E,τ) is an ad point of a F∗.B∗. $\mathfrak{I}$ on E if $\exists\, a$ F*.B*. $\mathfrak{I}$. larger than $\mathfrak{I}$ such that $\mathfrak{I}*$ −−conv.→ e.

Proof. (⇒)Assume that e is an ad point of a F∗.B∗. $\mathfrak{I}$. on E, then it is an C. point of every number of $\mathfrak{I}$. This returns, for each τ-open $\eta\mathbb{P}\mathbb{d}$ $\mathfrak{U}$ of h, we have cl($\mathfrak{U}$)∩ $\mathbb{F} \neq \emptyset$ for every number $\mathbb{F}$ in $\mathfrak{I}$. Consequently, cl($\mathfrak{U}$) contains a some member of any F∗.B∗. $\mathfrak{I}*$ larger than $\mathfrak{I}$ such that $\mathfrak{I}*$ −−−conv.→ e.
(⇐) Assume that e is not an ad point of a F∗.B∗. $\mathfrak{I}$. on E, then ∃ $\mathbb{F} \in \mathfrak{I}$ such that e is not an contact of $\mathbb{F}$. So, ∃ $\tau-$ open- $\eta\mathbb{P}\mathbb{d}$ $\mathfrak{U}$ of e such that cl($\mathfrak{U}$) ∩ $\mathbb{F}$ = ∅. Denote by $\mathfrak{I}*$ the family of sets $\mathbb{F}* =$ $\mathbb{F}$ ∩ cl($\mathfrak{U}$) for $\mathbb{F} \in \mathfrak{I}$, so the sets in which $\mathbb{F}* \neq \emptyset$. Additionally, is a $\mathcal{F}^*.\mathcal{B}^*$. and really is $\mathbb{F}*$ from $\mathfrak{I}$. This is, given $\mathbb{F}_1^* = \mathbb{F}_1 \cap (E \setminus cl(\mathfrak{U}))$ and $\mathbb{F}_2^* = \mathbb{F}_2 \cap (E \setminus cl(\mathfrak{U}))$, $\exists\ \mathbb{F}_3 = \mathbb{F}_1 \cap \mathbb{F}_2$, and this gives $\mathbb{F}_2^* = \mathbb{F}_3 \cap \left(E \setminus cl(\mathfrak{U})\right) \subset \mathbb{F}_1 \cap \mathbb{F}_2 \cap \left(E \setminus cl(\mathfrak{U})\right) = \mathbb{F}_1 \cap (E \setminus cl(\mathfrak{U})) \cap \mathbb{F}_2 \cap (E \setminus cl(\mathfrak{U}))$. Since F∗ is not conv. to e. So, lead to a C!!!, and h is an ad point of a F∗.B∗. $\mathfrak{I}$. on E.

Lemma 2.3. Assume that $\mathfrak{I}$ is a F∗.B∗. $\mathfrak{I}$ on a topological space (E,τ). Suppose that e ∈ E, so $\mathfrak{I}$ −−conv.→ e if $\mathfrak{I}$ −− d.t→e.

Proof . (⇐) If $\mathfrak{I}$ does not conv. to e, then, ∃τ-open $\eta\mathbb{P}\mathbb{d}$ $\mathfrak{U}$ of e such that cl($\mathfrak{U}$)) ⊄ $\mathbb{F}$ = ∅ for every $\mathbb{F} \in \mathfrak{I}$. Then, $\mathfrak{Q}$ = {cl($\mathfrak{U}$) ∩ $\mathbb{F}$: $\mathbb{F} \in \mathfrak{I}$ } is a $\mathfrak{I}$ be a F∗.B∗. $\mathfrak{I}$. on E larger than $\mathfrak{I}$, and e ∉ ad of $\mathfrak{Q}$. Thus, $\mathfrak{I}$ cannot be d.t. e, so lead to a then C!!!,. Then, $\mathfrak{I}$ is conv. to e. (⇒). It is clear

Definition 2.3. The $\mathbb{F}.\mathbb{W}$.T.S. (E,τ) on a topological space (D,ρ) is named to be $\mathbb{F}.\mathbb{W}$. upper perfect (briefly, $\mathbb{F}.\mathbb{W}$.U.P.) if the projection X is U.P.

Definition 2.4. The F.W.T.S. (E,τ) on topological space (D,ρ) is named to be $\mathbb{F}.\mathbb{W}$. lower perfect (briefly, $\mathbb{F}.\mathbb{W}$..L.P.) if the projection X is L.P.

The F.W.T.S. (E, τ) on topological space (D,ρ) is named to be $\mathbb{F}.\mathbb{W}$. multi-perfect (briefly, $\mathbb{F}.\mathbb{W}$.M.P.) if it is $\mathbb{F}.\mathbb{W}$.U.P. and $\mathbb{F}.\mathbb{W}$.L.P.

In the next theory we prove that just points of D can be enough for the subset A in Definition (1.15), and so direction. Since converge can be replaced in view of Lemma (2.2.)

Theorem 2.1. Assume that (E,τ) is a $\mathbb{F}.\mathbb{W}$.T.S. on a topological space (D,ρ). So, the next are equivalent:

i. (E,τ) is $\mathbb{F}.\mathbb{W}$.U.P.T.S. (resp., $\mathbb{F}.\mathbb{W}$.L.P.T.S.).
ii. F∗.B∗. $\mathfrak{I}$ on X(E), where conv. to a point d in D,$E_{\mathfrak{I}}^+$ −−d.t→ E_d(resp., $E_{\mathfrak{I}}^-$−−d.t→ E_d).
iii. ∀ F∗.B∗. $\mathfrak{I}$ on E, ad X($\mathfrak{I}$) ⊂ X(ad $\mathfrak{I}$)

Proof . (i) ⇒(ii) By Lemma 2.2.
(ii) ⇒(iii) Assume that d ∈ ad X($\mathfrak{I}$). Thereafter, by Lemma (2.2.), ∃ F∗.B∗. $\mathfrak{Q}$ on X(E) larger from X($\mathfrak{I}$).s.t $\mathfrak{Q}$ –conv.→ d. Let $\mathfrak{U}$ = {$E_{\mathfrak{Q}}$ ∩ $\mathfrak{I}$: G ∈ $\mathfrak{Q}$ and $\mathbb{F} \in \mathfrak{I}$ } Thereafter, $\mathfrak{U}$ is a F∗.B∗. on E larger from $E_{\mathfrak{Q}}$. Since $\mathfrak{Q}$ −−d.t.→ d, by Lemma (2.3.) and X is P., $E_{\mathfrak{Q}}^+$−−d.t.→ Ed(resp., $E_{\mathfrak{Q}}^-$−−d.t.→ Ed). $\mathfrak{U}$ being larger than $E_{\mathfrak{Q}}$, we have Ed ∩ Ω^+(ad $\mathfrak{U}$) ≠ ∅(resp.$, Ed\ \cap\ \Omega^-(ad\ \mathfrak{U})\ \neq \emptyset.$). Hence it is obvious that $E_d\Omega(\mathfrak{I}) \neq \emptyset$. So, d ∈ X(ad $\mathfrak{I}$).
(iii)⇒(i) Let $\mathfrak{I}$ be a F∗.B∗. on X(E) such that it is d.t. some subset $\mathcal{A}$ of X(E). Assume that $\mathfrak{Q}$ is a F∗.B∗. on E larger than $E_{\mathfrak{I}}$. Thereafter, X($\mathfrak{Q}$) is a F∗.B∗. on X(E) larger than $\mathfrak{I}$ and so $\mathcal{A}$ ∩ (ad

X(𝔔)) ≠ ∅. Then, by (c), $\mathcal{A}$ ∩ X(ad (𝔔)) ≠ ∅ such that $E_{\mathcal{A}}^{+}$ ∩ (ad (𝔔)) ≠ ∅(resp., $E_{\mathcal{A}}^{-}$ ∩ (ad (𝔔)) ≠ ∅). Then, $E_{\mathfrak{I}}$ is d.t. $E_{\mathcal{A}}$. So, X is U.P.(resp., L.P.).

Corollary 2.1. Assume that (E,τ) is a F.W.T.S. on a topological space (D,ρ). So, the next are equivalent:

i. (E,τ) is 𝔽.𝕎.M.P.T.S..
ii. F∗.B∗. 𝔍 *on* X(E), where conv. to a point d in D,$E_{\mathfrak{I}}^{+}$ —d.t→ Ed(resp., $E_{\mathfrak{I}}^{-}$ —d.t→ E_d).
iii. ∀ F∗.B∗. 𝔍 on E,ad X(𝔍) ⊂ X(ad 𝔍)

Theorem 2.2. If the 𝔽.𝕎.T.S. (E,τ) on (D,ρ) is U.P.(resp., L.P.), then it is closed.
Proof. Suppose that E is a 𝔽.𝕎.U.P.T.S. (resp., 𝔽.𝕎.L.P.T.S.) on D, then the projection XE: E → D is U.P. (resp., L.P.) to show that it is closed, by Theorem (4.1.16.) (a) ⇒ (c) for any F∗.B∗. 𝔍 on E ad X(𝔍) ⊂ X(ad (D)), by Lemma (4.1.11.), Ω is closed if cl(Ω($\mathcal{A}$)) ⊂ (cl($\mathcal{A}$)) for every $\mathcal{A}$ ⊂E, so X is closed in which 𝔍 = { $\mathcal{A}$ }.
Corollary 2.2. If the 𝔽.𝕎.T.S. (E,τ) on (D,ρ) is M.P., then it is closed.

3. Fibrewise Multi-Perfect and multi-Rigidity Topological Spaces.

In this segment, we present the idea of multi-perfect topological, upper rigidity spaces lower rigidity spaces, multi-rigidity spaces and make sure of some of its base characteristics.

Definition 3.1. A subset $\mathcal{A}$ of a topological space (E,τ) is named to be upper rigid in E (briefly, U.R.) if for every F∗.B∗. 𝔍 on E $ad\ X^{+}(\mathfrak{I}) \cap \mathcal{A} = \emptyset, \exists \mathfrak{U} \in \tau$ and 𝔽 ∈ 𝔍 such that $\mathcal{A} \subset \mathfrak{U}$ or equivalently, if for every F∗.B∗. 𝔍 on E, whenever $\mathcal{A}$ ∩ (ad 𝔍) = ∅, thereafter for some F ∈ 𝔍, $\mathcal{A}$ ∩ (cl(𝔍)) = ∅.
Definition 4.2. A subset $\mathcal{A}$ of topological space (E,τ) is named to be lower rigid in E (briefly, L.R.) if for every F∗.B∗. 𝔍 on E $ad\ X^{-}(\mathfrak{I}) \cap \mathcal{A} = \emptyset, \exists \mathfrak{U} \in \tau$ and 𝔽 ∈ 𝔍 such that $\mathcal{A} \subset \mathfrak{U}$ or equivalently, if for every F∗.B∗. 𝔍 on E, whenever $\mathcal{A}$ ∩ (ad 𝔍) = ∅, thereafter for some F ∈ 𝔍, $\mathcal{A}$ ∩ (cl(𝔍)) = ∅.
A subset $\mathcal{A}$ of topological space (E,τ) is named to be multi-rigid in E (briefly, M.R.) if it is U.R. and L.R.
Theorem 3.1. If (E,τ) is a 𝔽.𝕎. closed topological space on (D,ρ) such that every E_d^{+}(resp., E_d^{-}). in which d ∈ D is U.R.(resp., L.R.) in E, then (E,τ) is a 𝔽.𝕎.U.P. (resp., 𝔽.𝕎.L.P.).
Proof. Suppose that E is a 𝔽.𝕎. closed topological space on D, thereafter X_E : E →D exists T.P. it is U.P.(resp., L.P.), assume that 𝔍 is a F∗.B∗. on $X_E$such that D –conv.→ d in D, for some d in D. If 𝔔 is a F∗.B∗ on E larger than the F∗.B∗.$E_{\mathfrak{I}}$, then X(𝔔) is a F∗.B∗. on D, larger than 𝔍. Because 𝔍 —d.t.→ d by Lemma (2.3.), d ∈ adX(𝔔), i.e, d ∈ ∩{ad X(G;G ∈ 𝔔)}, and hence, d ∈ ∩{X(ad G;G ∈ 𝔔)} by Lemma 1.1.). By X *is* closed, so E_d^{+} ∩ ad (G) ≠ ∅(resp., E_d^{-} ∩ ad (G) ≠ ∅), for every G ∈ 𝔔. So, for every 𝔘 ∈ τ *with* E_d^{+}(resp., E_d^{-})⊂ 𝔘, cl(𝔘) ∩G ≠ ∅ for every G ∈ 𝔔. Since, E_d^{+}(resp., E_d^{-}) is U.R.(resp., L.R.), it then follows that E_d^{+}∩ ad (𝔔) ≠ ∅(resp., E_d^{-}∩ ad (𝔔) ≠ ∅) . Thus, $E_{\mathfrak{I}}$ —d.t.→Ed. S*o* by Theorem [(2.1.), (b) ⇒(a)], X is U.P.(resp., L.R.)
Corollary 3.1. If (E,τ) is a 𝔽.𝕎. closed topological space on (D,ρ) such that every Ed in which d ∈ D *is* M.R. in E, then (E,τ) is a 𝔽.𝕎.M.P.
Theor*e*m 3.2. If the 𝔽.𝕎.T.S. (E,τ) on (D,ρ) is U.P. (resp., L.P.), then, it is closed and for every d ∈ B, E_d^{+}(resp., E_d^{-}). is U.R.(resp., L.R.) in E.

Proof. Let E be a 𝔽.𝕎.T.S. on D, so the projection X_E : E⟶ D exists and it is U. cont.(resp., L. cont.). X_E is an U.P.(resp., L.P.) so it is closed. T.P. is closed and for every d ∈ D, E_d^+(resp., E_d^-) is U.R.(L.R) in E. Let d ∈ D and suppose 𝔍 is a 𝔍 ∗.B*. on E such that (ad 𝔍)∩ E_d^+= ∅(resp., (ad 𝔍) ∩ E_d^- = ∅). Therefore, d ∉ X_E (ad 𝔍) By X_E is U.P. (resp., L.P.), by Theorem [(2.1.) (a) ⇒ c)], d ∉ adX_E (𝔍). Thus, ∃ an 𝔽∈ 𝔍 such that d ∉ adX_E (𝔽).∃an ρ−open a ηℙ𝕕 V of d such that cl(V)∩ X_E (𝔽) = ∅. Since X_E is cont., for every e ∈E_d^+(resp., E_d^-). We shall get a τ-open a ηℙ𝕕 𝔘e of e such that X_E (cl(𝔘e)) ⊂ cl(V) ⊂ D − X_E (𝔽). So X_E (cl(𝔘e)) ∩ X_E (𝔽) = ∅, so that cl(𝔘e)) ∩ 𝔽 = ∅. Then h ∉ cl(𝔽), for every e ∈ E_d^+(resp., E_d^-), so E_d^+(resp., E_d^-) ∩ cl(𝔽) = ∅, So E_d^+(resp., E_d^-) is U.R.(resp., L.R.) in E.

Corollary 3.2. If the 𝔽.𝕎.T.S. (E,τ) on (D,ρ) is M.P. then it is closed and for every d ∈ B, Ed is M.R. in E.

Definition 3.3. The function Ω : (E,τ) ⟶ (F,σ) is named to be weakly upper closed (briefly, W.U. closed) if ∀f ∈ Ω^+(E) and ∀ 𝔘 ∈ τ containing η−1(f) in E,∃ a ρ −open a ηℙ𝕕 V of d such that Ω−1(V) ⊂ cl(𝔘).

Definition 3.4. The function Ω : (E,τ) ⟶ (F,σ) is named to be weakly lower closed (briefly, W.L. closed) if ∀f ∈ Ω^-(E) and ∀ 𝔘 ∈ τ containing Ω−1(f) in E,∃ a ρ −open a ηℙ𝕕 V of d such that Ω−1(V) ⊂ cl(𝔘).

The function Ω : (E,τ) ⟶ (F,σ) is named to be weakly multi-closed (briefly, W.M. closed) if it is W.U. closed and W.L. closed.

Definition 3.5. The 𝔽.𝕎.T.S. (E,τ) on (D,ρ) is named to be 𝔽.𝕎. upper weakly closed (briefly, 𝔽.𝕎.U.W. closed) if the projection X is W.U. closed.

Definition 3.6. The 𝔽.𝕎.T.S. (E,τ) on (D,ρ) is named to be 𝔽.𝕎. lower weakly closed (briefly, 𝔽.𝕎.L.W. closed) if the projection X is W.L. closed.

The 𝔽.𝕎.T.S. (E,τ) on (D,ρ) is named to be 𝔽.𝕎. multi-weakly closed (briefly, 𝔽.𝕎.M.W. closed) if it is 𝔽.𝕎.U.W. closed and 𝔽.𝕎.U.W. closed.

Theorem 3.3. The 𝔽.𝕎. closed topological space (E,τ) on (D,ρ) is W.U. closed (resp., W.L. closed).

Proof. Assume that E is a 𝔽.𝕎. closed topological space on D, then the projection XE: E⟶ D exists, and to prove its W.U. closed (resp., W.L. closed). Let d ∈ X_E and le$t$ 𝔘 ∈ τ containing E_d^+(resp., E_d^-) in E. Currently, by Theorem (4.1.18.) cl(E−cl(𝔘)) = cl(E−cl(𝔘)), and, hence by Lemma, (4.1.11.) and since X_E is closed, we have cl(X_E (E−cl(𝔘))) ⊂ X_E [cl(E−cl(𝔘))].. Currently, since d ∉ X_E [cl(E−cl(𝔘))], d ∉ cl(X_E (E−cl(U))), and thus, ∃an ρ−open a ηℙ𝕕 V of d ∈ D such that cl(V)∩ X_E (E−cl(𝔘)) = ∅ which means that $E_{cl(V)}^+$∩(E−cl(𝔘)) = ∅(resp., $E_{cl(V)}^-$∩(E−cl(𝔘)) = ∅), and so X_E is W.U. closed (resp., W.L. closed).

Corollary 3.3. The 𝔽.𝕎. closed topological space (E,τ) on (D,ρ) is W.M. closed.

Theorem 3.4. Let (E,τ) be 𝔽.𝕎.T.S. on (D,ρ). Then (E,τ) is 𝔽.𝕎.U.P.(resp., 𝔽.𝕎.L.P.), if:

i. (E,τ) is 𝔽.𝕎.U.W. closed (resp., 𝔽.𝕎.L.W. closed) topological space.
ii. E_d^+(resp., E_d^-) is U.R.(resp., L.R.), for every d ∈ D.

Proof. Assume that E is a 𝔽.𝕎. space on D satisfying the conditions (i) and (ii), then the projection X_E: E⟶ D exists. To prove that X_E is U.R.(resp., L.R.), we have to show in view of

Theorem (3.1.) that X_E is closed. Let d ∈ $X_E(\mathcal{A})$, for some not empty subset $\mathcal{A}$ of E, but d ∉ $X_E(cl(\mathcal{A}))$. Then, E = { $\mathcal{A}$ } is a F∗.B∗ on E and (ad(E))∩ E_d^+(resp., E_d^-) = ∅. By U.R.(resp., L.R.) of E_d^+(resp., E_d^-), a ∃ 𝔘 ∈ τ containing E_d^+(resp., E_d^-) such that cl(𝔘)∩ $\mathcal{A}$ = ∅. By W.U. closed (resp., W.L. closed) of X_E ∃ an ρ−open a ηℙ𝕕 D of d such that, $E_{cl(V)}^+ \cap \mathcal{A}$ = ∅(resp., $E_{cl(V)}^- \cap \mathcal{A}$ = ∅), i.e., cl(V) ∩ XE ($\mathcal{A}$) = ∅, which is impossible since d ∈ $X_E(\mathcal{A})$. So Ω is closed.

Corollary 3.4. Let (E,τ) be 𝔽.𝕎.T.S. on (D,ρ). Then, (E,τ) is 𝔽.𝕎.M.P, if

i. (E,τ) is 𝔽.𝕎.M.W. closed topological space.

ii. E_d is M.R, for every d ∈ D.

Lemma 3.1. [11]A subset A of a topological space (E,τ) is 𝔼. set if for every F∗.B∗ on ℑ on $\mathcal{A}$; (ad(ℑ)) ∩ $\mathcal{A}$ ≠ ∅.

Theorem 3.5. If (E,τ) is 𝔽.𝕎.U.P.T.S.(resp., 𝔽.𝕎.L.P.T.S.) on (D, ρ) and D∗ ⊂ D is an 𝔼 set in D, so $E_{D^*}^+$(resp., $E_{D^*}^-$) is an 𝔼 set in E.

Proof. Suppose that E is a 𝔽.𝕎.U.P.T.S. (resp., 𝔽.𝕎.L.P.T.S.) on D, therefore X_E: E → D exist. Let ℑ be a F∗.B∗. on D∗. By D∗ is an 𝔼 set in D, D∗ ∩ ad X_E(ℑ) ≠ ∅, by Lemma (3.1.). By Theorem [(2.1.) (i) ⇒ (iii)], D∗ ∩ X_E(ad (ℑ) ≠ ∅, so $E_{D^*}^+$∩ ad (ℑ) ≠ ∅(resp., $E_{D^*}^-$∩ ad (ℑ) ≠ ∅). Hence, by Lemma (3.1.), $E_{D^*}^+$(resp., $E_{D^*}^-$) is an 𝔼 set in E.

Corollary 3.5. If (E,τ) is 𝔽.𝕎.M.P.T.S on (D, ρ) and D∗ ⊂ D is an 𝔼 set in D, so $\mathbb{E}_{D^*}$ is an 𝔼 set in E.

Definition 3.7. The function Ω : (E,τ) → (F,σ) is named to be almost U.P. if for every E set K in F, Ω+(K) is an 𝔼 set in E.

Definition 3.8. The function Ω : (E,τ) → (F,σ) is named to be almost L.P. if for every 𝔼 set K in F, Ω-(K) is an 𝔼 set in E.

The function Ω : (E,τ) → (F,σ) is named to be almost M.P. if almost U.P. and almost L.P.

Definition 3.9. The 𝔽.𝕎.T.S. almost U.P. on (D,ρ) is named to be 𝔽.𝕎. almost U.P. if the projection X is almost perfect.

Definition 3.10. The 𝔽.𝕎.T.S. almost L.P. on (D,ρ) is named to be 𝔽.𝕎. almost L.P. if the projection X is almost perfect.

The 𝔽.𝕎.T.S. almost M.P. on (D,ρ) is named to be 𝔽.𝕎. almost M.P. if it is 𝔽.𝕎. almost U.P. and 𝔽.𝕎. almost L.P.

Theorem 3.6. Let (E,τ) be 𝔽.𝕎.T.S. on (D,ρ) such that:

i. For every d ∈ D, E_d^+(resp., E_d^-) is U.R.(resp., L.R.) and

ii. (E,τ) be 𝔽.𝕎.U.W. closed (resp., 𝔽.𝕎.L.W. closed) topological space. Then, (E,τ) is 𝔽.𝕎. almost U.P.T.S.(resp., 𝔽.𝕎. almost L.P.T.S.).

Proof. Let E be 𝔽.𝕎.T.S. on D, so XE : E→ D exist and it is U. cont. (resp., L. cont.). Assume that D∗ is an 𝔼 set in D and let ℑ be a F∗.B∗. on ED∗. Currently, XE(ℑ) is a F∗.B∗. on D∗ and so by Lemma (4.2.15.), (ad XE(ℑ)) ∩ D∗ ≠ ∅. Let d ∈ (ad XE(ℑ)) ∩ D∗. Let ℑ has no ad point in $E_{D^*}^+$(resp., $E_{D^*}^-$), so that (ad (ℑ))∩ E_d^+(resp., E_d^-) = ∅. By E_d^+(resp., E_d^-) is U.R.(resp., U.R.),∃ an 𝔽 ∈ ℑ and τ−open se$t$ 𝔘 containing $E_{D^*}^+$(resp., $E_{D^*}^-$), such that 𝔽 ∩cl(𝔘) = ∅. Since W.U. closed (resp., W.L. closed) of XE, ∃ ρ− closed a ηℙ𝕕 V of d such that E (ρ−cl(V)) ⊂ τ − cl(𝔘) which means that $E_{(\rho-cl(V))}^+ \cap \mathbb{F}$ = ∅ (resp., $E_{(\rho-cl(V))}^- \cap \mathbb{F}$ = ∅) i.e., ρ − cl(V) ∩ X(𝔽) = ∅, which is a contradiction. Thus, by Lemma (4.2.15.), $E_{D^*}^+$(resp., $E_{D^*}^-$) is an 𝔼 set in E and so XE is almost U.P. (resp., almost L.P.).

Corollary 3.6. Let (E,τ) be 𝔽.𝕎.T.S. on (D,ρ) such that:

i. For every d ∈ D, Ed is M.R. and

(E,τ) be 𝔽.𝕎.M.W. closed topological space. Then, (E,τ) is 𝔽.𝕎. almost M.P.T.S.

4. Some Result on Multi Topological Spaces

We Currently give some results of 𝔽.𝕎.U.P.T.S.(resp., 𝔽.𝕎.L.P.T.S. and 𝔽.𝕎.M.P.T.S.). The following characterization theorem for an U. cont. (resp., L. cont. and M. cont.) function is recalled to this end.

Theorem 4.1. A topological space (E,τ) is 𝔽.𝕎.U.T.S.(resp., 𝔽.𝕎.L.T.S.) on (D,ρ) if XE(cl($\mathcal{A}$)) ⊂ cl(XE($\mathcal{A}$)), for each $\mathcal{A}$ ⊂ E .

Proof. (⇒) Assume that E is 𝔽.𝕎.U.T.S.(resp., 𝔽.𝕎.L.T.S.) on D then the projection XE : E→ D exist and it is U. cont. (resp., L. cont.). Suppose that e ∈ cl($\mathcal{A}$) and D is ρ−open a $\eta\mathbb{P}$𝕕 of Ω(e). Since XE is U. cont. (resp., L. cont.), ∃ an τ-open a $\eta\mathbb{P}$𝕕 $\mathfrak{U}$ of e such that X(cl($\mathfrak{U}$)) ⊂ cl(V). Since cl($\mathfrak{U}$) ∩ $\mathcal{A} \neq \emptyset$, then cl(V) ∩ X($\mathcal{A}$) ≠ ∅. So, XE($\mathcal{A}$) ∈ cl(XE($\mathcal{A}$)). This shows that XE(cl($\mathfrak{U}$)) ⊂ cl(XE(V)).

(⇐) It is clear.

Corollary 4.1. A topological space (E,τ) is F.W.M.T.S on (D,ρ) if XE(cl($\mathcal{A}$)) ⊂ cl(XE($\mathcal{A}$)).

Theorem 4.2. Let (E,τ) is 𝔽.𝕎.U.P.T.S.(resp., 𝔽.𝕎.L.P.T.S.) on (D,ρ). So $E^{+}_{\mathcal{A}}$(resp., $E^{-}_{\mathcal{A}}$) preserves U.R. (resp., L.R.).

Proof. Assume that E is a 𝔽.𝕎.U.P.T.S.(resp., 𝔽.𝕎.L.P.T.S.) on D, then the projection XE : E → D exist and it is U. cont. (resp., L. cont.). Let $\mathcal{A}$ be an U.R. set(resp., L.R. set) in D and let $\mathfrak{I}$ be a F∗.B∗. on 𝔼 such that $E_{\mathcal{A}}$∩ (ad ($\mathfrak{I}$)) = ∅. By XE is U.R. (resp., L.R.). and $\mathcal{A}$ ∩ XE(ad ($\mathfrak{I}$)) = ∅, by Theorem [(2.1.) (i) ⇒ (iii)] we get $\mathcal{A}$ ∩ (ad (X_E($\mathfrak{I}$))) = ∅. Currently, a being an U.R.(resp., L.R.) set in D, ∃ an 𝔽∈ $\mathfrak{I}$ such that $\mathcal{A}$ ∩ (cl(XE($\mathfrak{I}$))) = ∅. Because XE is U. cont.(resp., L. cont.) and by Theorem (4.1.) it follows that $\mathcal{A}$ ∩ XE(cl($\mathfrak{I}$)) = ∅. Then $E^{+}_{\mathcal{A}}$∩ (cl($\mathfrak{I}$)) = $\emptyset(resp., E^{-}_{\mathcal{A}}$∩ (cl($\mathfrak{I}$)) = ∅). T$hen$ T.P. $E^{+}_{\mathcal{A}}(resp., E^{-}_{\mathcal{A}})$ is U.R.(resp., L.R.).

We p$resen$t the f$ollowin$g d$efinitio$n to study the co$nditio$ns under which an F.W. almost p$erfec$t t$opolo$gical space can be an 𝔽.𝕎.U.P.T.S.(r$e$sp., 𝔽.𝕎.L.P.T.S.).

Corollary 4.2. Let (E,τ) be 𝔽.𝕎.M.P.T.S on (D,ρ). So $E_{\mathcal{A}}$ preserves M.R.

Definition 4.1. The function Ω : (E,τ) → (F,σ) is named to be upper∗ continuous (briefly, U∗. cont.) if for any τ-open a $\eta\mathbb{P}$𝕕 V of Ω^{+}(e), ∃ an τ−open a $\eta\mathbb{P}$𝕕 $\mathfrak{U}$ of e such that Ω(cl($\mathfrak{U}$)) ⊂ cl(V).

Definition 4.2. The function Ω : (E,τ) → (F,σ) is named to be lower* continuous (briefly, L*. cont.) if for any τ-open a $\eta\mathbb{P}$𝕕 V of Ω^{-}(e), ∃ an τ−open a $\eta\mathbb{P}$𝕕 $\mathfrak{U}$ of e such that Ω(cl($\mathfrak{U}$)) ⊂ cl(V).

The function Ω : (E,τ) → (F,σ) is named to be multi∗ -cont. (briefly, M∗. cont.) if it is L∗. cont. and U∗. cont.

Definition 4.3. The 𝔽.𝕎.T.S. (E,τ) on (F,σ) is named 𝔽.𝕎.U*.T.S. if the projection X is U∗.cont.

Definition 4.4. The 𝔽.𝕎.T.S. (E,τ) on (F,σ) is named 𝔽.𝕎.L*.T.S. if the projection X is L∗.cont.

The 𝔽.𝕎.T.S. (E,τ) on (F,σ) is named 𝔽.𝕎.M*.T.S. if it is 𝔽.𝕎.L*.T.S. and 𝔽.𝕎.U*.T.S.

Importance of the above definition for characterization of 𝔽.𝕎.U.P.T.S.(resp., 𝔽.𝕎.L.P.T.S. and 𝔽.𝕎.M.P.T.S.). It is quite clear from the next result.

Lemma 4.1.[27] In a Urysohn topological space 𝔼 set is closed set.

Theorem 4.3. If (E,τ) is $\mathbb{F}.\mathbb{W}$.U∗.T.S.(resp., $\mathbb{F}.\mathbb{W}$.L∗.T.S.) on a Te (F,σ), so it is $\mathbb{F}.\mathbb{W}$.U.P.T.S.(resp., $\mathbb{F}.\mathbb{W}$.L.P.T.S.) if ∀ F∗.B∗ on E, if $X_{\mathfrak{I}}$ ––conv. →d ; d ∈ D, then ad $\mathfrak{I} \neq \emptyset$.

Proof. (⇒) Assume that (E,τ) is a $\mathbb{F}.\mathbb{W}$.U∗.T.S.(resp., $\mathbb{F}.\mathbb{W}$.L∗.T.S.) on a Te (D,ρ), then ∃U∗. cont.(resp., L∗. cont.) projection function XE : (E,τ) → (D,ρ) and $X_{\mathfrak{I}}$ ––conv.→ d in which d ∈ D, for a F∗.B∗ on $\mathfrak{I}$ on E. So $E^{+}_{X_{\mathfrak{I}}}$ –– dir.–→E^{+}_{d}(resp., $E^{-}_{X_{\mathfrak{I}}}$ –– dir.–→E^{-}_{d}). By $\mathfrak{I}$ is larger than $E^{+}_{X_{\mathfrak{I}}}$(resp.$, E^{-}_{X_{\mathfrak{I}}}$), E^{+}_{d}(resp., E^{-}_{d}) ∩ ad $\mathfrak{I} \neq \emptyset$, so ad $\mathfrak{I} \neq \emptyset$.

(⇐) Assume that $\forall\, F *. B *. \mathfrak{I}. on\ E, X_{\mathfrak{I}}$ $--$conv.→ d in which d ∈ D, implies ad $\mathfrak{I} \neq \emptyset$. Let $\mathfrak{Q}$ be a F∗.B∗. on D such that $\mathfrak{Q}$ –conv.→ d, and let $\mathfrak{Q}\,*$ be a F∗.B∗ on E, such that $\mathfrak{Q}\,*$ is larger than $E_{\mathfrak{Q}}$. Then $X_{\mathfrak{Q}^*}$ is larger than $\mathfrak{Q}$. So $X_{\mathfrak{Q}^*}$––conv.→ d. So, ad $\mathfrak{Q} * \neq \emptyset$. Let z ∈ D such that z ≠ d. So, by D is U.(resp., L.) Te, ∃ ρ−open a $\eta\mathbb{P}\mathbb{d}$ $\mathfrak{U}$ of d and ρ−open a $\eta\mathbb{P}\mathbb{d}$ V of z such that (ρ − cl($\mathfrak{U}$)) ∩ (ρ − cl(V)) = $\emptyset$. Since $X_{\mathfrak{Q}^*}$ ––conv.→ d, ∃ a G ∈ $\mathfrak{Q}\,*$ such that XG ⊂ ρ − cl($\mathfrak{U}$). Currently, by X is U∗. cont. (resp., L∗. cont.), corresponding to every e ∈ Ez, ∃ τ−open a $\eta\mathbb{P}\mathbb{d}$ W of e such that X(τ − cl(V)). Thus, ρ − cl(W ∩G) = $\emptyset$. It follows that E^{+}_{Z}(resp.$, E^{-}_{Z}$) ∩ $\mathfrak{Q}\,*$ = $\emptyset$, ∀ z ∈ D −{d}. Consequently, E^{+}_{d} ∩ ad $\mathfrak{Q} * \neq \emptyset$(resp.$, E^{-}_{d}$ ∩ ad $\mathfrak{Q} * \neq \emptyset$), and X is U.P.(resp., L.P.) and so (E,τ) is $\mathbb{F}.\mathbb{W}$.U∗.T.S.(resp., $\mathbb{F}.\mathbb{W}$.L∗.T.S.).

C*oro*llary 4.3. If (E,τ) is $\mathbb{F}.\mathbb{W}$.M∗.T.S on a Te (F,σ), so it is $\mathbb{F}.\mathbb{W}$.M.P.T.S if ∀F∗.B∗ on E, if $X_{\mathfrak{I}}$ ––conv. →d ; d ∈ D, then ad $\mathfrak{I} \neq \emptyset$.

Corollary 4.4. Let (E,τ) be $\mathbb{F}.\mathbb{W}$.M∗.T.S on (QHC) on a Urysohn topological space (D,ρ), so (E,τ) is $\mathbb{F}.\mathbb{W}$.M.T.S..

Theorem 4.4. Let (E,τ) be $\mathbb{F}.\mathbb{W}$.U∗.T.S.(resp., $\mathbb{F}.\mathbb{W}$.L∗.T.S.) on locally QHC on a Te(D,ρ), then (D,ρ) is $\mathbb{F}.\mathbb{W}$.U∗.T.S.(resp., $\mathbb{F}.\mathbb{W}$.L∗.T.S.) if it is $\mathbb{F}.\mathbb{W}$. almost U.P.(resp., $\mathbb{F}.\mathbb{W}$. almost L.P.).

Proof. (⇐) Let (E,τ) is $\mathbb{F}.\mathbb{W}$. almost U.P.(resp., $\mathbb{F}.\mathbb{W}$. almost L.P.), so ∃ almost U.P.(resp., almost L.P.) projection function XE : E → D and let D be any F∗.B∗. on E and let $X_{\mathfrak{I}}$ ––conv.→ d in which d ∈ D. There are an E set D∗ in D and ρ−open a $\eta\mathbb{P}\mathbb{d}$V of d such that, d ∈ V ⊆ D∗. Let E = {ρ − cl($\mathfrak{U}$)) ∩ $X_{\mathbb{F}}$ ∩ D∗; $\mathbb{F}$ ∈ $\mathfrak{I}$ and $\mathfrak{U}$ is a ρ−open a $\eta\mathbb{P}\mathbb{d}$ of d}. By L$emma$ (4.1.), D∗ is closed and hence no member of E is void. Reality, if not, let for some ρ−open a $\eta\mathbb{P}\mathbb{d}$ $\mathfrak{U}$ of d and some $\mathbb{F}$ ∈ $\mathfrak{I}$, ρ − cl($\mathfrak{U}$) ∩ $X_{\mathbb{F}}$ ∩D∗ = $\emptyset$. Then W = $\mathfrak{U}$ ∩V since d ∈ $\mathfrak{U}$ ∩V ∈ ρ and ρ − cl(W = cl(W) ⊂ cl(D∗) = D∗, by Lemma (4.1.). Currently $\emptyset$ = ρ−cl(W)∩ $X_{\mathbb{F}}$∩D∗ = ρ −cl(W)∩ $X_{\mathbb{F}}$, which is not possible, since $X_{\mathbb{F}}$—conv.→ d. So E is F∗.B∗. on D, and is obviously larger than $X_{\mathfrak{I}}$, so that E –conv.→ d. Also $\mathfrak{Q}$ = {E^{+}_{H}(resp.$, E^{-}_{H}$) ∩ $\mathbb{F}$: $\mathcal{H}$ ∈ E and $\mathbb{F}$ ∈ $\mathfrak{I}$ } is obviously a filter on $E^{+}_{D^*}$(resp.$, E^{-}_{D^*}$). Because X is almost U.P.(resp., almost L.P.), $E^{+}_{D^*}$(resp.$, E^{-}_{D^*}$) is an $\mathbb{H}$.set and so ad $\mathfrak{Q}$ ∩ $E^{+}_{D^*} \neq \emptyset$(resp., $\mathfrak{Q}$ ∩ $E^{-}_{D^*} \neq \emptyset$).Thus X is U.P.(resp., L.R.) and by Theorem (4.3.) (E,τ) be $\mathbb{F}.\mathbb{W}$.U∗.T.S.(resp., $\mathbb{F}.\mathbb{W}$.L∗.T.S.).

C*oro*llary 4.5. Let (E,τ) be $\mathbb{F}.\mathbb{W}$.M∗.T.S on locally QHC on a Te(D,ρ), then (D,ρ) is $\mathbb{F}.\mathbb{W}$.M∗.T.S if it is $\mathbb{F}.\mathbb{W}$. almost M.P..

Lemma 4.2. [10] A topological space (E,τ) is T_2 ⇐⇒ {e} = cl(e) ∀ e ∈E.

Theorem 4.5. If (E,τ) is a $\mathbb{F}.\mathbb{W}$.U.P.(resp., $\mathbb{F}.\mathbb{W}$.U.P.) injection and surjective topological space with E is a U.T_2 space(resp., L.T_2 space) on (D,ρ), Then D is U.T_2 space (resp., L.T_2 space).

Proof. Let d1, d2 ∈ D such that d1 ≠ d2. By X is surjective, so d1, d2 ∈ E and p is injection, then $E^{+}_{d1} \neq E^{+}_{d2}$(resp., $E^{-}_{d1} \neq E^{-}_{d2}$. Since X is U.P.(resp., L.P.), so by Theorem (2.2.) it is closed. By Lemma (4.2.) we have {E^{+}_{d1}} = cl{d1} (resp., {E^{-}_{d1}} = cl{d1)) and {E^{+}_{d2}} = cl{d2} (resp., {E^{-}_{d2}}

= cl{d2}) Because X is U.T2 space (resp., U.T2 space). Currently, X(cl{E_{d1}^{+}}) = cl{d1}(resp., X(cl{E_{d1}^{-}}) = cl{d1}) and X(cl{E_{d2}^{+}}) = cl{d2}(resp., X(cl{E_{d2}^{-}}) = cl{d2}), since X is closed. This mean {d1} = cl{d1} and {d2} = cl{d2}. Hence D is U.T2 space(re$s$p., U.T2 space).

Our following theory gives a description of an important class of $\mathbb{F}.\mathbb{W}$.U.TS.(resp., $\mathbb{F}.\mathbb{W}$.L.TS.) meaning the QHC spaces in terms of $\mathbb{F}.\mathbb{W}$.U.P.T.S. (resp., $\mathbb{F}.\mathbb{W}$.L.P.T.S.).

Corollary 4.6. If (E,τ) is a $\mathbb{F}.\mathbb{W}$.M.P. injection and surjective topological space with E is a M.T2 space on (D,ρ), Then D is M.T2. space.

Theorem 4.6. For a topological space (E,τ), the next are equivalent:

i. H is QHC.

ii. A $\mathbb{F}.\mathbb{W}$.U. (E,τ) is P.T.(resp., $\mathbb{F}.\mathbb{W}$.L. (E,τ) is P.T.) space with constant projection on D∗ in which D∗ is a singleton with two equal topologies meaning the unique topology on D∗.

iii. The $\mathbb{F}.\mathbb{W}$.. (B×H,Q) is U.P.T.S.(resp., L.P.T.S.) on (D,ρ), in which $\mathfrak{Q}$ = $\rho \times \tau$.

Proof . (i) ⇒ (ii) Suppose that XE : E → D is a constant projection on D∗ where D∗ is a singleton with two equal topologies meaning the unique topology on D∗. X is obviously closed. Additionally, $E_{D^*}^{+}$(resp.$,E_{D^*}^{-}$), i.e. E is obviously U.R.(resp., L.R.) by D∗ is QHC. Then by Lemma (3.1.) X is U.P.(resp., L.R.)

(ii) ⇒ (i) From Theorem (4.1.).

(i) ⇒ (iii) Let that (D×E, $\mathfrak{Q}$) is $\mathbb{F}.\mathbb{W}$.U.T.S.(resp., $\mathbb{F}.\mathbb{W}$.L.T.S.) on (D,ρ) in which $\mathfrak{Q}$ = $\rho \times \tau$, then there is a projection X = π; (D×E, $\mathfrak{Q}$) → (D,ρ). We show that π is closed and ∀d ∈ D, E_{D}^{+}(resp., E_{D}^{-}) is U.R.(resp., L.R.) in D×E. So, the result will be based on Theorem (3.1.). Let $\mathcal{A}$ ⊂ D×E and a ∉ π(cl($\mathcal{A}$)). ∀ e ∈E,(a,e) ∉ cl($\mathcal{A}$), so that ∃ a ρ−open $a\ \eta\mathbb{P}\mathbb{d}$ G of a and a τ-open a $\eta\mathbb{P}\mathbb{d}$ $\mathbb{E}$e of e such that [$\mathfrak{Q}$ − cl(Ge × E_{e}^{+}(resp., E_{e}^{-}))] ∩ $\mathcal{A}$ = ∅. Since E is QHC,{a}×E is a $\mathbb{E}$.set in D ×E. So that ∃ finitely many elements e1,e2,e3,...,en with, {a}×E⊂$\cup_{k=1}^{n}$ $\mathfrak{Q} - cl(G_{ek} \times E_{ek}^{+}$(resp., E_{ek}^{-})). Currently, a ∈ ∩nk=1Ghk = G, which is a ρ -open a $\eta\mathbb{P}\mathbb{d}$ of a ∫.t.(ρ −cl(G)∩π($\mathcal{A}$) = ∅. So a ∉ clπ($\mathcal{A}$) and thus clπ($\mathcal{A}$) ⊂ π(cl($\mathcal{A}$)). So π is closed by Lemma (2.1.). Next, let d ∈ D T.P. $(D\times E)_{d}^{+}$(resp.$,(D\times E)_{d}^{-}$) = π−1(d) to be U.R.(resp., L.R.) in D ×E. Let $\mathfrak{J}$ be a$\mathcal{F}^*.\mathcal{B}^*$. on D ×E such that π−1(d) ∩ ad $\mathfrak{J}$ = ∅. ∀e ∈ E,(d,e) ∉ ad $\mathfrak{J}$. So, ∃ρ−open $a\ \eta\mathbb{P}\mathbb{d}$ $\mathfrak{U}$e of d in D, a ρ -open $a\ \eta\mathbb{P}\mathbb{d}$ Ve of e in E and an $\mathbb{F}_{e}$ ∈ $\mathfrak{J}$ such that F− cl($\mathfrak{U}$e ×Ve) ∩$\mathbb{F}_{e}$ = ∅. As prove above, ∃finitely many elements e1,e2,e3,...,en of E such that {d}× E ⊂$\cup_{k=1}^{n}$ $\mathfrak{Q} - cl(G_{ek} \times V_{ek})$. Putting $\mathfrak{U}$ and choosing $\mathbb{F}$ ∈ $\mathfrak{J}$ with, $\mathbb{F}$ $\cap_{k=1}^{n}$ $\mathbb{F}_{\ ek}$, we get d ×E⊂ $\mathfrak{U}$ ×E⊂Q such that Q−cl($\mathfrak{U}$ ×E)∩ $\mathbb{F}$ = ∅. Thus cl($\mathbb{F}$)∩π−1(d) = ∅. So π−1(d) is U.R.(resp., L.R.)in D ×E.

(iii)⇒(i) Taking D∗ = D, we have that X = π : D∗ × D → D∗ is U.R.(resp., L.R.) Therefore by (Theorem (3.5.)), D∗ ×E is an $\mathbb{E}$.set and hence is QHC.

Corollary 4.7. For a topological space (E,τ), the next are equivalent:

i. H is QHC.

ii. A $\mathbb{F}.\mathbb{W}$.M. (E,τ) is P.T space with constant projection on D∗ in which D∗ is a singleton with two equal topologies meaning the unique topology on D∗.

iii. The $\mathbb{F}.\mathbb{W}$.. (B×H,Q) is M.P.T.S. on (D,ρ), in which $\mathfrak{Q}$ = $\rho \times \tau$.

5. Conclusion

The main purpose of the present work is to providethe starting point for some application of fibr*ewise* multi-p*e*r*fec*t t*o*p*o*l*o*gical spa*ces* structures in a falter base by using multi-topological spaces. Definitions of characterization theorems are used for multi-r*igi*d, fibr*ewise* multi-*we*akly cl*ose*d, $\mathbb{E}$ *se*t, fibr*ewis*e alm*os*t multi-p*e*r*fe*ct, multi*-*co*ntinu*o*us fibr*ewise* multi* -t*o*p*o*l*o*gical spa*ces*.

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