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Study Sstrong Convergence and Acceleration of New Iteration Type Three – Step

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Abstract
I The paper aims to define a new iterative method called the three-step type in which Jungck resolvent CR-iteration and resolvent Jungck SP-iteration are discussed and study rate convergence and strong convergence in Banach space to reach the fixed point which is differentially solved of nonlinear equations. The studies also expanded around it to find the best solution for nonlinear operator equations in addition to the varying inequalities in Hilbert spaces and Banach spaces, as well as the use of these iterative methods to approximate the difference between algorithms and their images, where we examined the necessary conditions that guarantee the unity and existence of the solid point. Finally, the results show that resolvent CR-iteration is faster than resolvent Jungck SP-iteration using Jungck resolvent estimation.

Keywords: Jungck mapping, Rate of convergence, Jungck $CR-$ iteration, Rsolvent Jungck $\mathcal{S}$P – iteration.

1.Introduction and Preliminary

In the last three decades, a lot of literature has been published on the iterative approximation of fixed −points for certain classes of operators, using the methods of the Mann and Ishikawa algorithm. Fixed−point theorems were developed for single-valued or definite-value assignments of metric spaces and Banach. Among the topics of fixed−point theorem, the topic of fixed−point approximations for assignments is important because it is useful for proving the existence of fixed−point assignments. However, once one knows that there is a fixed− point for some map, then finding the value of that fixed point is not an easy task, which is the reason of why we use iterative operations to calculate it. Over time, many iterative processes have been developed. These iterative processes have been used, to solve various types of differential equations in recent years. The well-known Bannack deflationary theorem uses the process of Picard's algorithm to approximate the fixed− point [1]. Some other well-known iterative processes are Agarwal [2], Noor [3], SP [4], Piccard S [5], and so on. Recently, Qing and Qhou [6] extended their results to a random period and the Ishikawa algorithm. They gave some control conditions for the convergence of the Ishikawa algorithm over a random period. In 2006, [7] presented that Mann's algorithm converges faster than Ishikawa's algorithm. Subsequently, many studies were conducted on this subject see [8, 9, 10].

This paper aims to propose new iterative schemes for a fixed point and proves the convergence and speediness using resolvent ZA-Jungck mapping

Now, let $\mathcal{D}$ be a nonempty subset closed -convex of a Banach space N ,we recall the following:

Definition 1.1 [11]: Let $\{v_n\}_{n=0}^{\infty}$ be a bounded sequence in N for all $v \in N$and
$r(v, \langle v_n \rangle) = \limsup_{n\to\infty} \|v_n - v\|$. Then

1. the asymptotic radius of $\langle v_n \rangle$ relative to $\mathcal{D}$ is given by:
$r(\mathcal{D}, \langle v_n \rangle) = inf\,\{r(v, \langle v_n \rangle) : v \in \mathcal{D}\}$.
2. The asymptotic center $A(\mathcal{D}, \langle v_n \rangle)$ of $\langle v_n \rangle$ is defined as:
$A\,(\mathcal{D}, \langle u_n \rangle) = \{v \in \mathcal{D}: r(v, \langle v_n \rangle) = r(\mathcal{D}, \langle v_n \rangle)\}$.

Definition 1.2 [12]: Let $\langle y_n \rangle, \langle o_n \rangle$ be a real sequence such that $\langle y_n \rangle$ converges to y , $\langle o_n \rangle$ Converges to o. And $E = \lim_{n\to\infty} \frac{|y_n - y|}{|o_n - o|}$

1. If $E = 0 \mapsto$ the sequence $\langle y_n \rangle$ is converge to y Faster then $\langle o_n \rangle$ converge to o
2. if $0 < E < \infty \mapsto \langle Y_n \rangle$ and $\langle O_n \rangle$ have the same rate of convergence.

Lemma 1.3 [13]: Let $\mathcal{B}$ be a uniformly convex - space and $\langle u_n \rangle$ be any sequence such that $0 < q \leq u_n < 1$ For some $q \in \mathbb{R}$ and$\forall\, n \geq 1$. Let $\langle v_n \rangle$ and $\langle l_n \rangle$ be any two sequences of $\mathcal{B}$ such that $\lim_{n\to\infty}\sup\|v_n\| \leq c$, $\lim_{n\to\infty}\sup\|l_n\| \leq c$and $c = \lim_{n\to\infty}\sup\|u_n v_n + (1 - u_n) l_n\|$ for some $c \geq 0$.

$Then\ \lim_{n\to\infty}\|v_n - l_n\| = 0$.

2.Main Results

In this part, we will present a new iteration and study the existence of fixed points and the convergence of these iterations

Definition 2.1: A resolvent Jungck $\mathcal{CR}$ – iteration
$Se_{n+1} = \mathcal{J}_{rn}\big((1 - a_n)TSw_n + a_n\, \mathcal{J}_{rn}TTw_n\big)$,
$Sw_n = (1 - c_n)\mathcal{J}_{rn}TSz_n + c_n\, TTz_n$,
$Sz_n = (1 - b_n)\mathcal{J}_{rn}TSe_n + b_n\mathcal{J}_{rn}\, Te_n$, where $z_0 \in \mathcal{C}$, S, T commute mappings are on $\mathcal{C}$ and $\langle a_n \rangle, \langle b_n \rangle$ and $\langle c_n \rangle$ are sequences in [0,1]

Definition 2.2: A resolvent Jungck $\mathcal{SP}$ – iteration
$Su_{n+1} = (1 - a)\mathcal{J}_{rn}TSx_n + a_n\, Tx_n,$
$Sx_n = \mathcal{J}_{rn}\big((1 - c_n)Sv_n + c\, Tv_n\big)$,
$Sv_n = (1 - b_n)Su_n + b_n\mathcal{J}_{rn}\, Tu_n$, where $u_0 \in \mathcal{C}, \langle a_n \rangle, \langle b_n \rangle$ and $\langle c_n \rangle$ are sequences in [0,1]

Definition 2.3 [14]: A self-mapping, $T: \mathcal{D} \to \mathcal{D}$ is called a resolvent $\mathcal{ZA}$-Jungck mapping if

$$\|Tn - Th\| \leq \varepsilon\psi(a_1\|Sn - Sh\| + a_2\|Sn - \mathcal{J}_{rn}h\|) + \min\left\{\|\mathcal{J}_{rn}n - Tn\|, \frac{\|Sn - \mathcal{J}_{rn}n\| + \|\mathcal{J}_{rn}h - Sh\|}{2}, \|Th - \mathcal{J}_{rn}h\|\right\}$$

$\forall n, h \in C, a_1, a_2, \varepsilon \in [0,1]and\ \ a_1 + a_2 \leq 1$

Now, we give new results.

Lemma 2.4: Let($\boldsymbol{N}$, ‖ . ‖) be a Banach - space, and T be a resolvent $\mathcal{ZA}$-jungck self-maping on $\mathcal{D}$ with $\mathrm{b} \in [0,1)$.Suppose that $\langle \mathcal{S}U_n \rangle$ be a resolvent Jungck $\mathcal{SP}$ – iteration scheme in $\mathcal{D}$. There exists $p \in F(s) \cap F(\mathcal{T}) \cap A^{-1}(0) \neq \emptyset$, then $\lim_{n\to\infty}\|Su_n - t\|$ exists, for all $n \in N$.

Proof: Since t be a fixed point of S , T and $A^{-1}(0)$. Then the following inequalities hold

$$\|Su_{n+1} - p\| = \|(1- a_n)\mathcal{J}_{rn}TSx_n + a_n Tx_n - p\|$$
$$\le (1- a_n)\|\mathcal{J}_{rn}TSx_n - p\| + a_n \| Tx_n - p\|$$
$$\le (1- a_n)\|TSx_n - p\| + a_n \| T x_n - p\| \le$$
$$(1- a_n)\mathfrak{b}\psi(a_1\|SSx_n - SP\| + a_2\|SSx_n - \mathcal{J}_{rn}p\|) + \min\left\{ \|\mathcal{J}_{rn}Sx_n - TSx_n\|, \frac{\|SSx_n - \mathcal{J}_{rn}Sx_n\| + \|\mathcal{J}_{rn}p - Sp\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\}$$
$$+ a_n\mathfrak{b}\psi(a_1\|Sx_n - Sp\| + a_2\|Sx_n - \mathcal{J}_{rn}p\|)$$
$$+ \min\left\{\|\mathcal{J}_{rn}x_n - Tx_n\|, \frac{\|Sx_n - \mathcal{J}_{rn}x_n\| + \|\mathcal{J}_{rn}p - Sp\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\}$$
$$= (1- a_n)\mathfrak{b}\psi\|Sx_n - p\| + a_n\mathfrak{b}\psi(\|Sx_n - p\|)$$
$$\le (1- a_n)\mathfrak{b}\|Sx_n - p\| + a_n\mathfrak{b}\|Sx_n - P\|$$
$$\le \big((1- a_n) + a_n\mathfrak{b}\big)\|Sx_n - P\|$$
$$\le \mathfrak{b}\|Sx_n - p\|. \qquad (2.1)$$

$$\|Sx_n - P\| = \|\mathcal{J}_{rn}[(1- c_n)Sv_n + c_n Tv_n] - p\|$$
$$\le \|(1- c_n)Sv_n + c_n Tv_n - p\|$$
$$\le (1- c_n)\|Sv_n - t\| + c_n\| Tv_n - p\|$$
$$\le (1- c_n)\|Sv_n - p\| + c_n\mathfrak{b}\psi(a_1\|Sv_n - Sp\| + a_2\|Sv_n - \mathcal{J}_{rn}p\|)$$
$$+ \min\left\{ \|\mathcal{J}_{rn}v_n - Tv_n\|, \frac{\|Sv_n - \mathcal{J}_{rn}v_n\| + \|\mathcal{J}_{rn}p - Sp\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\}$$
$$\le (1- c_n)\|Sv_n - p\| + c_n\mathfrak{b}\psi(a_1\|Sv_n - p\| + a_2\|Sv_n - p\|)$$
$$\le (1- c_n)\|Sv_n - p\| + c_n\mathfrak{b}\psi(\|Sv_n - p\|)$$
$$\le (1- c_n)\|Sv_n - p\| + c_n\mathfrak{b}\|Sv_n - p\|$$
$$= \big((1- c_n) + c_n\mathfrak{b}\big)\|Sv_n - p\|. \qquad (2.2)$$

$$\|Sv_n - p\| = \|(1- b_n)Su_n + b_n\mathcal{J}_{rn} Tu_n - p\|$$
$$\le (1- b_n)\|Su_n - p\| + b_n\|Tu_n - p\|$$
$$\le (1- b_n)\|Su_n - P\| + b_n \|Tu_n - P\|$$
$$\le (1- b_n)\|Su_n - p\| + b_n\mathfrak{b}\psi(a_1\|Su - Sp\| + a_2\|Su_n - \mathcal{J}_{rn}p\|)$$
$$+ \min\left\{ \|\mathcal{J}_{rn}u_n - Tu_n\| , \frac{\|Su_n - \mathcal{J}_{rn}u_n\| + \|\mathcal{J}_{rn}p - Sp\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\}$$
$$= (1- b_n)\|Su_n - P\| + b_n\mathfrak{b}\psi(a_1\|Su_n - P\| + a_2\|Su_n - P\|)$$
$$\le (1- b_n)\|Su_n - p\| + b_n\mathfrak{b}\psi(\|Su_n - p\|)$$
$$\le (1- b_n)\|Su_n - t\| + b_n\mathfrak{b}\|Su_n - p\|$$
$$= ((1- b_n) + b_n\mathfrak{b}) \|Su_n - p\| \qquad (2.3)$$

We substitute (2.3) in (2.2) as follows

$$\|Sx_n - p\| \le ((1- c_n) + c_n\mathfrak{b})((1- b_n) + b_n\mathfrak{b})) \|Su_n - p\| \qquad (2.4)$$

We substitute (2.4) in (2.1) as follows

$$\|Su_{n+1} - p\| \le \mathfrak{b}(1- c_n(1-\mathfrak{b})(1- b_n(1-\mathfrak{b})) \|Su_n - p\|$$
$$\le (1- c_n(1-\mathfrak{b}) \|Su_n - p\|$$
$$\|Su_{n+1} - p\| \le \|Su_n - p\|$$
$$\|Su_{n+1} - p\| \le \|Su_n - p\| \le \cdots \ldots \ldots \le \|Su_0 - p\|.$$

So, $\{\|Su_n - p\|\}$ is decreasing and bounded, for each p ∈ f(S) ∩ f(T) ∩ $A^{-1}(0)$, this implies that $\lim_{n\to\infty}\|Su_n - p\|$ exists.

Theorem 2.2: Let(N , ‖ . ‖) be a Banach-space, and T be a resolvent $\mathcal{ZA}$-jungck self-mapping on. If $\langle Su_n\rangle$ is resolvent Jungck $\mathcal{SP}$ – iteration scheme. Then the resolvent Jungck $\mathcal{SP}$ – iteration scheme converge-strongly to a unique common fixed- point of S , T and $A^{-1}(0)$

Proof: Let $P \in F(S) \cap F(T) \cap A^{-1}(0)$. By Lemma. (2.1), we have

$\|Su_{n+1} - p\| \leq (1 - c_n(1 - \mathfrak{b})\ \|Su_n - p\|$

$\|Su_{n+1} - p\| \leq \prod_{j=0}^{n}(1 - c_j(1 - \mathfrak{b})\ \|Su_0 - p\|$ (2.5)

$\|Su_n - p\| \leq e^{-(1-\mathfrak{b})\sum_{j=0}^{\infty} c_j}\|Su_0 - p\|$

Since $c_j \in [0, 1]$ and $\mathfrak{b} \in [0, 1]$ and $\sum_{j=0}^{\infty} c_j = \infty$

Then $e^{-(1-\mathfrak{b})\sum_{j=0}^{\infty} c_j} \to 0$ as $n \to \infty$

from (2.5), $\lim_{n \to \infty} \|Su_n - p\| = 0$. Therefore, $\langle Se_n \rangle$ converges strongly to t

Now, we prove that the unique common fixed point of S, T and $A^{-1}(0)$. Let there exist another point $p^* \in C$ such that $Tp^* = \mathcal{J}_{rn}\, p^* = Sp^* = p^*$

$$
\begin{aligned}
0 \leq \|p - p^*\| &= \|Tp - T\,p^*\| \\
&\leq b\psi(a_1\|Sp - Sp^*\| + a_2\|Sp - \mathcal{J}_{rn}\, p^*\|) + \\
&\min\left\{\|\mathcal{J}_{rn}p - \mathcal{T}p\|, \frac{\|Sp - \mathcal{J}_{rn}\, p^*\| + \|\mathcal{J}_{rn}\, p^* - Sp^*\|}{2}, \|Tp^* - \mathcal{J}_{rn}\, p^*\|\right\} \\
&\leq b(a_1\|Sp - S\rho^*\| + a_2\|Sp - \mathcal{J}_{rn}\, p^*\|) \\
&\leq b\|Sp - Sp^*\| \\
p &= p^*
\end{aligned}
$$

Theorem 2.3: Let $\mathcal{N}$ be a uniformly convex Banach-space, $\mathcal{D}$ be a nonempty convex-closed subset of N and T be a resolvent $\mathcal{ZA}$-jungck self-mapping on $\mathcal{D}$. If $\langle Su_n \rangle$ the resolvent Jungck $\mathcal{SP}$ – iteration scheme in $\mathcal{D}$ then $F(\mathcal{S}) \cap F(T) \cap A^{-1}(0) \neq \phi$ if and only if $\lim_{n\to\infty}\|Tu_n - \mathcal{J}_{rn}Su_{n+1}\| = 0$ ..

Proof: Since $F(S) \cap F(\mathcal{G}) \cap A^{-1}(0) \neq \emptyset \Longrightarrow \exists p \in F(S) \cap F(T) \cap A^{-1}(0) \Longrightarrow p = F(S) = f(t) = A^{-1}(0)$.

By Lemma (2.1) $We\ have$, $\lim_{n\to\infty}\|Su_n - p\| = \delta$

We get, $\lim_{n\to\infty}\sup\|\mathcal{J}_{rn}Su_{n+1} - p\| \leq \delta$ (2.6)

Now, To prove that $\lim_{n\to\infty}\sup\|Tu_n - p\| \leq \delta$

$$
\begin{aligned}
\lim_{n\to\infty}\sup\|Tu_n - p\| &\leq \lim_{n\to\infty}\sup \mathfrak{b}\psi(a_1\|Su_n - Sp\| + a_2\|Su_n - \mathcal{J}_{rn}p\|) \\
&+ \min\left\{\|\mathcal{J}_{rn}u_n - Tu_n\|, \frac{\|Su - \mathcal{J}_{rn}u_n\| + \|\mathcal{J}_{rn}p - Sp\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\} \\
&= \lim_{n\to\infty}\sup \mathfrak{b}\psi(a_1\|Su_n - Sp\| + a_2\|Su_n - \mathcal{J}_{rn}p\|) \\
&\leq \lim_{n\to\infty}\sup\|Su_n - Sp\|. \\
&= \delta
\end{aligned}
$$

$We\ get$, $\lim_{n\to\infty}\sup\|Tu_n - p\| \leq \delta$ (2.7)

Now by Lemma (2.1) we have

$$
\begin{aligned}
\delta &= \lim_{n\to\infty}\sup\|Su_{n+1} - p\| = \lim_{n\to\infty}\sup\|(1 - a_n)\mathcal{J}_{rn}TSx_n + a_n Tx_n - p\| \\
&\leq \lim_{n\to\infty}\sup\,[(1 - a_n)\|\mathcal{J}_{rn}TSx_n - p\| + a_n\|Tx_n - p\|] \\
&\leq \lim_{n\to\infty}\sup\,[(1 - a_n)\|TSx_n - p\| + a_n\|Tx_n - p\|] \\
&\leq \lim_{n\to\infty}\sup\,[(1 - a_n)\mathfrak{b}\psi(a_1\|SSx_n - \mathcal{S}p\| + a_2\|SSx_n - \mathcal{J}_{rn}p\|) + \\
&\quad \min\left\{\|\mathcal{J}_{rn}Sx_n - TSx_n\|, \frac{\|SSx_n - \mathcal{J}_{rn}Sx_n\| + \|\mathcal{J}_{rn}p - Sp\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\} \\
&\quad + a_n\mathfrak{b}\psi(a_1\|Sx_n - Sp\| + a_2\|Sx_n - \mathcal{J}_{rn}p\|) \\
&\quad + \min\left\{\|\mathcal{J}_{rn}x_n - Tx_n\|, \frac{\|Sx_n - \mathcal{J}_{rn}x_n\| + \|\mathcal{J}_{rn}p - Sp\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\} \\
&\leq \lim_{n\to\infty}\sup\,[(1 - a_n)\mathfrak{b}\psi\|Sx_n - p\| + a_n\mathfrak{b}\psi(\|Sx_n - p\|)] \\
&\leq \lim_{n\to\infty}\sup\,[(1 - a_n)\mathfrak{b}\|Sx_n - p\| + a_n\mathfrak{b}\|Sx_n - p\|] \\
&= \lim_{n\to\infty}\sup\,[\big((1 - a_n) + a_n\mathfrak{b}\big)\|Sx_n - p\|]
\end{aligned}
$$

$$\leq \lim_{n\to\infty}\sup \mathfrak{b}\|Sx_n - p\| . \qquad (2.8)$$

$$\|Sx_n - P\| = \|\mathcal{J}_{rn}[(1- c_n)Sv_n + c_n Tv_n] - p\|$$
$$\leq \|(1- c_n)Sv_n + c_n Tv_n - p\|$$
$$\leq (1- c_n)\|Sv_n - t\| + c_n\| Tv_n - p\|$$
$$\leq (1- c_n)\|Sv_n - p\| + c_n\mathfrak{b}\psi(a_1\|Sv_n - Sp\| + a_2\|Sv_n - \mathcal{J}_{rn}p\|)$$
$$+ \min\left\{ \|\mathcal{J}_{rn}v_n - Tv_n\|, \frac{\|Sv_n - \mathcal{J}_{rn}v_n\| + \|\mathcal{J}_{rn}p - Sp\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\}$$
$$\leq (1- c_n)\|Sv_n - p\| + c_n\mathfrak{b}\psi(a_1\|Sv_n - p\| + a_2\|Sv_n - p\|)$$
$$\leq (1- c_n)\|Sv_n - p\| + c_n\mathfrak{b}\psi(\|Sv_n - p\|)$$
$$\leq (1- c_n)\|Sv_n - p\| + c_n\mathfrak{b}\|Sv_n - p\|$$
$$= ((1- c_n) + c_n\mathfrak{b})\|Sv_n - p\| \qquad (2.9)$$

$$\|Sv_n - p\| = \|(1- b_n)Su_n + b_n\mathcal{J}_{rn} Tu_n - p\|$$
$$\leq (1- b_n)\|Su_n - p\| + b_n\|Tu_n - p\|$$
$$\leq (1- b_n)\|Su_n - P\| + b_n \|Tu_n - P\|$$
$$\leq (1- b_n)\|Su_n - p\| + b_n\mathfrak{b}\psi(a_1\|Su - Sp\| + a_2\|Su_n - \mathcal{J}_{rn}p\|)$$
$$+ \min\left\{ \|\mathcal{J}_{rn}u_n - Tu_n\| , \frac{\|Su_n - \mathcal{J}_{rn}u_n\| + \|\mathcal{J}_{rn}p - Sp\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\}$$
$$= (1- b_n)\|Su_n - P\| + b_n\mathfrak{b}\psi(a_1\|Su_n - P\| + a_2\|Su_n - P\|)$$
$$\leq (1- b_n)\|Su_n - p\| + b_n\mathfrak{b}\psi(\|Su_n - p\|)$$
$$\leq (1- b_n)\|Su_n - t\| + b_n\mathfrak{b}\|Su_n - p\|$$
$$= ((1- b_n) + b_n\mathfrak{b}) \|Su_n - p\| \qquad (2.10)$$

We substitute (2.10) in (2.9) as follows:

$$\|Sx_n - p\| \leq ((1- c_n) + c_n\mathfrak{b})((1- b_n) + b_n\mathfrak{b})) \|Su_n - p\| \qquad (2.11)$$

We substitute (2.11) in (2.8) as follows:

$$\|Su_{n+1} - p\| \leq \lim_{n\to\infty} sup\,\mathfrak{b}(1- c_n(1-\mathfrak{b})(1- b_n(1-\mathfrak{b})) \|Su_n - p\|$$
$$\leq \lim_{n\to\infty} sup(1- c_n(1-\mathfrak{b}) \|Su_n - p\|.$$

$$\lim_{n\to\infty}\sup\|Se_{n+1} - t\| \leq \lim_{n\to\infty}\sup\|Se_n - t\| = \delta \qquad (2.12)$$

By (2.6), (2.7), (2.12) and use Lemma (1.3) we get

$\lim_{n\to\infty}\|Te_n - \mathcal{J}_{rn}Se_{n+1}\| = 0$

Now, to prove that $f(T) \neq \emptyset$

Let p $\in$ $A(\mathcal{D}, \langle \mathcal{J}_{rn}Su_{n+1}\rangle) \Rightarrow r(\mathcal{D}, \langle Su_n\rangle) = r(\mathcal{T}p, \langle \mathcal{J}_{rn}Su_{n+1}\rangle)$

Now, $r(T\rho, \langle \mathcal{J}_{rn}Su_{n+1}\rangle) = \lim_{n\to\infty} sup \left\| \mathcal{J}_{rn}Su_{n+1} - T\rho \right\| \leq \lim_{n\to\infty} sup[\left\|\mathcal{J}_{rn}Su_{n+1} - T\mathcal{O}_n\right\| + \left\|Tu_n - T\rho\right\|]$

$$\leq \lim_{n\to\infty} sup[\, b\psi(a_1\|Su_n - \rho\| + a_2\|Su_n - \rho\|)$$
$$+ \quad \min\left\{\|\mathcal{J}_{rn}u_n - Tu_n\|, \frac{\|Su_n - \mathcal{J}_{rn}\mathcal{Y}_n\| + \|\mathcal{J}_{rn}\rho - Sp\|}{2}, \|T\mathcal{P} - \mathcal{J}_{rn}\rho\|\right\}$$
$$= \lim_{n\to\infty}\sup b\psi(a_1\|Su_n - \rho\| + a_2\|S\mathcal{Y}_n - \rho\|)]$$
$$\leq \lim_{n\to\infty} sup\|Su_n - \rho\| = r(\rho, \langle Su_n\rangle) = r(\mathcal{D}, \langle Su_n\rangle)$$
$$r(\mathcal{D}, \langle Su_n\rangle) = r(T\rho, \langle \mathcal{J}_{rn}Su_{n+1}\rangle)$$

$Tp \in A(\mathcal{D}, \langle Su_n\rangle)$ $\mathcal{D}$ a uniformly convex $\Rightarrow A(\mathcal{D}, \langle Su_n\rangle)$ is singleton

$\Rightarrow p = Tp \Rightarrow \mathcal{P} \in f(T) \Rightarrow f(T) \neq \emptyset$

In the same way, we get, $\rho \in f(T)$ & $\rho \in A^{-1}(0)$

$\Longrightarrow f(T) \cap f(S) \cap A^{-1}(0) \neq \emptyset$

Lemma 2.7: Let$(\boldsymbol{N}, \|.\|)$ be a Banach-space, and T be a resolvent $\mathcal{ZA}$-jungck self-mapping on $\mathcal{D}$ with $\mathfrak{b} \in [0,1)$. Suppose that $\langle \mathcal{S}e_n\rangle$ be a resolvent Jungck $\mathcal{CR}$ – iteration scheme in $\mathcal{D}$. There exists $p \in F(\mathcal{S}) \cap F(\mathcal{G}) \cap A^{-1}(0) \neq \emptyset$, then $\lim_{n\to\infty}\|\mathcal{S}e_n - t\|$ exists, for all $n \in N$.

Proof: As the same proof of lemma 2.1

Theorem 2.8: Let$(\boldsymbol{N}, \|.\|)$ be a Banch-space, and T be a resolvent $\mathcal{ZA}$-jungck self-mapping on $\mathcal{D}$ .If $\langle \mathcal{S}e_n\rangle$ is resolvent Jungck $\mathcal{CR}$ – iteration scheme . Then the resolvent Jungck $\mathcal{CR}$ – iteration scheme converges-strongly to a unique common fixed− point of $\mathcal{S}, T\ and\ A^{-1}$

Proof: As the same proof of Theorem 2.2

Theorem 2.9: Let $\boldsymbol{N}$ be a uniformly convex Banach-space, $\mathcal{D}$ be a nonempty convex-closed subset of $\boldsymbol{N}$ and T be a resolvent $\mathcal{ZA}$-jungck self-maping on $\mathcal{D}$. If $\langle \mathcal{S}e_n\rangle$ the resolvent Jungck $\mathcal{CR}$ – Iteration scheme in $\mathcal{D}$ then $F(\mathcal{S}) \cap F(T) \cap A^{-1}(0) \neq \phi$ if and only if $\lim_{n\to\infty}\|\mathcal{G}e_n - \mathcal{J}_{rn}\mathcal{S}e_{n+1}\| = 0$

Proof: the same proof Theorem 2.3

Theorem 2.10: Let$(\boldsymbol{N}, \|.\|)$ be a normed−space, and T be a resolvent $\mathcal{ZA}$-jungck self-mapping on $\mathcal{D}$ if $P \in f(s) \cap f(T) \cap A^{-1}(0) \neq \phi$ where $0 < \mathfrak{r} \leq a_n, b_n, c_n < 1$ then the resolvent Jungck $\mathcal{CR}$ – iteration scheme is faster the resolvent Jungck $\mathcal{SP}$ – iteration scheme

Proof: Let $p \in f(\mathcal{S}) \cap f(T) \cap A^{-1}(0)$

The resolvent Jungck $\mathcal{CR}$ – iteration scheme, we have

$$\begin{aligned}
\|se_{n+1} - p\| &= \left\|\mathcal{J}_{rn}\big((1-a_n)T\mathcal{S}w_n + a_n\mathcal{J}_{rn}TTw_n\big) - p\right\| \\
&\leq \|(1-a_n)T\mathcal{S}w_n + a_n TTw_n - p\| \\
&\leq (1-a_n)\|T\mathcal{S}w_n - p\| + a_n\|TTw_n - p\| \\
&\leq (1-a_n)\mathfrak{b}\psi(a_1\|\mathcal{S}\mathcal{S}w_n - \mathcal{S}p\| + a_2\|\mathcal{S}\mathcal{S}w_n - \mathcal{J}_{rn}p\|) \\
&+ \min\left\{\|\mathcal{J}_{rn}\mathcal{S}w_n - T\mathcal{S}w_n\|, \frac{\|\mathcal{S}\mathcal{S}w_n - \mathcal{J}_{rn}\mathcal{S}w_n\| + \|\mathcal{J}_{rn}p - \mathcal{S}p\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\} \\
&+ a_n\mathfrak{b}\psi(a_1\|\mathcal{S}Tw_n - \mathcal{S}p\| + a_2\|\mathcal{S}Tw_n - \mathcal{J}_{rn}p\|) + \\
&\min\left\{\|\mathcal{J}_{rn}Tw_n - TTw_n\|, \frac{\|\mathcal{S}w_n - \mathcal{J}_{rn}m_n\| + \|\mathcal{J}_{rn}p - \mathcal{S}p\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\} \\
&\leq (1-a_n)\mathfrak{b}\psi\|\mathcal{S}w_n - p\| + a_n\mathfrak{b}\|T\mathcal{S}w_n - p\| \\
&\leq (1-a_n)\mathfrak{b}\|\mathcal{S}w_n - p\| + a_n\mathfrak{b}\mathfrak{b}\psi(a_1\|\mathcal{S}\mathcal{S}w_n - \mathcal{S}p\| + a_2\|\mathcal{S}\mathcal{S}w_n - \mathcal{J}_{rn}p\|) \\
&+ \min\left\{\|\mathcal{J}_{rn}\mathcal{S}w_n - \mathcal{G}\mathcal{S}w_n\|, \tfrac{\|\mathcal{S}\mathcal{S}w_n - \mathcal{J}_{rn}\mathcal{S}w_n\| + \|\mathcal{J}_{rn}p - \mathcal{S}p\|}{2}, \|\mathcal{G}p - \mathcal{J}_{rn}p\|\right\} \\
&\leq (1-a_n)\mathfrak{b}\|\mathcal{S}w_n - p\| + a_n\mathfrak{b}^2\|\mathcal{S}w_n - p\| \qquad (2.13) \\
&\leq \mathfrak{b}(1 - a_n(1-\mathfrak{b}))\|\mathcal{S}w_n - p\|
\end{aligned}$$

Now, to find $\|\mathcal{S}w_n - p\|$.

$$\begin{aligned}
\|\mathcal{S}w_n - P\| &= \|(1-c_n)\mathcal{J}_{rn}TSz_n + c_n TTz_n - p\| \\
&\leq (1-c_n)\|TSz_n - p\| + c_n\|TTz_n - p\| \\
&\leq (1-c_n)\|TSz_n - p\| + c_n\|T\,Tz_n - p\| \\
&\leq (1-c_n)\mathfrak{b}\psi(a_1\|SSz_n - \mathcal{S}p\| + a_2\|SSz_n - \mathcal{J}_{rn}p\|) + \\
&\min\left\{\|\mathcal{J}_{rn}Sz_n - TSz_n\|, \frac{\|SSz_n - \mathcal{J}_{rn}Sz_n\| + \|\mathcal{J}_{rn}p - Sp\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\} \\
&+ c_n\mathfrak{b}\psi(a_1\|STz_n - Sp\| + a_2\|STz_n - \mathcal{J}_{rn}p\|) + \\
&\min\left\{\|\mathcal{J}_{rn}Tz_n - TTz_n\|, \frac{\|STz_n - \mathcal{J}_{rn}Tz_n\| + \|\mathcal{J}_{rn}p - Sp\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\} \\
&\leq (1-c_n)\mathfrak{b}\|Sz_n - p\| + c_n\mathfrak{b}(\|TSz_n - p\|) \\
&\leq (1-c_n)\mathfrak{b}\|Sz_n - p\| + c_n\mathfrak{b}\mathfrak{b}\psi(a_1\|Sz_n - \mathcal{S}p\| + a_2\|Sz_n - \mathcal{J}_{rn}p\|) \\
&+ \min\left\{\|\mathcal{J}_{rn}z_n - \mathcal{G}z_n\|, \frac{\|Sz_n - \mathcal{J}_{rn}z_n\| + \|\mathcal{J}_{rn}p - Sp\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\} \\
&\leq (1-c_n)\mathfrak{b}\|Sz_n - p\| + c_n\mathfrak{b}^2\|Sz_n - p\| \\
&= \mathfrak{b}\big(1 - c_n(1-\mathfrak{b})\big)\|Sz_n - p\| \ . \qquad (2.14)
\end{aligned}$$

$$\|Sz_n - p\| = \|(1- b_n)\mathcal{J}_{rn}TSe_n + b_n\mathcal{J}_{rn}Te_np\|$$
$$\leq (1- b_n)\|\mathcal{J}_{rn}TSe_n - p\| + b_n\|\mathcal{J}_{rn}Te_n - p\|$$
$$\leq (1- b_n)\|TSe_n - p\| + b_n\|Te_n - p\|$$
$$\leq (1- b)\mathfrak{b}\psi(a_1\|SSe_n - Sp\| + a_2\|SSe_n - \mathcal{J}_{rn}p\|) + \min\left\{\|\mathcal{J}_{rn}Se_n - TSe_n\|, \frac{\|SSe_n - \mathcal{J}_{rn}Se_n\| + \|\mathcal{J}_{rn}p - Sp\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\}$$
$$+ b_n\mathfrak{b}\psi(a_1\|Se_n - Sp\| + a_2\|Se_n - \mathcal{J}_{rn}p\|)$$
$$+ \min\left\{\|\mathcal{J}_{rn}e_n - Te_n\|, \frac{\|Se_n - \mathcal{J}_{rn}e_n\| + \|\mathcal{J}_{rn}p - Sp\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\}$$
$$\leq (1- b_n)\mathfrak{b}\|Se_n - p\| + b_n\mathfrak{b}(\|Se_n - p\|)$$
$$\leq \mathfrak{b}\|Se_n - p\|$$
$$\leq 1\|Se_n - p\| \tag{2.15}$$

We substitute (2.15) in (2.14) as follows

$$\|Sw_n - p\| \leq \mathfrak{b}^2\big(1- c_n(1-\mathfrak{b})\big)\|Se_n - \mathcal{t}\| \tag{2.16}$$

We substitute (2.16) in (2.13) as follows

$$\|Se_{n+1} - p\| \leq \mathfrak{b}^2(1- a_n(1- \mathfrak{b})\big(1- c_n(1-\mathfrak{b})\big)\|Se_n - \mathcal{t}\|$$
$$\leq \mathfrak{b}^2\big(1- \mathfrak{r}(1-\mathfrak{b})\big)^2\|Se_n - \mathcal{t}\|$$

.

$$\|Se_{n+1} - \mathcal{t}\| \leq \mathfrak{b}^{2n}\big(1- \mathfrak{r}(1-\mathfrak{b})\big)^{2n}\|Se_0 - \mathcal{t}\|$$

The resolvent Jungck $\mathcal{SP}$ – Iteration scheme, we have

$$\|Su_{n+1} - p\| = \|(1- a_n)\mathcal{J}_{rn}TSx_n + a_nTx_n - p\|$$
$$\leq (1- a_n)\|\mathcal{J}_{rn}TSx_n - p\| + a_n\|Tx_n - p\|$$
$$\leq (1- a_n)\|TSx_n - p\| + a_n\|Tx_n - p\|$$
$$\leq (1- a_n)\mathfrak{b}\psi(a_1\|SSx_n - SP\| + a_2\|SSx_n - \mathcal{J}_{rn}p\|) + \min\left\{\|\mathcal{J}_{rn}Sx_n - TSx_n\|, \frac{\|SSx_n - \mathcal{J}_{rn}Sx_n\| + \|\mathcal{J}_{rn}p - Sp\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\}$$
$$+ a_n\mathfrak{b}\psi(a_1\|Sx_n - \mathcal{S}p\| + a_2\|Sx_n - \mathcal{J}_{rn}p\|)$$
$$+ \min\left\{\|\mathcal{J}_{rn}x_n - Tx_n\|, \frac{\|Sx_n - \mathcal{J}_{rn}x_n\| + \|\mathcal{J}_{rn}p - Sp\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\}$$
$$= (1- a_n)\mathfrak{b}\psi\|Sx_n - p\| + a_n\mathfrak{b}\psi(\|Sx_n - p\|)$$
$$\leq (1- a_n)\mathfrak{b}\|Sx_n - p\| + a_n\mathfrak{b}\|Sx_n - P\|$$
$$\leq \big((1- a_n) + a_n\mathfrak{b}\big)\|Sx_n - P\|$$
$$\leq \mathfrak{b}\|Sx_n - p\| \tag{2.17}$$

$$\|Sx_n - P\| = \|\mathcal{J}_{rn}[(1- c_n)Sv_n + c_nTv_n] - p\|$$
$$\leq \|(1- c_n)Sv_n + c_nTv_n - p\|$$
$$\leq (1- c_n)\|Sv_n - \mathcal{t}\| + c_n\|Tv_n - p\|$$
$$\leq (1- c_n)\|Sv_n - p\| + c_n\mathfrak{b}\psi(a_1\|Sv_n - Sp\| + a_2\|Sv_n - \mathcal{J}_{rn}p\|)$$
$$+ \min\left\{\|\mathcal{J}_{rn}v_n - Tv_n\|, \frac{\|Sv_n - \mathcal{J}_{rn}v_n\| + \|\mathcal{J}_{rn}p - Sp\|}{2}, \|Tp - \mathcal{J}_{rn}p\|\right\}$$
$$\leq (1- c_n)\|Sv_n - p\| + c_n\mathfrak{b}\psi(a_1\|Sv_n - p\| + a_2\|Sv_n - p\|)$$
$$\leq (1- c_n)\|Sv_n - p\| + c_n\mathfrak{b}\psi(\|Sv_n - p\|)$$
$$\leq (1- c_n)\|Sv_n - p\| + c_n\mathfrak{b}\|Sv_n - p\|$$
$$= ((1- c_n) + c_n\mathfrak{b})\|Sv_n - p\| \tag{2.18}$$

$$\|Sv_n - p\| = \|(1- b_n)Su_n + b_n\mathcal{J}_{rn}Tu_n - p\|$$
$$\leq (1- b_n)\|Su_n - p\| + b_n\|Tu_n - p\|$$
$$\leq (1- b_n)\|Su_n - P\| + b_n\|Tu_n - P\|$$

$$\leq (1- b_n)\|Su_n - p\| + b_n \mathfrak{b}\psi(a_1\|Su - Sp\| + a_2\|Su_n - \mathcal{J}_{rn}p\|)$$
$$+ \min\left\{ \|\mathcal{J}_{rn}u_n - Tu_n\| \ , \frac{\|Su_n - \mathcal{J}_{rn}u_n\| + \|\mathcal{J}_{rn}p - Sp\|}{2}, \|Tp - \mathcal{J}_{rn}p\| \right\}$$
$$= (1- b_n)\|Su_n - P\| + b_n \mathfrak{b}\psi(a_1\|Su_n - P\| + a_2\|Su_n - P\|)$$
$$\leq (1- b_n)\|Su_n - p\| + b_n \mathfrak{b}\psi(\|Su_n - p\|)$$
$$\leq (1- b_n)\|Su_n - t\| + b_n \mathfrak{b}\|Su_n - p\|$$
$$= ((1- b_n) + b_n \mathfrak{b})\ \|Su_n - p\| \tag{2.19}$$

We substitute (2.19) in (2.18) as follows

$$\|Sx_n - p\| \leq ((1- c_n) + c_n \mathfrak{b})((1- b_n) + b_n \mathfrak{b}))\ \|Su_n - p\| \tag{2.20}$$

We substitute (2.20) in (2.17) as follows

$$\|Su_{n+1} - p\| \leq \mathfrak{b}(1- c_n(1-\mathfrak{b})(1- b_n(1-\mathfrak{b}))\ \|Su_n - p\|$$
$$\leq (1- c_n(1-\mathfrak{b})\ \|Su_n - p\|$$
$$\|Su_{n+1} - t\| \ \leq \mathfrak{b}^n(1-\mathfrak{r}(1-\mathfrak{b}))^{2n}\ \|Se_0 - t\|$$

$$\frac{\|Se_{n+1} \ - t\|}{\|Su_{n+1} - t\|} \leq \frac{\mathfrak{b}^{2n}\left(1- \mathfrak{r}\,(1-\mathfrak{b})\right)^{2n}\|Se_0 - t\|}{\mathfrak{b}^n\left(1-\mathfrak{r}(1-\mathfrak{b})\right)^{2n}\ \|Su_0 - t\|} \to 0 \quad n \to \infty$$

3. Conclusion

In this section, we will provide some conclusions. New iterations were performed and the convergence and acceleration of these iterations to the common fixed point were demonstrated. We also install the resolvent Jungck CR – iteration is faster than resolvent Jungck $\mathcal{S}$P – iteration.

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