IHJPAS. 36(1)2023

Ibn Al-Haitham Journal for Pure and Applied Sciences

Journal homepage: jih.uobaghdad.edu.iq

On the Stability and Acceleration of Projection Algorithms

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Article history: Received 29 June 2022, Accepted 21 Augest 2022, Published in January 2023.

doi,org/10.30526/36.1.2923

Abstract

The focus of this paper is the presentation of a new type of mapping called projection Jungck zn- Suzuki generalized and also defining new algorithms of various types (one-step and two-step algorithms) (projection Jungck-normal $\mathcal{N}$ algorithm, projection Jungck-Picard algorithm, projection Jungck-Krasnoselskii algorithm, and projection Jungck-Thianwan algorithm). The convergence of these algorithms has been studied, and it was discovered that they all converge to a fixed point. Furthermore, using the previous three conditions for the lemma, we demonstrated that the difference between any two sequences is zero. These algorithms' stability was demonstrated using projection Jungck Suzuki generalized mapping. In contrast, the rate of convergence of these algorithms was demonstrated by contrasting the rates of convergence of the various algorithms, leading us to conclude that the projection Jungck-normal $\mathcal{N}$ algorithm is the fastest of all the algorithms mentioned above.

Keywords: metric projection, Jungck-Picard algorithm, Jungck-normal $\mathcal{N}$ algorithm, Jungck-Thianwan algorithm, Jungck-Krasnoselskii algorithm, Fixed Point.

1.Introduction and Preliminary

There are a lot of published studies that included new algorithms and studied their strong convergence and stability. In addition, they proved the rate of convergence of these algorithms, see [1-9]. These algorithms are valuable tools used to find the value of the fixed point and to resolve some problems. For example, they were used in solving nonlinear differential equations, integration problems, etc.

The algorithms presented by the authors are varied (one-step, two-step, etc.). In 1967[10], scientist Jungck introduced a new algorithm called Jungck Picard algorithm, but sometimes it is called Jungck algorithm, as it consists of one step

$\Psi k_{n+1} = Tk_n$, where $k_0 \in C, n \in \mathbb{N}$. In 2011[11], the author Alfred Olufemi Bosede presented an algorithm called Jungck-krasnoselskii. This algorithm is a special case of Jungck-Mann. The Jungck-Krasnoselskii algorithm is defined as follows:

$\Psi n_{n+1} = (1-\delta)\Psi n_n + \delta\, Tn_n$, where $n_0 \in C, n \in \mathbb{N}$, and $\delta \in (0,1)$.He also proved the stability of Jungck-Mann and Jungck-Krasnoselskii algorithms. On the other hand, V. Brined proved in 2004[12] that the Picard algorithm converges faster than the Mann algorithm. In 2008, [13] presented a new two-step algorithm named after him. The Thianwan algorithm is defined as follows:

$z_{n+1} = (1-a_n)r_n + a_n\, Tr_n$

$r_n = (1-\beta_n)z_n + \beta_n\, Tz_n$, where $z_0 \in C, n \in \mathbb{N}$,

and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are the real sequence in [0,1].

In addition, it has been proven the strong and weak convergence of this algorithm in the uniformly convex Banach space. Now, we will mention some of the priorities we need:

Definition (1.1):[12] Let $\{m_n\}_{n=0}^{\infty}, \{n_n\}_{n=0}^{\infty}$ are two sequence lies in R such that $\{m_n\}_{n=0}^{\infty}$ converge to m, $\{n_n\}_{n=0}^{\infty}$ converge to n, and $\mathcal{W} = \lim_{n\to\infty} \frac{|m_n - m|}{|n_n - n|}$

1. If $\mathcal{W} = 0 \longrightarrow$ the sequence $\{m_n\}_{n=0}^{\infty}$is converge to m faster then $\{n_n\}_{n=0}^{\infty}$ converge to n.
2. If $0 < \mathcal{W} < \infty \rightarrow \{m_n\}_{n=0}^{\infty}$ and $\{n_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Lemma (1.2):[14] Let $\mathcal{U}$ be a uniformly convex Banach space and $\{m_n\}_{n=0}^{\infty}$ be any sequence such that $0 < \mathfrak{p} \le \alpha_n \le \mathfrak{q} < 1$, for some $\mathfrak{p},\mathfrak{q} \in \mathbb{R}^+$ and for all $n \ge 1$. Let $\{m_n\}_{n=0}^{\infty}$and $\{n_n\}_{n=0}^{\infty}$ are two sequences of $\mathcal{U}$ such that:

$\lim_{n\to\infty} \sup\|m_n\| \le c$, $\lim_{n\to\infty} \sup\|n_n\| \le c$ and

$\lim_{n\to\infty} \sup\|\alpha_n m_n + (1-\alpha_n)n_n\| = c$ for some $c \ge 0$ Then

$\lim_{n\to\infty} \|m_n - n_n\| = 0$

Definition (1.3): [15] Let $\Psi, T : C \to C$ such that $T(C) \subseteq \Psi(C)$ and p a coincidence point of Ψ and T, that is, $\Psi p = Tp = p$. For $any\ c_0 \in C$, let the sequence $\{\Psi m_n\}_{n=0}^{\infty}$ generated by the algorithm procedure $\Psi m_n = f(T, m_n)\ \ n \ge 0$ converge to p. Let $\{\Psi y_n\}_{n=0}^{\infty} \subset C$ be an arbitrary sequence and set

$\varepsilon_n = d\big(\Psi y_{n+1}, f(T, y_n)\big), n = 0, 1, \cdots$. Then, the algorithm Ψc_n will be called $(\Psi, T) - stable$ if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} \Psi y_n = p$.

Lemma (1.4): [12] If η is a real number such that $0 < \eta < 1$ and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers, such that $\lim_{n\to\infty} \epsilon_n = 0$ then, for any sequence of positive numbers $\{m_n\}_{n=0}^{\infty}$ satisfying $m_{n+1} \le \eta m_n + \epsilon_n$

Definition (1.5): [16] the mapping $T: C \to C$ is said Suzuki if satisfying the following condition:

$\frac{1}{2}\|m - T(m)\| \le \|m - n\| \Longrightarrow \|T(m) - T(n)\| \le \|m - n\|, \forall m, n \in C$

2 .Main Results

We introduce a new type of mapping called projection Jungck zn-Suzuki generalized and by using this type of mapping, we will propose new algorithms and analyse their convergence and rate of convergence.

Definition (2.1):

Let $\mathcal{X}$ be a normed space, C be a nonempty closed convex subset of $\mathcal{X}$. A mapping $T, \Psi: C \to C$ and $\mathcal{P}_c$ are called projection Jungck zn-Suzuki generalized mapping if

$\frac{1}{2}\|x - T(x)\| \le \|\Psi x - \Psi y\|$ Implies that

$\|T(x) - T(y)\| \le L\|\Psi x - \Psi y\| + \frac{\phi(\|x - \mathcal{P}_c(x)\| + \|x - \Psi x\|)}{1 + max\,\{\|\mathcal{P}_c(x) - \mathcal{P}_c(y)\|, \|\Psi x - \Psi y\|\}}$

$\phi: \mathcal{R}^+ \to \mathcal{R}^+$ is a monotone increasing function such that $\phi(0) = 0$

and $L \le 1$.

Definition (2.2):
The projection Jungck-Picard algorithm is defined as follows:
$\Psi k_{n+1} = \mathcal{P}_c T k_n \ , k_0 \in \mathrm{C}$.
Definition (2.3):
The projection Jungck-Krasnoselskii is defined as follows:
$\Psi n_{n+1} = (1-\delta)\Psi\mathcal{P}_c(n_n) + \delta\mathcal{P}_c T n_n \ , n_0 \in \mathrm{C}$ where $\mathcal{P}_c$ is metric projection and $\delta \in (0,1)$.
Definition (2.4):
The projection Jungck-normal $\mathcal{N}$ algorithm is defined as follows:
$\Psi u_{n+1} = \mathcal{P}_c T\big((1-\alpha_n)\Psi u_n + \alpha_n \mathcal{P}_c(u_n)\big), u_0 \in \mathrm{C}$ where $\alpha_n \in [0,1]$
and Ψ has property $\mathcal{N}$, i.e., $\Psi\Psi(x) \le \Psi(x),\ x \in C$ & Ψ is a linear map
Definition (2.5):
The projection Jungck-Thianwan algorithm is defined as follows:
$\Psi z_{n+1} = (1-\alpha_n)\Psi\mathcal{P}_c(r_n) + \alpha_n \mathcal{P}_c T r_n$
$\Psi r_n = (1-\beta_n)\Psi\mathcal{P}_c(z_n) + \beta_n \mathcal{P}_c T z_n \ , z_0 \in C$.
And this mapping is commute if $\Psi\mathcal{P}_c(x_n) = \mathcal{P}_c\Psi(x_n)$.
Now, we talk about convergence, stability and rate of convergence.

Lemma (2.6):
Let C be a non-empty closed convex subset of a uniformly convex Banach space $\mathcal{U}$. The mappings $T, \Psi: C \to C$ are a projection Jungck zn-Suzuki generalized if $\{\Psi u_n\}$ generated by projection Jungck-normal $\mathcal{N}$ algorithm, such that

1. $\lim_{n\to\infty} \|\Psi u_n - p\|$ exists for all $p \in \mathcal{CF}(\mathcal{P}_c, T, \Psi)$, where $\mathcal{CF}(\mathcal{P}_c, T, \Psi)$ is the family of a common fixed point.
2. $\lim_{n\to\infty} \|\Psi\Psi u_n - \Psi\mathcal{P}_c(u_n)\| = 0$

Proof:
Let $p \in \mathcal{CF}(\mathcal{P}_c, T, \Psi)$,
$\|\Psi u_{n+1} - p\| = \big\|\mathcal{P}_c T\big((1-\alpha_n)\Psi u_n + \alpha_n\mathcal{P}_c(u_n)\big) - p\big\|$
$\le \big\|T\big((1-\alpha_n)\Psi u_n + \alpha_n\mathcal{P}_c(u_n)\big) - p\big\|$
$\le L\big\|\Psi\big((1-\alpha_n)\Psi u_n + \alpha_n\mathcal{P}_c(u_n)\big) - p\big\| + \dfrac{\phi(\|p-\mathcal{P}_c(p)\| + \|p-\Psi p\|)}{1+\max\{\|\mathcal{P}_c(p)-\mathcal{P}_c\big((1-\alpha_n)\Psi u_n+\alpha_n\mathcal{P}_c(u_n)\big)\|, \|\Psi p-\Psi\big((1-\alpha_n)\Psi u_n+\alpha_n\mathcal{P}_c(u_n)\big)\|\}}$
$\le [(1-\alpha_n)\|\Psi\Psi u_n - p\| + \alpha_n\|\Psi\mathcal{P}_c(u_n) - p\|]$
$\le [(1-\alpha_n)\|\Psi u_n - p\| + \alpha_n\|\mathcal{P}_c\Psi(u_n) - p\|]$
$\le \|\Psi u_n - p\|$
So, we have

$$\|\Psi u_{n+1} - p\| \le \|\Psi u_n - p\| \tag{2.1}$$
$$\le \|\Psi u_{n-1} - p\|$$
$$\vdots$$
$$\le \|\Psi u_0 - p\| \tag{2.2}$$

From (2.1) and (2.2) $\lim_{n\to\infty}\|\Psi u_n - p\|$ is exist
Now to prove
$\lim_{n\to\infty}\|\Psi\Psi u_n - \Psi\mathcal{P}_c(u_n)\| = 0$
Since, $\lim_{n\to\infty}\|\Psi u_n - p\| = c$

$\Longrightarrow \lim_{n\to\infty} sup\|\Psi u_n - p\| = c$

Now,

$\lim_{n\to\infty} sup\,\|\Psi\Psi u_n - p\|$

$\leq \lim_{n\to\infty} sup\,\|\Psi u_n - p\| = c$

So, $\lim_{n\to\infty} sup\,\|\Psi\Psi u_n - p\| \leq c$ (2.3)

To proof $\lim_{n\to\infty} sup\,\|\Psi\mathcal{P}_c(u_n) - p\| \leq c$

$\lim_{n\to\infty} sup\,\|\Psi\mathcal{P}_c(u_n) - p\|$

$\leq \lim_{n\to\infty} sup\,\|\mathcal{P}_c\Psi(u_n) - p\|$

$\leq \lim_{n\to\infty} sup\|\Psi u_n - p\| = c$

So, $\lim_{n\to\infty} sup\,\|\Psi\mathcal{P}_c(u_n) - p\| \leq c$ (2.4)

Since $c = \lim_{n\to\infty} sup\|\Psi u_{n+1} - p\|$

$= \lim_{n\to\infty} sup\left\|\mathcal{P}_c T\big((1-\alpha_n)\Psi u_n + \alpha_n\mathcal{P}_c(u_n)\big) - p\right\|$

$\leq \lim_{n\to\infty} sup\left\|T\big((1-\alpha_n)\Psi u_n + \alpha_n\mathcal{P}_c(u_n)\big) - p\right\|$

$\leq \lim_{n\to\infty} sup\|(1-\alpha_n)(\Psi\Psi u_n - p) + \alpha_n(\Psi\mathcal{P}_c(u_n) - p)\|$ (2.5)

$\leq \lim_{n\to\infty} sup[(1-\alpha_n)\|\Psi u_n - p\| + \alpha_n\|\mathcal{P}_c\Psi(u_n) - p\|]$

$\leq \lim_{n\to\infty} sup[(1-\alpha_n)\|\Psi u_n - p\| + \alpha_n\|\Psi u_n - p\|]$

$= \lim_{n\to\infty} sup\|\Psi u_n - p\| = c$

So, $\lim_{n\to\infty} sup\|(1-\alpha_n)(\Psi\Psi u_n - p) + \alpha_n(\Psi\mathcal{P}_c(u_n) - p)\| = c$

From (2.3), (2.4), (2.5) and by using lemma (1.2) we get

$\lim_{n\to\infty}\|\Psi\Psi u_n - \Psi\mathcal{P}_c(u_n)\| = 0.$

Lemma (2.7):

Let $T, \Psi: C \to C$ are a projection Jungck zn-Suzuki generalized if $\{\Psi k_n\}$ generated by the projection Jungck-Picard algorithm, such that

$\lim_{n\to\infty}\|\Psi k_n - p\|$ exists for all $p \in \mathcal{CF}(\mathcal{P}_c, T, \Psi)$

Proof: Let, $p \in \mathcal{CF}(\mathcal{P}_c, T, \Psi)$

$\|\Psi k_{n+1} - p\| = \|\mathcal{P}_c T k_n - p\|$

$\leq \|T k_n - p\|$

$\leq L\|\Psi k_n - p\| + \frac{\phi(\|p-\mathcal{P}_c(p)\|+\|p-\Psi p\|)}{1+max\{\|\mathcal{P}_c(p)-\mathcal{P}_c(k_n)\|,\|\Psi p-\Psi k_n\|\}}$

$\leq \|\Psi k_n - p\|$

So, we have

$\|\Psi k_{n+1} - p\| \leq \|\Psi k_n - p\| \Longrightarrow \{\Psi k_n\}$ is non-increasing sequence (2.6)

$\leq \|\Psi k_{n-1} - p\|$

$\vdots$

$\leq \|\Psi k_0 - p\| \Longrightarrow \{\Psi k_n\}$ is bounded sequence (2.7)

From (2.6) and (2.7) the $\lim_{n\to\infty}\|\Psi k_n - p\|$ is exist.

Lemma (2.8):

Let $T, \Psi: C \to C$ be a projection Jungck zn-Suzuki generalized mapping if $\{\Psi n_n\}$ is generated by the projection Jungck-Krasnoselskii algorithm, such that

1. $\lim_{n\to\infty} \|\Psi n_n - p\|$ exists for all $p \in \mathcal{CF}(\mathcal{P}_c, T, \Psi)$
2. $\lim_{n\to\infty} \|\Psi \mathcal{P}_c(n_n) - \mathcal{P}_c T n_n\| = 0$

Proof: By following the same steps for the proof of theorem (2.6), we get the wanted results.

Lemma (2.9):

Let $T, \Psi: C \to C$ are a projection Jungck zn-Suzuki generalized mapping if $\{\Psi z_n\}$ is generated by the projection Jungck-Thianwan algorithm, such that:

1. $\lim_{n\to\infty} \|\Psi z_n - p\|$ exists for all $p \in \mathcal{CF}(\mathcal{P}_c, T, \Psi)$
2. $\lim_{n\to\infty} \|\Psi \mathcal{P}_c(r_n) - \mathcal{P}_c T(r_n)\| = 0$

Proof:

Proof in the same way as lemma proof (2.6)

Theorem (2.10): Let $T, \Psi: C \to C$ are a projection Jungck zn-Suzuki generalized mapping with $L \in (0,1)$. Let $\{\Psi n_n\}$ be projection Jungck-Krasnoselskii algorithm converging to p where $\delta \in (0,1)$. Then, the projection Jungck-Krasnoselskii algorithm is $(\Psi, T, \mathcal{P}_c)$-stable.

Proof:

Let $\{y_n\} \subset C$ and $\varepsilon_n = \|\Psi y_{n+1} - f(T, y_n)\|$

$$= \|\Psi y_{n+1} - (1-\delta)\Psi \mathcal{P}_c(y_n) + \delta \mathcal{P}_c T y_n\|$$

So, $\varepsilon_n = \|\Psi y_{n+1} - (1-\delta)\Psi \mathcal{P}_c(y_n) + \delta \mathcal{P}_c T y_n\|$

If $\lim_{n\to\infty} \varepsilon_n = 0$, we get $\lim_{n\to\infty} \Psi y_{n+1} = p$

Then, the projection Jungck-Krasnoselskii algorithm is $(\Psi, T, \mathcal{P}_c)$-stable.

Theorem (2.11): Let $T, \Psi: C \to C$ are a projection Jungck zn-Suzuki generalized mapping with $L \in (0,1)$. Let $\{\Psi u_n\}$ be a projection Jungck-normal $\mathcal{N}$ algorithm converging to p , where $\{\alpha_n\}$ are sequences in [0,1], such that $0 < \alpha \le \alpha_n$. Then, the projection Jungck-normal $\mathcal{N}$ algorithm is $(\Psi, T, \mathcal{P}_c)$-stable.

Proof:

Let $\{y_n\} \subset C$ and $\varepsilon_n = \|\Psi y_{n+1} - f(T, y_n)\|$

$$= \left\|\Psi y_{n+1} - \mathcal{P}_c T\big((1-\alpha_n)\Psi y_n + \alpha_n \mathcal{P}_c(y_n)\big)\right\|$$

So, $\varepsilon_n = \left\|\Psi y_{n+1} - \mathcal{P}_c T\big((1-\alpha_n)\Psi y_n + \alpha_n \mathcal{P}_c(y_n)\big)\right\|$

If $\lim_{n\to\infty} \varepsilon_n = 0$, we get $\lim_{n\to\infty} \Psi y_{n+1} = p$

Then, the projection Jungck-normal $\mathcal{N}$ algorithm is $(\Psi, T, \mathcal{P}_c)$-stable.

Theorem (2.12): Let $T, \Psi: C \to C$ are projection Jungck zn-Suzuki generalized mapping with $L \in (0,1)$. Let $\{\Psi k_n\}$ be a projection Jungck-Picard algorithm converging to p, where $\{\alpha_n\}$ are sequences in [0,1]. Then, the projection Jungck-Picard algorithm is $(\Psi, T, \mathcal{P}_c)$-stable.

Theorem (2.13): Let $T, \Psi: C \to C$ are projection Jungck zn-Suzuki generalized mapping with $L \in (0,1)$. Let $\{\Psi z_n\}$ be a projection Jungck- Thianwan algorithm converging to p, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1] such that $0 < \alpha \le \alpha_n$ and , $0 < \beta \le \beta_n$ then, the projection Jungck-Thianwan algorithm is $(\Psi, T, \mathcal{P}_c)$-stable.

Proof: By following the same steps of the proof of theorem (2.9), we get the wanted results.

Theorem (2.14): Let T, Ψ are projection Jungck zn-Suzuki generalized mapping and $\mathcal{CF}(\mathcal{P}_c, T, \Psi) \neq \phi$. Then, the projection Jungck-Picard algorithm converges faster than projection Jungck-Krasnoselskii algorithm.

Proof:
For projection Jungck-Picard algorithm

$$\|\Psi k_{n+1} - p\| = \|\mathcal{P}_c T k_n - p\|$$
$$\leq \|T k_n - p\|$$
$$\leq L\|\Psi k_n - p\| + \frac{\phi(\|p-\mathcal{P}_c(p)\| + \|p-\Psi p\|)}{1+max\{\|\mathcal{P}_c(p)-\mathcal{P}_c(k_n)\|, \|\Psi p - \Psi k_n\|\}}$$
$$\vdots$$
$$\leq L^n\|\Psi k_0 - p\|$$

Put $\mathcal{P}.\mathcal{J}.P.\mathcal{A} = L^n\|\Psi k_0 - p\|$

For projection Jungck-Krasnoselskii algorithm.

$$\|\Psi n_{n+1} - p\| = \|(1-\delta)\Psi\mathcal{P}_c(n_n) + \delta\mathcal{P}_c T n_n - p\|$$
$$\leq (1-\delta)\|\Psi\mathcal{P}_c(n_n) - p\| + \delta\|\mathcal{P}_c T n_n - p\|$$
$$= (1-\delta)\|\mathcal{P}_c\Psi(n_n) - p\| + \delta\|\mathcal{P}_c T n_n - p\|$$
$$\leq (1-\delta)\|\Psi n_n - p\| + L\delta\|\Psi n_n - p\| + \frac{\phi(\|p-\mathcal{P}_c(p)\| + \|p-\Psi p\|)}{1+\max\{\|\mathcal{P}_c(p)-\mathcal{P}_c(n_n)\|, \|\Psi p - \Psi n_n\|\}}$$
$$\leq (1-\delta(1-L))\|\Psi n_n - p\|$$
$$\vdots$$
$$\leq [1-\delta(1-L)]^n\|\Psi n_0 - p\|$$

Put:

$$\mathcal{P}.\mathcal{J}.\mathcal{K}.\mathcal{A} = [1-\delta(1-L)]^n\|\Psi n_0 - p\|$$

Now since:

$$\frac{\mathcal{P}.\mathcal{J}.P.\mathcal{A}}{\mathcal{P}.\mathcal{J}.\mathcal{K}.\mathcal{A}} = \frac{L^n\|\Psi k_0 - p\|}{[1-\delta(1-L)]^n\|\Psi n_0 - p\|} \to 0 \quad \text{as } n \to \infty.$$

Hence, the projection Jungck-Picard algorithm converges to p faster than the projection Jungck-Krasnoselskii algorithm.

Theorem (2.15): Let T, Ψ be a projection Jungck zn-Suzuki generalized mapping and $\mathcal{CF}(\mathcal{P}_c, T, \Psi) \neq \phi$. Then, the projection Jungck-normal $\mathcal{N}$ algorithm converges faster than the projection Jungck- Picard algorithm.

Proof: Let $p \in \mathcal{CF}(\mathcal{P}_c, T, \Psi)$ and suppose that there exists $\lambda; 0 \leq \lambda \leq \alpha_n \leq 1$

For projection Jungck –Picard algorithm

We have,

$$\mathcal{P}.\mathcal{J}.P.\mathcal{A} = L^n\|\Psi k_0 - p\|$$

From projection Jungck-normal $\mathcal{N}$ algorithm

$$\|\Psi u_{n+1} - p\| = \left\|\mathcal{P}_c T\big((1-\alpha_n)\Psi u_n + \alpha_n\mathcal{P}_c(u_n)\big) - p\right\|$$
$$\leq \left\|T\big((1-\alpha_n)\Psi u_n + \alpha_n\mathcal{P}_c(u_n)\big) - p\right\|$$
$$\leq L\left\|\Psi\big((1-\alpha_n)\Psi u_n + \alpha_n\mathcal{P}_c(u_n)\big) - p\right\| + \frac{\phi(\|p-\mathcal{P}_c(p)\| + \|p-\Psi p\|)}{1+max\{\left\|\mathcal{P}_c(p)-\mathcal{P}_c\big((1-\alpha_n)\Psi u_n + \alpha_n\mathcal{P}_c(u_n)\big)\right\|, \left\|\Psi p - \Psi\big((1-\alpha_n)\Psi u_n + \alpha_n\mathcal{P}_c(u_n)\big)\right\|\}}$$
$$\leq L\left\|\Psi\big((1-\alpha_n)\Psi u_n + \alpha_n\mathcal{P}_c(u_n)\big) - p\right\|$$
$$\leq L\|(1-\alpha_n)\Psi\Psi u_n + \alpha_n\Psi\mathcal{P}_c(u_n) - (1-\alpha_n+\alpha_n)p\|$$
$$\leq L\|(1-\alpha_n)(\Psi\Psi u_n - p) + \alpha_n(\Psi\mathcal{P}_c(u_n) - p)\|$$
$$\leq L[(1-\alpha_n)\|\Psi\Psi u_n - p\| + \alpha_n\|\Psi\mathcal{P}_c(u_n) - p\|]$$
$$\leq L[(1-\alpha_n)\|\Psi u_n - p\| + \alpha_n\|\Psi\mathcal{P}_c(u_n) - p\|]$$
$$= L[(1-\alpha_n)\|\Psi u_n - p\| + \alpha_n\|\mathcal{P}_c\Psi(u_n) - p\|]$$
$$\leq L[(1-\alpha_n)\|\Psi u_n - p\| + \alpha_n\|\Psi u_n - p\|]$$
$$\leq L[1-\lambda(1-L)]\|\Psi u_n - p\|$$
$$\vdots$$

$\leq L^n[1-\lambda(1-L)]^n\|\Psi u_0-p\|$

Let,

$\mathcal{P}.\mathcal{J}.\mathcal{N}.\mathcal{A}=L^n[1-\lambda(1-L)]^n\|\Psi u_0-p\|$

Now, since:

$\frac{\mathcal{P}.\mathcal{J}.\mathcal{N}.\mathcal{A}}{\mathcal{P}.\mathcal{J}.\mathcal{P}.\mathcal{A}}=\frac{L^n[1-\alpha_n(1-L)]^n\|\Psi u_0-p\|}{L^n\|\Psi k_0-p\|}\to 0$ as $n\to\infty$

Hence, the projection Jungck-normal $\mathcal{N}$ algorithm converges to p faster than the projection Jungck-Picard algorithm.

Theorem (2.16)

Let $\mathcal{X}$ be a normed space and C be a nonempty closed convex subset of $\mathcal{X}$ if T is a projection Jungck zn-Suzuki generalized mapping and $\mathcal{CF}(\mathcal{P}_c,T,\Psi)\neq\phi$. Then, the projection Jungck-normal $\mathcal{N}$ algorithm converges faster than the projection Jungck-Thianwan algorithm.

Proof: Let $p\in\mathcal{CF}(\mathcal{P}_c,T,\Psi)$ and suppose that there exists $\Lambda; 0\leq\Lambda\leq\beta_n,\alpha_n\leq 1$

For projection Jungck Thianwan-algorithm

$\|\Psi z_{n+1}-p\|=\|(1-\alpha_n)\Psi\mathcal{P}_c(r_n)+\alpha_n\mathcal{P}_cTr_n-(1-\alpha_n+\alpha_n)p\|$

$\leq(1-\alpha_n)\|\Psi\mathcal{P}_c(r_n)-p\|+\alpha_n\|\mathcal{P}_cTr_n-p\|$

$=(1-\alpha_n)\|\mathcal{P}_c\Psi(r_n)-p\|+\alpha_n\|\mathcal{P}_cTr_n-p\|$

$\leq(1-\alpha_n)\|\Psi r_n-p\|+\alpha_n\|Tr_n-p\|$

$\leq(1-\alpha_n)\|\Psi r_n-p\|+L\alpha_n\|\Psi r_n-p\|+\frac{\phi(\|p-\mathcal{P}_c(p)\|+\|p-\Psi p\|)}{1+max\{\|\mathcal{P}_c(p)-\mathcal{P}_c(r_n)\|,\|\Psi p-\Psi r_n\|\}}$

$\leq(1-\alpha_n(1-L)\|\Psi r_n-p\|$ (2.8)

Now,

$\|\Psi r_n-p\|=\|(1-\beta_n)\Psi\mathcal{P}_c(z_n)+\beta_n\mathcal{P}_cTz_n-(1-\beta_n+\beta_n)p\|$

$\leq(1-\beta_n)\|\Psi\mathcal{P}_c(z_n)-p\|+\beta_n\|\mathcal{P}_cTz_n-p\|$

$=(1-\beta_n)\|\mathcal{P}_c\Psi(z_n)-p\|+\beta_n\|\mathcal{P}_cTz_n-p\|$

$\leq(1-\beta_n)\|\Psi z_n-p\|+\beta_n\|Tz_n-p\|$

$\leq(1-\beta_n)\|\Psi z_n-p\|+L\beta_n\|\Psi z_n-p\|+\frac{\phi(\|p-\mathcal{P}_c(p)\|+\|p-\Psi p\|)}{1+max\{\|\mathcal{P}_c(p)-\mathcal{P}_c(z_n)\|,\|\Psi p-\Psi z_n\|\}}$

$=\left(1-\beta_n(1-L)\right)\|\Psi z_n-p\|$

$\leq\left(1-\lambda(1-L)\right)\|\Psi z_n-p\|$ (2.9)

Substitute Equation (2.9) in to Equation (2.8)

$\|\Psi z_{n+1}-p\|\leq(1-\alpha_n(1-L)\left[\left(1-\lambda(1-L)\right)\|\Psi z_n-p\|\right]$

$=[1-\lambda(1-L)-\alpha_n(1-L)+\alpha_n\lambda(1-L)]\|\Psi z_n-p\|$

$\leq[1-\lambda(1-L)-\lambda(1-L)+\lambda^2(1-L)]\|\Psi z_n-p\|$

$\leq[1-2\lambda(1-L)+\lambda^2(1-L)]\|\Psi z_n-p\|$

$\leq[1-\lambda(1-L)]^2\|\Psi z_n-p\|$

:

$\leq[1-\lambda(1-L)]^{2n}\|\Psi z_0-p\|$

Put:

$\mathcal{P}.\mathcal{J}.\mathcal{T}.\mathcal{A}=[1-\lambda(1-L)]^{2n}\|\Psi z_0-p\|$

From projection Jungck-normal $\mathcal{N}$ algorithm

$\mathcal{P}.\mathcal{J}.\mathcal{N}.\mathcal{A}=L^n[1-\lambda(1-L)]^n\|\Psi u_0-p\|$

Now since:

$\frac{\mathcal{P}.\mathcal{J}.\mathcal{N}.\mathcal{A}}{\mathcal{P}.\mathcal{J}.\mathcal{T}.\mathcal{A}}=\frac{L^n[1-\lambda(1-L)]^n\|\Psi u_0-p\|}{[1-\lambda(1-L)]^{2n}\|\Psi z_0-p\|}$ as $n\to\infty$

Hence, the projection Jungck-normal $\mathcal{N}$ algorithm converges to p is faster than the projection Jungck- Thianwan algorithm.

3. Conclusion

This paper offered a new type of mapping as well as introduced new algorithms and demonstrated their convergence and stability. On the other hand, the acceleration and stability were examined. We discovered that the projection Jungck-normal algorithm is quicker than the techniques stated in this paper to achieve the fixed point.

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