IHJPAS. 36(1)2023

Ibn Al-Haitham Journal for Pure and Applied Sciences

Journal homepage: jih.uobaghdad.edu.iq

Cubic Ideals of TM-algebras

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Article history: Received 26 June 2022, Accepted 23 Augest 2022, Published in January 2023.

doi.org/10.30526/36.1.2920

Abstract

For the generality of fuzzy ideals in TM-algebra, a cubic ideal in this algebra has been studied, such as cubic ideals and cubic T-ideals. Some properties of these ideals are investigated. Also, we show that the cubic T-ideal is a cubic ideal, but the converse is not generally valid. In addition, a cubic sub-algebra is defined, and new relations between the level subset and a cubic sub-algebra are discussed. After that, cubic ideals and cubic T-ideals under homomorphism are studied, and the image (pre-image) of cubic T-ideals is discussed. Finally, the Cartesian product of cubic ideals in Cartesian product TM-algebras is given. We proved that the product of two cubic ideals of the Cartesian product of two TM-algebras is also a cubic ideal.

Key words: TM-algebra, cubic T-ideal, cubic ideal, fuzzy ideals.

1.Introduction

In 2010 the notion of TM-algebras was introduced by [1] as a generalization of BCK and BCI algebras. After that, many authors studied this structure differently; see [2-6].

The cubic set is an essential concept for generalizing the fuzzy set. So, Jun et al. [7- 8] introduced subalgebras and ideals in BCK/BCI-algebras and discussed the relationship between a cubic subalgebra and a cubic ideal. In [9], Yaqoob et al. introduced the cubic KU-algebra, a generalization of fuzzy KU-ideals of KU-algebras. After that, some authors introduced a cubic set of different structures. See [10-13].

This paper introduces the concept of cubic T-ideals in TM-algebra, and investigate some properties of these ideals. Also, a few relations between a cubic ideal and a cubic T-ideal are discussed. The Cartesian product of cubic T-ideals in Cartesian product TM-algebras is given.

2. Basic concepts

We will recall some concepts related to TM algebra and cubic sets.

Definition (1)[1]. A TM-algebra is a nonempty subset with a constant "0" and a binary operation "*" satisfying the following:

$(\text{tm}_1)\ \rho * 0 = \rho,$

$(\text{tm}_2)(\rho * \tau) * (\rho * \varepsilon) = \varepsilon * \tau, \forall\ \rho, \tau, \varepsilon \in \aleph$.

For $\aleph$ we can define a binary operation $\leq$ by $\rho \leq \tau$ if and only if $\rho * \tau = 0$.

For any TM-algebra($\aleph$,$*$,0),the following axioms hold.$\forall\ \rho, \tau , \varepsilon \in \aleph$

a) $\rho * \rho = 0,$
b) $(\rho * \tau) * \rho = 0 * \tau,$
c) $\rho * (\rho * \tau) = \tau,$
d) $(\rho * \varepsilon) * (\tau * \varepsilon) \leq \rho * \tau,$
e) $(\rho * \tau) * \varepsilon = (\rho * \varepsilon) * \tau,$
f) $\rho * 0 = 0 \Rightarrow \rho = 0,$
g) $\rho \leq \tau \Rightarrow \rho * \varepsilon \leq \tau * \varepsilon\ and\ \varepsilon * \tau \leq \varepsilon * \rho,$
h) $\rho * \big(\rho * (\rho * \tau)\big) = \rho * \tau,$
i) $0 * (\rho * \tau) = \tau * \rho = (0 * \rho) * (0 * \tau),$
j) $\big(\rho * (\rho * \tau)\big) * \tau = 0,$
k) If $\rho * \tau = 0$ and $\tau * \rho = 0$ imply $\rho = \tau$.

Example (2) [1]. Let$\aleph$= {0,1,2,3}be a set with the following table.

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then,($\aleph$,$*$,0) is a TM-algebra.

Definition (3) [2]. A non-empty subset S of a TM-algebra ($\aleph$,$*$,0) is called a TM-subalgebra of$\aleph$if$\rho * \tau \in S$whenever $\rho, \tau \in S$.

Definition (4) [2]. A non-empty subset ψ ofanTM-algebra ($\aleph$,$*$,0) is said to be an ideal of $\aleph$if it satisfies, for any $\rho, \tau \in \psi$

i) $0 \in \psi,$

ii) $\rho * \tau \in \psi$ and $\tau \in \psi$ implies that $\rho \in \psi$.

Example (5) [2]. Let$\aleph = \{0, 1, 2, 3\}$ be a set with a binary operation$*$defined inthe following **Table**:

*	0	1	2	3
0	0	0	3	2
1	1	0	3	2
2	2	2	0	3
3	3	3	b	0

Then $(\aleph, *, 0)$is a TM-algebra and $\psi = \{0,1\}$ is an ideal of $\aleph$.

Definition (6) [1]. A non-empty subset E of a TM-algebra $\aleph$ is a T-ideal, if
i) $0 \in E$
ii) $\forall\, \rho, \tau, \varepsilon \in \aleph, (\rho * \tau) * \varepsilon \in E$ and $\tau \in E$ imply $(\rho * \varepsilon) \in E$.

Definition (7) [5]. Let $(\aleph, *, 0)$ and $(\aleph', *', 0')$be a TM-algebras. A homomorphism is a map $f: \aleph \to \aleph'$ satisfying $f(\rho * \tau) = f(\rho) *' f(\tau)$, for all $\rho, \tau \in \aleph$.

Now, we review an interval-valued fuzzy set concepts.

Definition (8) [5]. Let $\tilde{a} = [a^L, a^U]$ be an interval number, where $0 \le a^L \le a^U \le 1$ and let $D[0,1]$be denoted the family of all closed subinterval of [0,1], that is ,
$D[0,1] = \{\tilde{a} = [a^L, a^U] : a^L \le a^U, for\ a^L \le a^U \in [0,1]\}$.
The operations$\ge, \le, =, rmin, and\ rmax$ of two elements in $D[0,1]$ is defined as follows: let $\tilde{a} = [a^L, a^U]$, $\tilde{b} = [b^L, b^U]$ in $D[0,1]$, then

(1) $\tilde{a} \ge \tilde{b}$ $if\ and\ only\ if\ a^L \ge b^L\ and\ a^U \ge b^U$,
(2) $\tilde{a} \le \tilde{b}$ $if\ and\ only\ if\ a^L \le b^L\ and\ a^U \le b^U$,
(3) $\tilde{a} = \tilde{b}$ $if\ and\ only\ if\ a^L = b^L\ and\ a^U = a^U$,
(4) $rmin\{\tilde{a}, \tilde{b}\} = [min\{a^L, b^L\}, min\{a^U, b^U\}]$,
(5) $rmax\{\tilde{a}, \tilde{b}\} = [max\{a^L, b^L\}, max\{a^U, b^U\}]$,
And if $\tilde{a}_i \in D[0,1]$ where $i \in \Lambda$. We define

$$r\inf_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} {a_i}^L, \inf_{i\in\Lambda} a_i^U\right], \qquad r\sup_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^L, \sup_{i\in\Lambda} a_i^U\right].$$

An interval-valued fuzzy set $V = <\rho, \tilde{\vartheta}(\rho)>$ on $\aleph$ is defined as $\tilde{\vartheta}(\rho) = \{\langle \rho, [\vartheta^L(\rho), \vartheta^L(\rho)] \rangle : \rho \in \aleph\}$, where $\vartheta^L(\rho) \le \vartheta^U(\rho)$, for all $\rho \in \aleph$. Then,$\vartheta^L(\rho) : \aleph \to [0,1]$ and $\vartheta^U : \aleph \to [0,1]$ are called a lower fuzzy set and an upper fuzzy set of $\widetilde{\vartheta}$, respectively.

Definition (9) [5]. Let $(\aleph, *, 0)$ be a TM-algebra and $\tilde{\vartheta}: \aleph \to D[0,1]$. Then,
$V = <\rho, \tilde{\vartheta}(\rho)>$ is called an interval valued fuzzy sub TM-algebra $\aleph$, if

$$\tilde{\vartheta}(\rho * \gamma) \geq rmin\{\tilde{\vartheta}(\rho), \tilde{\vartheta}(\gamma)\}, \forall \rho, \gamma \in \aleph.$$

Definition (10) [5]. Let$(\aleph, *, 0)$ be a TM-algebra and $\tilde{\vartheta}: \aleph \to D[0,1]$. Then $V =< \rho, \tilde{\vartheta}(\rho) >$ is said to be an interval valued fuzzy ideal if
(i_1) $\tilde{\vartheta}(0) \geq \tilde{\vartheta}(\rho), \forall \rho \in \aleph$,

(i_2) For all $\rho, \gamma \in \aleph, \tilde{\vartheta}(\rho) \geq rmin\{\tilde{\vartheta}(\rho * \gamma), \tilde{\vartheta}(\gamma)\}$.

Definition (11) [5]. Let$(\aleph, *, 0)$ be a TM-algebra and $\tilde{\vartheta}: \aleph \to D[0,1]$. Then $V =< \rho, \tilde{\vartheta}(\rho) >$is said to bean interval valued fuzzy T-ideal if
(i_1) $\tilde{\vartheta}(0) \geq \tilde{\vartheta}(\rho), \forall \rho \in \aleph$,

(i_2) For all $\rho, \gamma, \varepsilon \in \aleph, \tilde{\vartheta}(\rho * \varepsilon) \geq rmin\{\tilde{\vartheta}((\rho * \gamma) * \varepsilon), \tilde{\vartheta}(\gamma)\}$.

3. Cubic T-ideals of TM-Algebra

We recall that a cubic set δ in a set $\aleph$ is the structure $\delta = \{\langle \rho, \tilde{\vartheta}_\delta(\rho), \alpha_\delta(\rho) \rangle : \rho \in \aleph\}$, where $\tilde{\vartheta}_\delta : \aleph \to D[0,1]$ such that $\tilde{\vartheta}_\delta(\rho) = [\vartheta_\delta^L(\rho), \vartheta_\delta^U(\rho)]$ is an interval valued fuzzy set in $\aleph$ and α_δ is a fuzzy set in $\aleph$.We write a cubic set by as follows.

$\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$ and we can define the level subset of $\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$ which is denoted by $U(\delta, \tilde{t}, s)$ as follows$U(\delta, \tilde{t}, s) = \{\rho \in \aleph: \tilde{\vartheta}_\delta(\rho) \geq \tilde{t}, \alpha_\delta \leq s\}$, for every $[0,0] \leq \tilde{t} \leq [1,1]$ and $s \in [0,1]$.

Definition (12). Let $\aleph$ be a TM-algebra. A cubic set $\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$ in $\aleph$ is called a cubic sub-algebra if

(1) $\tilde{\vartheta}_\delta(\rho * \tau) \geq rmin\,\{\tilde{\vartheta}(\rho), \tilde{\vartheta}(\tau)\}$.
(2) $\alpha_\delta(\rho * \tau) \leq max\{\alpha_\delta(\rho), \alpha_\delta(\tau)\}$,$\forall\, \rho, \tau \in \aleph$.

Example (13). Let $\aleph$ = {0, a, b, c} be a set with the following **Table**:

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then $(\aleph, *, 0)$ is a TM-algebra. Define$\tilde{\vartheta}_\delta(\rho)$ and $\alpha_\delta(\rho)$ by

$$\tilde{\vartheta}_\delta(\rho) = \begin{cases} [0.2,0.9] & if \quad \rho = \{0, a, b\} \\ [0.1,0.3] & if \quad \rho = c \end{cases}, \quad \alpha_\delta(\rho) = \begin{cases} 0.2 & if \quad \rho = \{0, a, b\} \\ 0.4 & if \quad \rho = c \end{cases},$$

By apply definition(12), we can prove that$\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$is a cubic sub-algebra of$\aleph$.

Proposition (14). If $\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$is a cubic sub-algebra of$\aleph$, then $\tilde{\vartheta}_\delta(0) \geq \tilde{\vartheta}_\delta(\rho)$ and $\alpha_\delta(0) \leq \alpha_\delta(\rho), \forall \rho \in \aleph$

Proof. Since $\rho * \rho = 0$, then $\tilde{\vartheta}_\delta(0) = \tilde{\vartheta}_\delta(\rho * \rho) \geq rmin\{\tilde{\vartheta}_\delta(\rho), \tilde{\vartheta}_\delta(\rho)\} = \tilde{\vartheta}_\delta(\rho)$and $\alpha_\delta(0) = \alpha_\delta(\rho * \rho) \leq max\{\alpha_\delta(\rho), \alpha_\delta(\rho)\} = \alpha_\delta(\rho)$.

Theorem (15). Let $\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$ be a cubic set in $\aleph$, then $\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$ is a cubic sub-algebra of $\aleph$ if and only if for all $\tilde{t} \in D[0,1]$ and $s \in [0,1]$, the set $U(\delta; \tilde{t}, s)$ is either empty or a sub-algebra of $\aleph$.

Proof. Assumethat $\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$ is a cubic sub-algebra of $\aleph$, let $\tilde{t} \in D[0,1]$ and $s \in [0,1]$, be such that $U(\delta; \tilde{t}, s) \neq \emptyset$. Then, for any $\rho, \tau \in U(\delta, \tilde{t}, s)$ we have $\tilde{\vartheta}_\delta(\rho) \geq \tilde{t}$, $\tilde{\vartheta}_\delta(\tau) \geq \tilde{t}$ and $\alpha_\delta(\rho) \leq s$,$\alpha_\delta(\tau) \leq s$ and since $\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$ is a cubic sub-algebra, we have

$\tilde{\vartheta}_\delta(\rho * \tau) \geq rmin\{\tilde{\vartheta}(\rho), \tilde{\vartheta}(\tau)\} = \tilde{t}.$

$\alpha_\delta(\rho * \tau) \leq max\{\alpha_\delta(\rho), \alpha_\delta(\tau)\} = s,$

So that $\rho * \tau \in U(\delta; \tilde{t}, s)$. Hence, $U(\delta; \tilde{t}, s)$ is a sub-algebraof $\aleph$.

Conversely, suppose that $U(\delta; \tilde{t}, s)$ is a sub-algebraof $\aleph$and let $\rho, \tau \in \aleph$.

Take $\tilde{t} = rmin\{\tilde{\vartheta}(\rho), \tilde{\vartheta}(\tau)\}$ and $s = max\{\alpha_\delta(\rho), \alpha_\delta(\tau)\}$

By assumption $U(\delta; \tilde{t}, s)$ is sub algebra of $\aleph$ implies:

$\rho * \tau \in U(\delta; \tilde{t}, s)$, therefore $\tilde{\vartheta}_\delta(\rho * \tau) \geq \tilde{t} = rmin\{\tilde{\vartheta}(\rho), \tilde{\vartheta}(\tau)\}$ and

$\alpha_\delta(\rho * \tau) \leq s = max\{\alpha_\delta(\rho), \alpha_\delta(\tau)\}$. Hence $\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$ is a cubic sub-algebra of $\aleph$.

Definition(16). Let $\aleph$ be a TM-algebra. A cubic set$\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$ in $\aleph$ is said to be a cubic ideal if:

$(H_1)\tilde{\vartheta}_\delta(0) \geq \tilde{\vartheta}_\delta(\rho)$ and $\alpha_\delta(0) \leq \alpha_\delta(\rho)$.

$(H_2)\tilde{\vartheta}_\delta(\rho) \geq rmin\{\tilde{\vartheta}_\delta(\rho * \tau), \tilde{\vartheta}_\delta(\tau)\}$ and $\alpha_\delta(\rho) \leq max\{\alpha_\delta(\rho * \tau), \alpha_\delta(\tau)\}$, for all $\rho, \tau \in \aleph$.

Definition (17). Let $\aleph$ be a TM-algebra. A cubic set $\delta = \langle \check{\vartheta}_\delta, \alpha_\delta \rangle$ in $\aleph$ is said to be a cubic T-ideal if:

$(B_1)\tilde{\vartheta}_\delta(0) \geq \tilde{\vartheta}_\delta(\rho)\ and\ \alpha_\delta(0) \leq \alpha_\delta(\rho),$

$(B_2)\tilde{\vartheta}_\delta(\rho * \varepsilon) \geq rmin\{\tilde{\vartheta}_\delta((\rho * \tau) * \varepsilon), \tilde{\vartheta}_\delta(\tau)\}\ and$

$\alpha_\delta(\rho * \varepsilon) \leq max\{\alpha_\delta((\rho * \tau) * \varepsilon), \alpha_\delta(\tau)\}.$

Example (18). Let $\aleph = \{0, a, b, c\}$ in example(13). Define a cubic set $\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$ in $\aleph$ as follows:

$$\tilde{\vartheta}_\delta(\rho)=\begin{cases}[0.1,0.7], & \text{if } \rho=0,\\ [0.4,0.5], & \text{if } \rho\in\{a,b\},\\ [0.1,0.3], & \text{if } \rho\in c\end{cases}\quad \alpha_\delta(\rho)=\begin{cases}0.1, & \text{if } \rho=0,\\ 0.3, & \text{if } \rho\in\{a,b\},\\ 0.6, & \text{if } \rho\in c\end{cases}$$

Then, we can easy show that a cubic set $\delta=\langle\tilde{\vartheta}_\delta,\alpha_\delta\rangle$ is a cubic T-ideal of $\aleph$.

Proposition (19). If $\delta=\langle\tilde{\vartheta}_\delta,\alpha_\delta\rangle$ is a cubic T-ideal of TM-algebra $\aleph$, then

$\tilde{\vartheta}_\delta(\rho*(\rho*\tau))\geq\tilde{\vartheta}_\delta(\tau),\ \alpha_\delta(\rho*(\rho*\tau))\leq\alpha_\delta(\tau)$.

Proof. Taking $\varepsilon=\rho*\tau$ in Definition 3.6

We get $\tilde{\vartheta}_\delta(\rho*\varepsilon)\geq rmin\{\tilde{\vartheta}_\delta((\rho*\tau)*\varepsilon),\tilde{\vartheta}_\delta(\tau)\}$

$\tilde{\vartheta}_\delta(\rho*(\rho*\tau))\geq rmin\{\tilde{\vartheta}_\delta((\rho*\tau)*(\rho*\tau)),\tilde{\vartheta}_\delta(\tau)\}$

$=rmin\{\tilde{\vartheta}_\delta(0),\tilde{\vartheta}_\delta(\tau)\}=\tilde{\vartheta}_\delta(\tau)$ and

$\alpha_\delta(\rho*\varepsilon)\leq max\{\alpha_\delta((\rho*\tau)*\varepsilon),\alpha_\delta(\tau)\}$

$\alpha_\delta(\rho*(\rho*\tau))\leq max\{\alpha_\delta((\rho*\tau)*(\rho*\tau)),\alpha_\delta(\tau)\}$

$=max\{\alpha_\delta(0),\alpha_\delta(\tau)\}=\alpha_\delta(\tau)$.

Proposition (20). Let $\delta=\langle\tilde{\vartheta}_\delta,\alpha_\delta\rangle$ be a cubic T-ideal of TM-algebra $\aleph$. If the inequality $\rho*\tau\leq\varepsilon$ holds in $\aleph$, then $\tilde{\vartheta}_\delta(\rho)\geq rmin\{\tilde{\vartheta}_\delta(\varepsilon),\tilde{\vartheta}_\delta(\tau)\}$ and $\alpha_\delta(\rho)\leq max\{\alpha_\delta(\varepsilon),\alpha_\delta(\tau)\}$.

Proof. Assume that the inequality $\rho*\tau\leq\varepsilon$ holds in $\aleph$, then $(\rho*\tau)*\varepsilon=0$ and by $\tilde{\vartheta}_\delta(\rho*\varepsilon)\geq rmin\{\tilde{\vartheta}_\delta((\rho*\tau)*\varepsilon),\tilde{\vartheta}_\delta(\tau)\}$, if we put $\varepsilon=0$

Then, $\tilde{\vartheta}_\delta(\rho*0)\geq rmin\{\tilde{\vartheta}_\delta((\rho*\tau)*0),\tilde{\vartheta}_\delta(\tau)\}$

$\tilde{\vartheta}_\delta(\rho)\ \geq rmin\{\tilde{\vartheta}_\delta(\rho*\tau),\tilde{\vartheta}_\delta(\tau)\}\ \ldots\ldots\ldots(i)$

But $\tilde{\vartheta}_\delta(\rho*\tau)\geq rmin\{\tilde{\vartheta}_\delta((\rho*\varepsilon)*\tau),\tilde{\vartheta}_\delta(\varepsilon)\}$

$\tilde{\vartheta}_\delta(\rho*\tau)\geq rmin\{\tilde{\vartheta}_\delta((\rho*\tau)*\varepsilon),\tilde{\vartheta}_\delta(\varepsilon)\}$

$=rmin\{\tilde{\vartheta}_\delta(0),\tilde{\vartheta}_\delta(\varepsilon)\}=\tilde{\vartheta}_\delta(\varepsilon)\ \ldots\ldots(ii)$

From (i) and (ii), we get $\tilde{\vartheta}_\delta(\rho)\geq rmin\{\tilde{\vartheta}_\delta(\varepsilon),\tilde{\vartheta}_\delta(\tau)\}$.

Similarly, we can show that $\alpha_\delta(\rho)\leq max\{\alpha_\delta(\varepsilon),\alpha_\delta(\tau)\}$.

Proposition (21). If $\delta=\langle\tilde{\vartheta}_\delta,\alpha_\delta\rangle$ is a cubic T-ideal of TM-algebra $\aleph$ and $\rho\leq\tau$ then $\tilde{\vartheta}_\delta(\rho)\geq\tilde{\vartheta}_\delta(\tau)\ and\ \alpha_\delta(\rho)\leq\alpha_\delta(\tau)$.

Proof. If $\rho\leq\tau$ then $\rho*\tau=0$. This is together with $\rho*0=\rho$ and $\tilde{\vartheta}_\delta(0)\geq\tilde{\vartheta}_\delta(\tau)$ also $\alpha_\delta(0)\leq\alpha_\delta(\tau)$, we get

$$\tilde{\vartheta}_\delta(\rho * 0) = \tilde{\vartheta}_\delta(\rho) \geq rmin\{((\rho * \tau) * 0), \tilde{\vartheta}_\delta(\tau)\}$$

$= rmin\{\tilde{\vartheta}_\delta(0 * 0), \tilde{\vartheta}_\delta(\tau)\} = rmin\{\tilde{\vartheta}_\delta(0), \tilde{\vartheta}_\delta(\tau)\} = \tilde{\vartheta}_\delta(\tau)$, also

$$\alpha_\delta(\rho * 0) = \alpha_\delta(\rho) \leq max\{\alpha_\delta((\rho * \tau) * 0), \alpha_\delta(\tau)\} = max\{\alpha_\delta(0 * 0), \alpha_\delta(\tau)\}$$

$= max\{\alpha_\delta(0), \alpha_\delta(\tau)\} = \alpha_\delta(\tau)$.

Theorem (22). Let ℵ be a TM-algebra, a cubic set$\delta = \langle\tilde{\vartheta}_\delta, \alpha_\delta\rangle$of ℵ is a cubic T-ideal if δ is a cubic ideal of ℵ.

Proof. If we put $\varepsilon = 0$ in (B_2), then

$\tilde{\vartheta}_\delta(\rho) \geq rmin\{\tilde{\vartheta}_\delta((\rho * \tau)), \tilde{\vartheta}_\delta(\tau)\}\ and \alpha_\delta(\rho) \leq max\{\alpha_\delta((\rho * \tau)), \alpha_\delta(\tau)\}$.

Hence $\delta = \langle\tilde{\vartheta}_\delta, \alpha_\delta\rangle$is a cubic ideal of ℵ.

Remark (23). The converse of Theorem (22) is not true.

The following example shows the reverse direction of Theorem (22).

Example (24). Let ℵ = {0, a, b, c} be a set with the following **Table**:

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then, (ℵ,∗ ,0) is a TM-algebra. Define$\tilde{\vartheta}_\delta(\rho)$ and $\alpha_\delta(\rho)$ by

$$\tilde{\vartheta}_\delta(\rho) = \begin{cases} [0.1, 0.8] & if \quad \rho = \{0, a, b\} \\ [0.1, 0.3] & if \quad \rho = c \end{cases}, \quad \alpha_\delta(\rho) = \begin{cases} 0.1 & if \quad \rho = \{0, a, b\} \\ 0.8 & if \quad \rho = c \end{cases},$$

Then, it is easy to show that $\delta = \langle\tilde{\vartheta}_\delta, \alpha_\delta\rangle$ is a cubic ideal of ℵ. But not a cubic T-ideal since $\tilde{\vartheta}_\delta(a * b) \leq rmin\{\tilde{\vartheta}_\delta((a * c) * b), \tilde{\vartheta}_\delta(c)\}\ and$

$\alpha_\delta(a * b) \geq max\{\alpha_\delta((a * c) * b), \alpha_\delta(c)\}$.

4. Image and Pre-image of cubic T-ideals

Definition (25). Let $f: \aleph \to Y$ be a mapping. If $\delta = \langle\tilde{\vartheta}_\delta, \alpha_\delta\rangle$ is a cubic set of ℵ, then the cubic set $\omega = \langle\tilde{\vartheta}_\omega, \alpha_\omega\rangle$ of Y is define by

$$f(\tilde{\vartheta}_\delta)(\tau) = \tilde{\vartheta}_\omega(\tau) = \begin{cases} \underset{\rho \in f^{-1}(\tau)}{rsup}\ \tilde{\vartheta}_\delta(\rho), & if\ \ f^{-1}(\tau) = \{\rho \in \aleph, f(\rho) = \tau\} \neq \emptyset \\ 0 & otherwise \end{cases}$$

$$f(\alpha_\delta)(\tau) = \alpha_\omega(\tau) = \begin{cases} \inf\limits_{\rho \in f^{-1}(\tau)} \alpha_\delta(\rho) & if\ \ f^{-1}(\tau) = \{\rho \in \aleph, f(\rho) = \tau\} \neq \emptyset \\ 1 & otherwise \end{cases}$$

It is called the image of $\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$ under f. Similarly, if $\omega = \langle \tilde{\vartheta}_\omega, \alpha_\omega \rangle$ is a cubic subset of Y, then the cubic subset defined by $\tilde{\vartheta}_\delta(\rho) = \tilde{\vartheta}_\omega(f(\rho))$ and
$\alpha_\delta(\rho) = \alpha_\omega(f(\rho))$,for any $\rho \in \aleph$is said to be the pre-image of ωunder f.

Theorem (26). An epimorphism pre-image of a cubic T-ideal is also a cubic T-ideal.

Proof. Let $f : \aleph \to \aleph'$ be an epimorphism mapping of TM-algebra, $\omega = \langle \tilde{\vartheta}_\omega, \alpha_\omega \rangle$be a cubic T-ideal of $\aleph'$ and $\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$be the pre-image of ωunder f, then $\tilde{\vartheta}_\delta(\rho) = \tilde{\vartheta}_\omega(f(\rho))$ and $\alpha_\alpha(\rho) = \alpha_\omega(f(\rho))$ for any $\rho \in \aleph$, then

$\tilde{\vartheta}_\delta(0) = \tilde{\vartheta}_\omega(f(0)) \geq \tilde{\vartheta}_\omega(f(\rho)) = \tilde{\vartheta}_\delta(\rho)$,

$\alpha_\delta(0) = \alpha_\omega(f(0)) \leq \alpha_\omega(f(\rho)) = \alpha_\delta(\rho)$.

Now, let $\rho, \tau, \varepsilon \in \aleph$, then

$$\begin{aligned} \tilde{\vartheta}_\delta(\rho * \varepsilon) &= \tilde{\vartheta}_\omega(f(\rho * \varepsilon)) = \tilde{\vartheta}_\omega(f(\rho) *' f(\varepsilon)) \\ &\geq rmin\left\{\tilde{\vartheta}_\omega\left(\left(f(\rho) *' f(\tau)\right) *' f(\varepsilon)\right), \tilde{\vartheta}_\omega(f(\tau))\right\} \\ &= rmin\left\{\tilde{\vartheta}_\omega\left(f\left((\rho * \tau) * \varepsilon\right)\right), \tilde{\vartheta}_\omega(f(\tau))\right\} \\ &= rmin\{\tilde{\vartheta}_\delta((\rho * \tau) * \varepsilon), \tilde{\vartheta}_\delta(\tau)\}, \end{aligned}$$

$$\begin{aligned} \alpha_\delta(\rho * \varepsilon) &= \alpha_\omega(f(\rho * \varepsilon)) = \alpha_\omega(f(\rho) *' f(\varepsilon)) \\ &\leq max\left\{\alpha_\omega\left(\left(f(\rho) *' f(\tau)\right) *' f(\varepsilon)\right), \alpha_\omega(f(\tau))\right\} \\ &= max\left\{\alpha_\omega\left(f\left((\rho * \tau) * \varepsilon\right)\right), \alpha_\omega(f(\tau))\right\} \\ &= max\{\alpha_\delta((\rho * \tau) * \varepsilon), \alpha_\delta(\tau)\}. \end{aligned}$$

Definition (27). A cubic subset $\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$ of $\aleph$has sup and inf properties if for any subset $T\ of\ \aleph$, there exist $t, s \in T\ such\ that\ \tilde{\vartheta}_\delta(t) = rsup_{t \in T} \tilde{\vartheta}_\delta(t)\ and\ \alpha_\delta(s) = inf_{t \in T} \alpha_\delta(s)$.

Theorem (28). Let $f: \aleph \to Y$ be anepimorphism between TM-algebra $\aleph$

and Y. For every cubic T-ideal $\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$ $in\ \aleph, then\ f(\delta)$is cubic T-ideal of Y.

Proof. By definition25$\tilde{\vartheta}_\omega(\tau') = f(\tilde{\vartheta}_\delta)(\tau') = rsup_{\rho \in f^{-1}(\tau')} \tilde{\vartheta}_\delta(\rho)$ and

$\alpha_\omega(\tau') = f(\alpha_\delta)(\tau') = inf_{\rho \in f^{-1}(\tau')} \alpha_\delta(\rho)$ for any $\tau' \in Y$ and $rsup\ \emptyset = [0,0] = 0$. We must prove that

$\tilde{\vartheta}_\omega(\rho' * \varepsilon') \geq rmin\{\tilde{\vartheta}_\omega((\rho' * \tau') * \varepsilon'), \tilde{\vartheta}_\omega(\tau')\}$ and

$$\alpha_\omega(\rho' * \varepsilon') \leq max\{\alpha_\omega((\rho' * \tau') * \varepsilon'), \alpha_\omega(\tau')\}\ for\ any\ \rho', \tau', \varepsilon' \in Y.$$

Let $f: \aleph \to Y$ be an epimorphism mappingof $\aleph$, $\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$be a cubic T-ideal of $\aleph$with sup and inf properties and $\omega = \langle \tilde{\vartheta}_\omega, \alpha_\omega \rangle$be the image of $\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$ under f. Since $\delta = \langle \tilde{\vartheta}_\delta, \alpha_\delta \rangle$ is a cubicT-ideal of $\aleph$, we have

$$\tilde{\vartheta}_\delta(0) \geq \tilde{\vartheta}_\delta(\rho), \qquad \alpha_\delta(0) \leq \alpha_\delta(\rho) \qquad \forall \rho \in \aleph\,.$$

Note that $0 \in f^{-1}(0')$ where $0,\ 0'$ are the zero of $\aleph\ and\ Y$, respectively. Thus,

$$\tilde{\vartheta}_\delta(0') = \underset{t \in f^{-1}(0)}{\text{rsup}}\ \tilde{\vartheta}_\delta(t) = \tilde{\vartheta}_\delta(0) \geq \tilde{\vartheta}_\delta(\rho) \qquad \forall \rho \in \aleph\,,$$

$$\alpha_\omega(0') \leq \underset{t \in f^{-1}(0')}{\inf} \alpha_\omega(t) = \alpha_\omega(0) \leq \alpha_\omega(\rho) \qquad \forall \rho \in \aleph\,,$$

Which implies that $\tilde{\vartheta}_\omega(0') \geq \underset{t \in f^{-1}(\rho')}{\text{rsup}}\ \tilde{\vartheta}_\omega(\rho')$ and

$$\alpha_\omega(0') \leq \underset{t \in f^{-1}(\rho')}{\inf} \alpha_\delta(t) = \alpha_\omega(\rho') \text{for any} \rho' \in Y.$$

For any $\rho', \tau', \varepsilon' \in Y$, let $\rho_0 \in f^{-1}(\rho'), \tau_0 \in f^{-1}(\tau'), \text{and} \varepsilon_0 \in f^{-1}(\varepsilon')$

be such that

$$\tilde{\vartheta}_\delta(\rho_0 * \varepsilon_0) = \underset{t \in f^{-1}(\rho' * \varepsilon')}{\text{rsup}}\ \tilde{\vartheta}_\delta(t), \tilde{\vartheta}_\delta(\tau_0) = \underset{t \in f^{-1}(\tau')}{\text{rsup}}\ \tilde{\vartheta}_\delta(t),$$

$$\tilde{\vartheta}_\delta((\rho_0 * \tau_0) * \varepsilon_0) = \tilde{\vartheta}_\omega\{f((\rho_0 * \tau_0) * \varepsilon_0)\}$$

$= \tilde{\vartheta}_\omega((\rho' * \tau') * \varepsilon')$

$= \underset{((\rho_0 * \tau_0) * \varepsilon_0) \in f^{-1}((\rho' * \tau') * \varepsilon')}{\text{rsup}}\ \tilde{\vartheta}_\delta\left((\rho_0 * \tau_0) * \varepsilon_0\right)$

$= \underset{t \in f^{-1}((\rho' * \tau') * \varepsilon')}{\text{rsup}}\ \tilde{\vartheta}_\delta(t)$ .Also

$\alpha_\delta(\rho_0 * \varepsilon_0) = \underset{t \in f^{-1}(\rho' * \varepsilon')}{\inf} \alpha_\delta(t).$

$\alpha_\delta(\tau_0) = \underset{t \in f^{-1}(\tau')}{\inf} \alpha_\delta(t),$

$\alpha_\delta((\rho_0 * \tau_0) * \varepsilon_0) = \alpha_\omega\{f((\rho_0 * \tau_0) * \varepsilon_0)\}$

$$= \alpha_\omega\big((\rho' * \tau') * \varepsilon'\big)$$

$$= \inf_{((\rho_0 * \tau_0) * \varepsilon_0) \in f^{-1}((\rho' * \tau') * \varepsilon')} \alpha_\delta\big((\rho_0 * \tau_0) * \varepsilon_0\big)$$

$= \inf_{t \in f^{-1}((\rho' * \tau') * \varepsilon')} \alpha_\delta(t).$ Then

$$\tilde{\vartheta}_\omega(\rho' * \tau') = \operatorname{rsup}_{t \in f^{-1}(\rho' * \tau')} \tilde{\vartheta}_\delta(t) = \tilde{\vartheta}_\delta(\rho_0 * \tau_0)$$

$$\geq rmin\{\tilde{\vartheta}_\delta\big((\rho_0 * \tau_0) * \varepsilon_0\big), \tilde{\vartheta}_\delta(\tau_0)\}$$

$$= rmin\left\{\operatorname{rsup}_{t \in f^{-1}((\rho' * \tau') * \varepsilon')} \tilde{\vartheta}_\delta(t)\ ,\ \operatorname{rsup}_{t \in f^{-1}(\tau')} \tilde{\vartheta}_\delta(t)\right\}$$

$$= rmin\{\tilde{\vartheta}_\omega\big((\rho' * \tau') * \varepsilon'\big), \tilde{\vartheta}_\omega(\tau')\},$$

$$\alpha_\omega(\rho' * \varepsilon') = \inf_{t \in f^{-1}(\rho' * \varepsilon')} \alpha_\delta(t) = \alpha_\delta\,(\rho_0 * \tau_0)$$

$$\leq max\{\alpha_\delta\big((\rho_0 * \tau_0) * \varepsilon_0\big), \alpha_\delta(\tau_0)\}$$

$$= max\left\{\inf_{t \in f^{-1}((\rho' * \tau') * \varepsilon')} \alpha_\delta(t), \inf_{t \in f^{-1}(\tau')} \alpha_\delta(t)\right\}$$

$$= max\{\alpha_\omega\big((\rho' * \tau') * \varepsilon'\big), \alpha_\omega(\tau')\}.$$

Hence, $\omega$is a cubic T-ideal of Y.

5. Cartesian Product of Cubic T-Ideals

In this section, we provide some definitions of the Cartesian product of cubic T-ideals in TM-algebras.

Definition (29). Let$\delta_1 = \langle \tilde{\vartheta}_{\delta_1}, \alpha_{\delta_1} \rangle$ and $\delta_2 = \langle \tilde{\vartheta}_{\delta_2}, \alpha_{\delta_2} \rangle$ be two cubic subsets of TM-algebras $\aleph_1$ and $\aleph_2$, respectively. We definethe Cartesian product of two cubic sets δ_1 and δ_2 by $\delta_1 \times \delta_2 = \langle \tilde{\vartheta}_{\delta_1 \times \delta_2}, \alpha_{\delta_1 \times \delta_2} \rangle$ and

$$\tilde{\vartheta}_{\delta_1 \times \delta_2}(\rho, \tau) = rmin\{\tilde{\vartheta}_{\delta_1}(\rho), \tilde{\alpha}_{\delta_2}(\tau)\},$$

$\alpha_{\delta_1 \times \delta_2}(\rho, \tau) = max\{\alpha_{\delta_1}(\rho), \alpha_{\delta_2}(\tau)\},$ for any $(\rho, \tau) \in \aleph_1 \times \aleph_2$.

Remark (30). Let $\aleph$ and Y be TM-algebras. We define $*$ on $\aleph \times Y$ by
$(\rho, \tau) * (u, v) = (\rho * u, \tau * v)$ for every $(\rho, \tau), (u, v)$ belong to $\aleph \times Y$,
Then, clearly $\big(\aleph \times Y, *, (0,0)\big)$ is a TM-algebra.

Definition (31). A cubic subset $\delta_1 \times \delta_2 = \langle \tilde{\vartheta}_{\delta_1 \times \delta_2}, \alpha_{\delta_1 \times \delta_2} \rangle$ of $\aleph_1 \times \aleph_2$ is called a cubic ideal of $\aleph_1 \times \aleph_2$ if

($\mathbf{CP_1}$)$\tilde{\vartheta}_{\delta_1 \times \delta_2}(0,0) \geq \tilde{\vartheta}_{\delta_1 \times \delta_2}(\rho, \tau)$ and $\alpha_{\delta_1 \times \delta_2}(0,0) \leq \alpha_{\delta_1 \times \delta_2}(\rho, \tau)$,

$$(\mathbf{CP_2})\tilde{\vartheta}_{\delta_1\times\delta_2}(\rho_1,\tau_1) \geq rmin\{\tilde{\vartheta}_{\delta_1\times\delta_2}((\rho_1,\tau_1)*(\rho_2,\tau_2)),\tilde{\vartheta}_{\delta_1\times\delta_1}(\rho_2,\tau_2)\}$$

$$(\mathbf{CP_3})\alpha_{\delta_1\times\delta_2}(\rho_1,\tau_1) \leq max\{\alpha_{\delta_1\times\delta_2}((\rho_1,\tau_1)*(\rho_2,\tau_2)),\alpha_{\delta_1\times\delta_2}(\rho_2,\tau_2)\},$$

For any $(\rho_1,\tau_1),(\rho_2,\tau_2) \in \aleph_1 \times \aleph_2$.

Definition (32). A cubic subset $\delta_1 \times \delta_2 = \langle \tilde{\vartheta}_{\delta_1\times\delta_2}, \alpha_{\delta_1\times\delta_2}\rangle$ of $\aleph_1 \times \aleph_2$ is called a cubic T-ideal of $\aleph_1 \times \aleph_2$ if

$$(\mathbf{CP_1})\tilde{\vartheta}_{\delta_1\times\delta_2}(0,0) \geq \tilde{\vartheta}_{\delta_1\times\delta_2}(\rho,\tau)\ and\ \alpha_{\delta_1\times\delta_2}(0,0) \leq \alpha_{\delta_1\times\delta_2}(\rho,\tau),$$

$$(\mathbf{CP_2})\tilde{\vartheta}_{\delta_1\times\delta_2}((\rho_1,\tau_1)*(\rho_3,\tau_3)) \geq rmin\left\{\tilde{\vartheta}_{\delta_1\times\delta_2}\left(((\rho_1,\tau_1)*(\rho_2,\tau_2))*(\rho_3,\tau_3)\right),\tilde{\vartheta}_{\delta_1\times\delta_2}(\rho_2,\tau_2)\right\}$$

$$(\mathbf{CP_3})\alpha_{\delta_1\times\delta_2}((\rho_1,\tau_1)*(\rho_3,\tau_3)) \leq max\left\{\alpha_{\delta_1\times\delta_2}\left(((\rho_1,\tau_1)*(\rho_2,\tau_2))*(\rho_3,\tau_3)\right),\alpha_{\delta_1\times\delta_2}(\rho_2,\tau_2)\right\},$$

For any $(\rho_1,\tau_1),(\rho_2,\tau_2),(\rho_3,\tau_3) \in \aleph_1 \times \aleph_2$.

Proposition (33). If $\delta_1 \times \delta_2 = \langle \tilde{\vartheta}_{\delta_1\times\delta_2}, \alpha_{\delta_1\times\delta_2}\rangle$ is a cubic T-ideal of TM-algebra $\aleph_1 \times \aleph_2$ and if $(\rho_1,\tau_1) \leq (\rho_2,\tau_2)$, we have $\tilde{\vartheta}_{\delta_1\times\delta_2}(\rho_2,\tau_2) \leq \tilde{\vartheta}_{\delta_1\times\delta_2}(\rho_1,\tau_1)$ and $\alpha_{\delta_1\times\delta_2}(\rho_2,\tau_2) \geq \alpha_{\delta_1\times\delta_2}(\rho_1,\tau_1)$. For all $(\rho_1,\tau_1),(\rho_2,\tau_2) \in \aleph_1 \times \aleph_2$.

Proof. Let $(\rho_1,\tau_1),(\rho_2,\tau_2) \in \aleph_1 \times \aleph_2$, such that $(\rho_1,\tau_1) \leq (\rho_2,\tau_2) \Rightarrow (\rho_2,\tau_2)*(\rho_1,\tau_1) = (0,0)$.This together with

$(0,0)*(\rho_1,\tau_1) = (\rho_1,\tau_1)$and $\tilde{\vartheta}_{\delta_1\times\delta_2}(\rho_2,\tau_2) \leq \tilde{\vartheta}_{\delta_1\times\delta_2}(0,0)$

Also $\alpha_{\delta_1\times\delta_2}(\rho_2,\tau_2) \geq \alpha_{\delta_1\times\delta_2}(0,0)$. Consider

$$\vartheta_{\delta_1\times\delta_2}((0,0)*(\rho_1,\tau_1)) = \tilde{\vartheta}_{\delta_1\times\delta_2}(\rho_1,\tau_1)$$

$$\geq rmin\left\{\tilde{\vartheta}_{\delta_1\times\delta_2}\left(((0,0)*(\rho_2,\tau_2))*(\rho_1,\tau_1)\right),\tilde{\vartheta}_{\delta_1\times\delta_2}(\rho_2,\tau_2)\right\}$$

$$= rmin\{\tilde{\vartheta}_{\delta_1\times\delta_2}((0,0)*(0,0)),\tilde{\vartheta}_{\delta_1\times\delta_2}(\rho_2,\tau_2)\}$$

$$= rmin\{\tilde{\vartheta}_{\delta_1\times\delta_2}(0,0),\tilde{\vartheta}_{\delta_1\times\delta_2}(\rho_2,\tau_2)\}$$

$$= \tilde{\vartheta}_{\delta_1\times\delta_2}(\rho_2,\tau_2),$$

$$\alpha_{\delta_1\times\delta_2}((0,0)*(\rho_1,\tau_1)) = \alpha_{\delta_1\times\delta_2}(\rho_1,\tau_1)$$

$$\leq max\left\{\alpha_{\delta_1\times\delta_2}\left(((0,0)*(\rho_2,\tau_2))*(\rho_1,\tau_1)\right),\alpha_{\delta_1\times\delta_2}(\rho_2,\tau_2)\right\}$$

$$= max\{\alpha_{\delta_1\times\delta_2}((0,0)*(0,0)),\alpha_{\delta_1\times\delta_2}(\rho_2,\tau_2)\}$$

$$= max\{\alpha_{\delta_1\times\delta_2}(0,0),\alpha_{\delta_1\times\delta_2}(\rho_2,\tau_2)\}$$

$= \alpha_{\delta_1 \times \delta_2}(\rho_2, \tau_2)$

This shows that $\tilde{\vartheta}_{\delta_1 \times \delta_2}(\rho_2, \tau_2) \leq \tilde{\vartheta}_{\delta_1 \times \delta_2}(\rho_1, \tau_1)$ and

$\alpha_{\delta_1 \times \delta_2}(\rho_2, \tau_2) \geq \alpha_{\delta_1 \times \delta_2}(\rho_1, \tau_2)$, for all $(\rho_1, \tau_1), (\rho_2, \tau_2) \in \aleph_1 \times \aleph_2$.

Theorem (34)..Let $\delta_1 = \langle \tilde{\vartheta}_{\delta_1}, \alpha_{\delta_1} \rangle$and $\delta_2 = \langle \tilde{\vartheta}_{\delta_2}, \alpha_{\delta_2} \rangle$ be two cubic ideal of TM-algebra $\aleph_1$ *and* $\aleph_2$, respectively. Then $\delta_1 \times \delta_2 = \langle \tilde{\vartheta}_{\delta_1 \times \delta_2}, \alpha_{\delta_1 \times \delta_2} \rangle$ is a cubic ideal of $\aleph_1 \times \aleph_2$.

Proof. For any $(\rho, \tau) \in \aleph_1 \times \aleph_2$,

$$\tilde{\vartheta}_{\boldsymbol{\delta_1} \times \boldsymbol{\delta_1}}(0,0) = rmin\{\tilde{\vartheta}_{\delta_1}(0), \tilde{\vartheta}_{\delta_2}(0)\}$$

$$\geq rmin\{\tilde{\vartheta}_{\delta_1}(\rho), \tilde{\vartheta}_{\delta_2}(\tau)\} = \tilde{\vartheta}_{\delta_1 \times \delta_2}(\rho, \tau),$$

$$\alpha_{\delta_1 \times \delta_2}(0,0) = max\{\alpha_{\delta_1}(0), \alpha_{\delta_2}(0)\}$$

$$\leq max\{\alpha_{\delta_1}(\rho), \alpha_{\delta_2}(\tau)\} = \alpha_{\delta_1 \times \delta_2}(\rho, \tau).$$

For any $(\rho_1, \tau_1), (\rho_2, \tau_2) \in \aleph_1 \times \aleph_2$. Then

$$\tilde{\vartheta}_{\delta_1 \times \delta_2}(\rho_1, \tau_1) = rmin\{\tilde{\vartheta}_{\delta_1}(\rho_1), \tilde{\vartheta}_{\delta_2}(\tau_1)\}$$

$$\geq rmin\left\{rmin\{\tilde{\vartheta}_{\delta_1}(\rho_1 * \rho_2), \tilde{\vartheta}_{\delta_1}(\rho_2)\}, rmin\{\tilde{\vartheta}_{\delta_2}(\tau_1 * \tau_2), \tilde{\vartheta}_{\delta_2}(\tau_2)\}\right\}$$

$$= rmin\left\{rmin\{\tilde{\vartheta}_{\delta_1}(\rho_1 * \rho_2), \tilde{\vartheta}_{\delta_2}(\tau_1 * \tau_2)\}, rmin\{\tilde{\vartheta}_{\delta_1}(\rho_2), \tilde{\vartheta}_{\delta_2}(\tau_2)\}\right\}$$

$$= rmin\{\tilde{\vartheta}_{\delta_1 \times \delta_2}(\rho_1 * \rho_2, \tau_1 * \tau_2), \tilde{\vartheta}_{\delta_1 \times \delta_2}(\rho_2, \tau_2)\}$$

$$\geq rmin\{\tilde{\vartheta}_{\delta_1 \times \delta_2}\big((\rho_1 * \tau_1)(\rho_2 * \tau_2)\big), \tilde{\vartheta}_{\delta_1 \times \delta_2}(\rho_2, \tau_2)\},$$

$$\alpha_{\delta_1 \times \delta_2}(\rho_1, \tau_1) = max\{\alpha_{\delta_1}(\rho_1), \alpha_{\delta_2}(\tau_1)\}$$

$$\leq max\left\{max\{\alpha_{\delta_1}(\rho_1 * \rho_2), \alpha_{\delta_2}(\tau_2)\}, max\{\alpha_{\delta_2}(\tau_1 * \tau_2), \alpha_{\delta_2}(\tau_2)\}\right\}$$

$$= max\left\{max\{\alpha_{\delta_1}(\rho_1 * \rho_2), \alpha_{\delta_2}(\tau_1 * \tau_2)\}, max\{\alpha_{\delta_2}(\rho_2), \alpha_{\delta_2}(\tau_2)\}\right\}$$

$$= max\{\alpha_{\delta_1 \times \delta_2}(\rho_1 * \rho_2, \tau_1 * \tau_2), \alpha_{\delta_1 \times \delta_2}(\rho_2, \tau_2)\}$$

Hence, for all $(\rho_1, \tau_1), (\rho_2, \tau_2) \in \aleph_1 \times \aleph_1$, $\delta_1 \times \delta_2 = \langle \tilde{\vartheta}_{\delta_1 \times \delta_2}, \alpha_{\delta_1 \times \delta_2} \rangle$ is a cubic idealof TM-algebra $\aleph_1 \times \aleph_2$.

6.Conclusion

The goal of this paper is to introduce the definition of a cubic ideal and a cubic T-ideal. The homomorphism of these ideals is defined, and the Cartesian product of cubic ideals in Cartesian product TM-algebras is given. Cubic ideals are presented and studied by more than one author on different algebraic structures. Also, new relations between a cubic ideal and a cubic T-ideal are discussed.

References

1. Megalai, K.; Tamilarasi, A. Classification of TM-Algebra, *Computer Aided Soft Computing Techniques for Imaging and Biomedical Applications*. Special Issue **2010**.

2. Ganeshkumar, T.; Chandramouleeswaran, M. t−DERIVATIONS ON TM-ALGEBRAS. *International Journal of Pure and Applied Mathematics*. **2013**,*85*,*1*, 95-107.

3. Megalai, K. ; Tamilarasi, A. Fuzzy Subalgebras and Fuzzy T-ideals in TM-Algebras. *Journal of Mathematics and Statistics*. **2011**, *7,2*, 107-111.

4. Thomas, J.; Indhira, K.; Chandrasekaran, V. M. T-Normed Fuzzy TM-Subalgebra of TM-Algebras. *International Journal of Computational Intelligence Systems*. **2019**,*12*,*2*, 706–712.

5. Ghlaim, F. M. ; Kareem, F. F. Interval valued fuzzy ideals of TM-algebra. Accepted in *Journal of Interdisciplinary Mathematics*.

6. Ganeshkumar, T. Generalized Derivation on TM –Algebras. *International Journal of Algebra*. **2013**, *7, 6*, 251 – 258.

7. Jun, Y. B. ; Kim, C. S. ; Kang, M. S. Cubic subalgebras and ideals of BCK/BCI-algebras. *Far East Journal of Mathematical Sciences*. **2010**, *2, 44*, 239–250.

8. Jun, Y. B. ; Kim, C. S. ; Yang, K.O. Cubic sets. *Annals of Fuzzy Mathematics and Informatics*. **2012**, *1*,*4*, 83–98.

9. Yaqoob, N.; Mostafa, S.M. ; Ansari, M.A. On cubic KU-ideals of KU-algebras.*ISRN Algebra*. **2013**. Article ID935905, 10 pages.

10. Kareem, F. F. ; Hasan, O. A. Cubic ideals of semigroup in KU-algebra. *J. Phys.: Conf. Ser*. 1804. **2021**, 012018.

11. Akram, M. ; Yaqoob, N.; Gulistan, M. cubic KU-subalgebras. *Int. J. Pure Appl. Math***. 2013**,*89 ,5*, 659- 665.

12. Jun, Y. B. ; Kim, C. S. ; Kang ; Kang, J. G. Cubic q -ideals of BCI-algebras. *Annals of Fuzzy Mathematics and Informatics*. **2011**, *1, 1*, 25- 34.

13. Janaa, C.; Senapati, T. Cubic G-subalgebras of G-algebras. *Annals of Pure and Applied Mathematics*. **2015**, *10*,*1*, 105-115.