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Quasi Semi and Pseudo Semi (p, E)-Convexity in Non-Linear Optimization Programming

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Abstract

The class of quasi semi (p, E)-convex functions and pseudo semi (p, E)-convex functions are presented in this paper by combining the class of p-convex functions with the class of quasi semi E-convex functions and pseudo semi E-convex functions, respectively. Various non-trivial examples are introduced to illustrate the new functions and show their relationships with (p, E)-convex functions recently introduced in the literature. Different general properties and characteristics of this class of functions are established. In addition, some optimality properties of generalized non-linear optimization problems are discussed. In this generalized optimization problems, we used, as the objective function, quasi semi (p, E)-convex (respectively, strictly quasi semi (p, E)-convex functions and pseudo semi (p, E)-convex functions), and the constraint set is (p, E)-convex set.

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1. Introduction and Preliminaries

Generalized convexity has drawn the attention of many researchers in recent years due to its vast applications in different areas, especially in optimization and applied sciences (see e.g., [1]-[16]). One of the well-known generalizations of convex sets and convex functions is the class of so-called E-convex sets and E-convex functions introduced by [1] where mapping $E: R^n \to R^n$ is employed in this type of the generalized convexity. Due to some erroneous appeared in Youness's first paper, a new class of E-functions called semi E-convex functions, is introduced by [2], and its properties is studied [3]. This class also includes quasi-semi E-convex and pseudo semi E-

convex functions. Youness motivated other researchers to extend some concepts from convex analysis into E-convexity and apply this concept to optimization problems (see, [4], [5], [6], and references therein). Another important recent generalization of convex sets and functions is p-convex sets [7] and p-convex functions [8], affecting the actual number $p \in (0,1]$. Very recently, [9] presented the class of $(p, E\ )$-convex sets and (p, E)-convex functions by combining E-convex sets (respectively, E-convex functions) with p-convex sets (respectively, p-convex functions).

Inspired by the above research works and due to the importance of studying non-convex functions close to the convex in some sense, the class of quasi semi (p, E)-convex functions and pseudo semi (p, E)-convex functions is introduced by combining p-convex functions with quasi semi E-convex and pseudo semi E-convex functions, respectively. These non-convex functions enrich the study of many real-life problems which are non-convex in nature by modeling them as optimization problems that are close to convex problems. The paper is presented as follows. The rest of this section contains preliminary material that makes this work self-contained. In section 2, the definitions of quasi semi (p, E)-convex and pseudo semi (p, E)-convex functions are presented, and various examples and relations related to the new functions with (p, E)-convex functions are provided. In section 3, we provide different properties of quasi semi (p, E)-convex and pseudo semi (p, E)-convex functions. Section 4 is specified to study some optimality properties of non-linear optimization problems in which the objective function is quasi semi (p, E)-convex or pseudo semi (p, E)-convex functions and the constraint set is (p, E)-convex set.

In all the definitions and results throughout this paper, let $p \in (0,1]$ and R^n is the n-dimensional Euclidean space. Assume that A is a non-empty subset of R^n, $f: A \subseteq R^n \longrightarrow R$ be a function, and $E\ : R^n \to R^n$ is a given mapping. Let us now recall the concepts related to E-convex set (respectively, E-convex function) and p-convex set and function.

Definition 1.1. [1], [7] For any $x, y \in A$, $r, s \in [0,1]$, and $p \in (0,1]$ such that $r^p + s^p = 1$. The set A is named as

1. E-convex if $rE(x) + (1 - r)E(y) \in A$.
2. p-convex if $rx + sy \in A$.

Definition 1.2. [1], [2], [8] For any $x, y \in A$, $r, s \in [0,1]$, and $p \in (0,1]$ such that $r^p + s^p = 1$. The function f is named as

1. $E\text{-}convex$ if A is E-convex set and
$f(rE(x) + (1 - r)E(y)) \leq rf(E(x)) + (1 - r)f(E(y))$.
2. Quasi semi $E\text{-}convex$ if A is E-convex set and
$$f(rE(x) + (1 - r)E(y)) \leq \max\{f(x), f(y)\}.$$
3. Pseudo semi E-convex on E-convex set A if there exists a strictly positive function $b: R^n \times R^n \longrightarrow R$ such that if $f(x) < f(y)$ then
$$f\big(rE(x) + (1 - r)E(y)\big) \leq\ f(y) + r(r - 1)b(x, y),$$
for $0 < r < 1$.
4. p-convex if A is $p\text{-}convex$ set and $f\ (rx + sy\ ) \leq rf(x) + sf(y)$.

Very recently, Hazim and Majeed [9] have extended the concepts of E-convexity and p-convexity defined above to (p, E)-convexity as follows.

Definition 1.3. [9] The set A is called (p, E)-convex set if for all $x, y \in A$ and for all $r, s \in [0,1], p \in (0,1]$ such that $r^p + s^p = 1$ we have $rE(x) + sE(y) \in A$.

Definition 1.4. [9] For any $x, y \in A$, $r, s \in [0,1]$, and $p \in (0,1]$ such that $r^p + s^p = 1$. The function f is named as (p, E)-*convex* if A is (p, E)-convex and

$$f(rE(x) + sE(y)) \leq rf(E(x)) + sf(E(y)).$$

Remark 1.5. From the definition of (p, E)-convexity, one observes that

i. In Definition 1.3, if $p = 1$, the definition of E-convex set is obtained. Also, if $E = I$ (identity mapping), then A is p-convex set;
ii. Likewise, from Definition 1.4, if $p = 1$ we have f is E-convex function and when $E = I$, then the definition of f is p-convex function is obtained.

For the rest of the paper, the next remark is needed.

Remark 1.6.

1. The mapping $E(x)$ will be written as Ex.
2. The set A is (p, E) convex set.

2. Quasi Semi and Pseudo Semi (p, E)-Convex Functions

In this section, a new class of functions, which includes quasi semi (p, E)-convex and pseudo semi (p, E)-convex functions, is introduced. This class generalizes each of quasi semi E-convex and pseudo semi E-convex functions [2]. Some properties and related examples are established for this class.

Definition 2.1. The function f is named as

i. Quasi semi (p, E)-convex on A if

$$f(rEx + sEy) \leq \{f(x), f(y)\} \ ,$$

and f is strictly quasi semi (p, E)-convex if

$$f(rEx + sEy) < \{f(x), f(y)\} \ ,$$

where $r, s \in (0,1)$.

ii. Pseudo semi (p, E)-convex if there exist a strictly positive function $b: R^n \times R^n \to R$ such that if $f(x) < f(y)$ *then*

$$f(rEx + sEy) \ \leq f(y) + (-rs)\, b(x, y),$$

for all $r, s \in (0,1)$.

Remark 2.2. In Definition 2.1(i), if $p = 1$ then f turned to be quasi semi E-convex. Likewise, f in Definition 2.1(ii) becomes pseudo semi E-convex function.

Quasi semi and pseudo semi (p, E)-convex functions are not necessarily (p, E)-convex function as the following example shows.

Example 2.3. Let $f, E: R \to R$ *such that* $\quad f(x) = \begin{cases} -3 & if\ x = 0 \\ 1 & if\ x \neq 0 \end{cases}$

$$\text{and} \quad Ex = \begin{cases} 0 & if\ x = 0 \\ 4 & if\ x \neq 0\,. \end{cases}$$

Let $x, y \in R$, $p \in (0,1]$ and $r, s \in [0,1]$ $such\ that$ $r^p + s^p = 1$. First, we show that f is quasi semi (p, E)-convex and pseudo semi (p, E)-convex function. For showing f is quasi semi (p, E)-convex, we consider three cases:

Case 1: If $x = y = 0$, we get $f(rEx + sEy) = f(0) = -3 = \{f(x), f(y)\}$.

Case 2: If $x \neq 0, y \neq 0$, we get

$f(rEx + sEy) = f(4r + 4s) = 1 = \{f(x), f(y)\}$.

Case 3: If $x = 0$, $y \neq 0$, we get

$$f(rEx + sEy) = f(4s) = \begin{cases} -3 & \text{if } s = 0 \\ 1 & \text{if } s \neq 0 \end{cases} \quad (1)$$

$$\leq max\,\{f(x), f(y)\} = \{-3,1\} = 1 \ .$$

In all cases, $f(rEx + sEy) \leq max\{f(x), f(y)\}$, and hence f is quasi semi (p, E)-convex. From case 3, we get $f(x) = -3 < 1 = f(y)$. Thus, from (1), one can choose a strictly positive function $b(x, y)$ such that

$$f(rEx + sEy) \leq f(y) + (-rs)b(x, y) \leq f(y) = 1.$$

Thus, f is pseudo semi (p, E)-convex function. Finally, to show that f is not $(\frac{1}{2}, E)$-convex, take $x \neq 0, y \neq 0$ and $p = \frac{1}{2}$ with $s = r = \frac{1}{4}$. Then, $f(rEx + sEy) = 1 > rf(Ex) + sf(Ey) = rf(4) + sf(4) = r + s = \frac{1}{2}$ as required.

The next example provides (p, E)-convex function which is neither quasi semi (p, E)-convex nor pseudo semi (p, E)-convex.

Example 2.4. Let $A = [-5, -\infty) \times [-5, -\infty) \subseteq R^2$ and $E\colon R^2 \rightarrow R^2$ such that

$$E(x_1, x_2) = \begin{cases} ((x_1 + 1)^2\ , (x_2 + 1)^2) & \text{if } x_1, x_2 < 0 \\ (0,0) & \text{o.w.} \end{cases}$$

Define $f\colon R^2 \rightarrow R$ such that $f(x_1, x_2) = \begin{cases} \frac{x_1 + x_2}{3} & \text{if } x_1, x_2 < 0 \\ 0 & \text{o.w.} \end{cases}$

First, we show that A is (p, E)-convex set. Let $x = (x_1, x_2), y = (y_1, y_2) \in A$. If $x_1, x_2, y_1, y_2 < 0$ then

$rE(x_1, x_2) + sE(y_1, y_2) = (\, r(x_1 + 1)^2 + s(y_1 + 1)^2,\ r(x_2 + 1)^2 + s(y_2 + 1)^2) \in [0, +\infty) \times [0, +\infty) \subseteq A$. Similarly, if $Ex = Ey = (0,0)$ then $rE(x_1, x_2) + sE(y_1, y_2) = (0,0) \in A$. Thus, A is (p, E)-convex set. To show that f is (p, E)-convex function on A, let $x = (x_1, x_2), y = (y_1, y_2) \in A$. If $x_1, x_2, y_1, y_2 < 0$, then

$$f(rEx + sEy) = 0 = rf(Ex) + sf(Ey).$$

Similarly, $f(rEx + sEy) = rf(Ex) + sf(Ey) \quad \forall x, y \in A$. Hence, f is (p, E)-convex function as required. Now, take $x = \left(\frac{-1}{2}, \frac{-1}{2}\right)$, $y = \left(\frac{-1}{4}, \frac{-1}{4}\right)$, $r = s = \frac{1}{4}$ and $p = \frac{1}{2}$. Then, $f(rEx + sEy) = f\left(r\left(\frac{1}{4}, \frac{1}{4}\right) + s\left(\frac{9}{16}, \frac{9}{16}\right)\right) = 0 > \max\left\{-\frac{1}{3}, -\frac{1}{6}\right\} = -\frac{1}{6}$.

Also, $f(x) < f(y)$ and

$f(rEx + sEy) = 0 > f(y) + (-rs)b(x,y) = -\frac{1}{6} + \left(-\frac{1}{16}\right) b(x,y)$,

for a strictly positive function $b(x,y)$. Hence, f is neither quasi semi $(\frac{1}{2}, E)$-convex nor pseudo semi $(\frac{1}{2}, E)$-convex.

The relation between pseudo semi (p, E)- convex function and quasi semi (p, E)-convex functions are given in the next proposition and example.

Proposition 2.5. Every pseudo semi (p, E)- convex function on A is quasi semi (p, E)- convex.

Proof. Let $x, y \in A$ such that $f(x) < f(y)$. Since f is pseudo semi (p, E)- convex, then we have $f(rEx + sEy) \leq f(y) + (-rs)b(x,y) \leq f(y) = max\{f(x), f(y)\}$. ■

Example 2.6. Let $A = \left\{ x = (x_1, \dots, x_n) \in R^n : \sum_{i=1}^{n} |x_i|^{\frac{1}{2}} \leq 1 \right\}$, and E: $R^n \rightarrow R^n$ such that $E(x) = E(x_1, \dots, x_n) = \left(\frac{x_1}{2}, \dots, \frac{x_n}{2}\right)$ for all $x \in R^n$. Define $f: R^n \rightarrow R$ *such that*

$$f(x) = \begin{cases} -1 & \text{if } x_i = 0 \ \forall i = 1, \dots, n \\ 0 & \text{o.w.} \end{cases}$$

From [9, Example 2.4], the set A is $\left(\frac{1}{2}, E\right)$-convex. Next, we show that f is quasi semi $\left(\frac{1}{2}, E\right)$-convex function on A. To this end, let $x, y \in A$ and we consider three cases:

Case 1: If $x_i = y_i = 0 \ \ \forall i = 1, \dots, n$, then

$f(rEx + sEy) = f(0, \dots, 0) = -1 = \{f(x), f(y)\}$.

Case 2: If $x_i \neq 0 \ \ \forall i = 1, \dots, n$ and $y_i \neq 0$ for some $i = 1, \dots, n$.

$$f(rEx + sEy) = \begin{cases} -1 & \text{if } r\frac{x_i}{2} + s\frac{y_i}{2} = 0 \ \forall i = 1, \dots, n \\ 0 & \text{o.w.} \end{cases}$$

$$\leq max\{f(x), f(y)\} = 0 .$$

Case 3: If $x_i \neq 0$ and $y_i = 0$ for some $i = 1, \dots, n$.

$$f(rEx + sEy) = 0 = max\{f(x), f(y)\} .$$

From all cases, we have f is quasi semi $\left(\frac{1}{2}, E\right)$-convex function on A. To show f is not pseudo semi $\left(\frac{1}{2}, E\right)$-convex function on A, take $x = (0, \dots, 1)$, $y = (0, \dots, 0)$ such that $f(x) < f(y)$. Let $r = s = \frac{1}{4}$ then there exist strictly positive function $b(x,y) = 3$ such that

$$\begin{aligned} f(rEx + sEy) = f\big(rE(0, \dots, 1), +sE(0, \dots, 0)\big) = f\left(0, \dots, \frac{1}{2}s\right) &= 0 \\ &> f(y) + (-rs)b(x,y) \\ &= -1 - \frac{3}{16} = -\frac{19}{16} \end{aligned}$$

Hence, $f$is not pseudo semi (p, E)-convex.

Proposition 2.7. The function f is quasi semi (p, E)-convex on A if and only if the level set $K_n = \{x \in A\colon f(x) \leq n\}$ is (p, E)-convex set for all $n \in R$.

Proof. Let f is quasi semi (p, E)-convex on (p, E)-convex set A. Then, for any $x, y \in K_n$, we have $rEx + sEy \in A$, $f(x) \leq n$, $f(y) \leq n$, and $f(rEx + sEy) \leq max\{f(x), f(y)\} \leq n$. It follows that $rEx + sEy \in K_n$. Conversely suppose that K_n is (p, E)-convex for all $n \in R$. Let $n = max\{f(x), f(y)\}$. Since A is (p, E)-convex set then $rEx + sEy \in A$ and $f(rEx + sEy) \leq n = max\{f(x), f(y)\}$. Hence, f is quasi semi (p, E)-convex on A. ■

Proposition 2.8. If f is pseudo semi (p, E)-convex function on A then the level set K_n is (p, E)-convex set.

Proof. let $x, y \in K_n$. We show that $rEx + sEy \in K_n$. Now, since f is pseudo semi (p, E)-convex. Then, we have a strictly positive function $b(x, y)$ such that if $f(x) < f(y)$ then $f(rEx + sEy) \leq f(y) + (-rs)b(x, y) \leq f(y)$. Hence, K_α is (p, E)-convex set. ■

The converse of the proceeding proposition does not satisfy as it is clarified in the next example.

Example 2.9. Let $f, E\colon R \to R$ such that

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \infty) \\ -1 & \text{if } x \in [-\infty, 0) \end{cases} \quad \text{and} \quad E(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

Then, for any $n \in R$, the level set

$$K_n = \begin{cases} [-\infty, 0) & \text{if } n \in [0,1) \\ \mathbb{R} & \text{if } n \geq 1 \end{cases}$$

To show that K_n is (p, E)-convex set, we consider the following cases:

Case 1: If $K_n = R$ (i.e., $n \geq 1$) then, $rEx + sEy \in K_n$ for all $x, y \in K_n$. Hence, K_n is (p, E)-convex set.

Case 2: If $K_n = [-\infty, 0)$ for $n \in [0,1)$. Then, $f(rEx + sEy) = f(-rx^2 - sy^2) = -1 \leq n$ for all $x, y \in K_n$. Thus, $rEx + sEy \in K_n$ which yields the (p, E)-convexity of K_n.

From both cases, we obtain the (p, E)-convexity of K_n. To confirm that f is not pseudo semi (p, E)-convex function. Let $x = -1$, $y = 1$, $r = s = \frac{1}{4}$, and $p = \frac{1}{2}$ then $f(x) < f(y)$ and there exists $b(x, y) = 3 > 0$ such that

$$f(rEx + sEy) = f\left(-\frac{1}{4}x^2 + \frac{1}{4}y^2\right) = f(0) = 1$$

$$> f(y) + (-rs)b(x, y)$$

$$= f(1) + \left(-\frac{1}{16}\right)(3) = \frac{13}{16}$$

Hence, f is not a pseudo semi (p, E)-convex function.

3. Some Properties of Quasi Semi and Pseudo Semi (p, E)-Convex Functions

In this section, we discuss some properties of quasi semi (p, E)-convex and pseudo semi (p, E)-convex functions. We start first by showing that the increasing quasi semi (p, E)-convex functions (respectively, strictly increasing pseudo quasi semi (p, E)-convex) functions defined on $A \subseteq R$ are closed under addition and nonnegative scalar multiplication.

Proposition 3.1. Let $f, g: A \subseteq R \longrightarrow R$ are two increasing quasi semi (p, E)-convex functions on A. Then, $\alpha f + \beta g$ is increasing quasi semi (p, E)-convex function for all $\alpha, \beta \geq 0$.

Proof. Let $x, y \in A$ then either $x \leq y$ or $y \leq x$. If $x \leq y$ and f and g are increasing functions, then $f(x) \leq f(y)$ and $g(x) \leq g(y)$ which yield

$$max\{(\alpha f + \beta g)(x), (\alpha f + \beta g)(y)\} = (\alpha f + \beta g)(y) \qquad (2)$$

Hence, $\alpha f + \beta g$ is increasing function. Let $z = rEx + sEy \in A$. Then

$$\begin{aligned}(\alpha f + \beta g)(z) = \alpha f(z) + \beta g(z) &\leq \alpha\, max\{f(x), f(y)\} + \beta\, max\{g(x), g(y)\} \\ &= \alpha\, f(y) + \beta g(y) = (\alpha f + \beta g)(y) \\ &= max\{(\alpha f + \beta g)(x), (\alpha f + \beta g)(y)\},\end{aligned}$$

where the last conclusion follows from (2). Hence, $\alpha f + \beta g$ is increasing quasi semi (p, E)-convex function. If $y \leq x$, we proceed similarly to obtain the required conclusion. ■

Proposition 3.2. Let $f, g: A \subseteq R \longrightarrow R$ are strictly increasing pseudo semi (p, E)-convex on A.Then, for all $\alpha, \beta \geq 0$, $\alpha f + \beta g$ is strictly increasing pseudo semi (p, E)-convex on A.

Proof. From the definition of f and g we have, if $f(x) < f(y)$ and $g(x) < g(y)$ then then there exist $b_1, b_2: R^n \times R^n \to R$ such that

$$(\alpha f + \beta g)(x) < (\alpha f + \beta g)(y), \qquad (3)$$

Thus, $\alpha f + \beta g$ is strictly increasing. Let $z = rEx + sEy \in A$ then

$f(z) \leq f(y) + (-rs)\, b_1(x, y)$ and $g(z) \leq g(y) + (-rs)\, b_2(x, y)$.

Now, $$\begin{aligned}(\alpha f + \beta g)(z) &= \alpha\, f(z) + \beta g(z) \\ &\leq \alpha\left(f(y) + (-rs)b_1(x, y)\right) + \beta(g(y) + (-rs)b_2(x, y)) \\ &= (\alpha f + \beta g)(y) + (-rs)[b_1(x, y) + b_2(x, y)] \\ &= (\alpha f + \beta g)(y) + (-rs)\, b(x, y), \qquad (4)\end{aligned}$$

where $b(x, y) = b_1(x, y) + b_2(x, y)$. Since b_1 and b_2 are strictly positive functions, then $b(x, y)$ is strictly positive. From (3) and (4), we obtain the required conclusion. ■

Next, we show the supremum property of an arbitrary non-empty finite collection of quasi semi (p, E)-convex functions.

Proposition 3.3. Let $f_i: R \longrightarrow R$ be bounded from above increasing quasi semi (p, E)-convex functions for each $i \in \Lambda = \{1, \ldots, n\}$. Define, $f: R^n \longrightarrow R$ such that $f = sup_{i\in\Lambda} f_i$. Then f is quasi semi (p, E)-convex.

Proof. Let $x, y \in R$ such that $x \leq y$ and f_i is quasi semi (p, E)-convex for each $i \in \Lambda = \{1, \ldots, n\}$. Then, $f_i(rEx + sEy) \leq \{f_i(x), f_i(y)\} = f_i(y)$ for each $i \in \Lambda$. Applying the supremum for the both sides of the above inequality respectively, we get

$$sup_{i\in\Lambda} f_i(rEx + sEy) \leq sup_{i\in\Lambda}\{\{f_i(x), f_i(y)\}\},$$

$$f((rEx + sEy) \leq sup_{i\in\Lambda} f_i(y) = f(y) = max\{f(x), f(y)\}.$$

The last inequalities yields f is quasi semi (p, E)-convex. If $y \leq x$, we proceed similarly to obtain the required conclusion ■

Two composite properties are held for quasi semi (respectively, pseudo semi) (p, E)-convex functions as shown next.

Proposition 3.4. Let f:$A \subseteq R^n \longrightarrow R$ and G: $R \longrightarrow R$ is an increasing function. Then

i. If f is quasi semi (p, E)-$convex$ on A, then gof : $A \to R$ is quasi semi (p, E)-convex function.

ii. If f is pseudo semi (p, E)-$convex$ on A and G is sublinear and strictly positive, then gof : $A \to R$ is pseudo semi (p, E)-convex function.

Proof. Let us show (i). Let $x, y \in A$ and f is quasi semi (p, E)-convex on A, then $rEx + sEy \in A$ and $f(rEx + sEy) \leq max\{f(x), f(y)\}$. Since G is an increasing function then, $G(f(rEx + sEy) \leq max\{f(x), f(y)\})$. That is,

$(Gof)(rEx + sEy) \leq max\{G(f(x)), G(f(y))\} = max\{ (Gof)(x), (Gof)(y\}$.

Hence, Gof is quasi semi (p, E)-convex on A. For proving (ii), if $f(x) < f(y)$ then $f(rEx + sEy) \leq f(y) + (-rs)b(x, y)$. Since G is an increasing function, then, using the last expression, if $(Gof)(x) < (Gof)(y)$ we get

$$(Gof)(rEx + sEy) \leq G[f(y) + (-rs)b(x, y)].$$

From the assumption, G is a sublinear mapping. Thus, the last inequality yields,

$$(Gof)(rEx + sEy) \leq (Gof)(y) + (-rs)(Gob)(x, y).$$

Since G and b are strictly positive functions, then $(Gob)(x, y)$ is strictly positive. Hence, we obtain the required conclusion. ■

Proposition 3.5. Let g: $A \subseteq R^n \to R^n$ be a linear mapping such that $Eog = goE$. Assume also that f: $V \subseteq R^n \to R$ such that $V = g(A)$. Then

i. If f quasi semi (p, E)-convex function, then fog: $A \to R$ is quasi semi (p, E)-convex function.

ii. If f pseudo semi (p, E)-convex function, then fog: $A \to R$ is pseudo semi (p, E)-convex function.

Proof. Let $x, y \in A$ then $rEx + s\,Ey \in A$. For proving (i), we need to show that

$$(fog)(rEx + s\,Ey) \leq max\{f(g(x)), f(g(y))\}.$$

Now, from the linearity of g and the fact that $Eog = goE$, we have

$$\begin{aligned}(fog)(rEx + s\,Ey) &= f\big(r\,g(Ex) + s\,g(Ey)\big)\\ &= f(r\,(goE)(x) + s(goE)(y))\\ &= f(r(Eog)(x) + s(Eog)(y))\\ &= f\Big(r\,\big(E\big(g(x)\big)\big) + s\,\big(E\big(g(y)\big)\big)\Big). \qquad (5)\end{aligned}$$

Note that, since $x, y \in A$ then $g(x), g(y) \in V = g(A)$. Also, we have $V = g(A)$ is (p, E)-convex set (see [9, Proposition 2.11]) thus

$$r\,\big(E\big(g(x)\big)\big) + s\,\big(E\big(g(y)\big)\big) \in V. \qquad (6)$$

From (5)-(6) and the fact that f is quasi semi (p, E)-convex function, we get

$$(fog)(rEx + s\,Ey) \le max\{f(g(x)), f(g(y))\} = max\{(fog)(x), (fog)(y)\} .$$

Hence, fog is quasi semi (p, E)-convex. Let us prove (ii), from the assumption $x, y \in A$, $g(x), g(y) \in V = g(A)$, and f is pseudo semi (p, E)-convex function. Hence,

$$f(g(x)) < f(g(y)) \qquad (7)$$

Again, we follow the steps of the proof of (i) to obtain the equality (5). Namely,

$$(fog)(rEx + s\,Ey) = f\left(r\,\left(E(g(x))\right) + s\left(E(g(y))\right)\right)$$

From definition of f there exist strictly positive $b: R^n \times R^n \to R$ such that the right-hand side expression above yields

$$\le f(g(y)) + (-rs)\,b(x, y) = (fog)(y) + (-rs)b(x, y) \qquad (8)$$

Thus, from (7) and (8), fog is pseudo semi (p, E)-convex function. ■

4. Applications to Non-Linear Optimization Programming

In this section, a non-linear optimization programming problem denoted by (NLP) and is defined as

$$f(x)$$
$$\text{subject to } x \in A,$$

where A is (p, E)-convex. The set of all optimal solutions (or global minimum) of problem (NLP) is defined as $argmin_A\, f = \{x^* \in A: f(x^*) \le f(x) \text{ for all } \in A\ \}$. A point x^* is called a local minimizer for problem (NLP) if there exists $\delta > 0$ such that $f(x^*) \le f(x)$ for all $x \in B(x^*, \delta) \cap A$ where $B_\delta(x^*)$ is an open ball.

The following optimality properties are satisfied under different conditions for the objective function f and the mapping E.

Proposition 4.1. Let f is pseudo semi (p, E)-convex function. Then, every local minimum $x^* = E(x^*) \in E(A)$ of problem (NLP) is a global minimum.

Proof. Suppose $x^* = E(x^*)$ is not global minimum, then there exists $u \in A$ with $f(u) < f(x^*) = f(Ex^*)$. From the assumptions on f, we have for all $r, s \in [0,1]$ with $r^p + s^p = 1\left(i.e., s = (1 - r^p)^{\frac{1}{p}}\right)$, if $f(u) < f(x^*)$. then , we have $f(rEu + sEx^*) \le f(x^*) + (-rs)b(x^*, u)$. Now, if $f(x^*) \le 0$ or $f(x^*) \ge 0$. then we have

$$f(rEu + sEx^*) \le f(x^*) + (-rs)b(x^*, u) \le f(x^*). \qquad (9)$$

Now, for sufficiently small, $r \in (0,1]$) then $rEu + (1 - r^p)^{\frac{1}{p}}x^*$ will be close enough to x^*. i.e., there exists $\delta > 0$ such that $rEu + (1 - r^p)^{\frac{1}{p}}x^* \in B_\delta(x^*) \cap A$. From the local minimality of x^*, one obtains $f(x^*) \le f(rEu + (1 - r^p)^{\frac{1}{p}}x^*)$ which contradicts (9). Thus, x^* is a global minimum. ■

Proposition 4.2. let f is quasi semi (p, E)-convex function. Then every local minimum $x^* = E(x^*) \in E(A)$ of problem (NLP) is a global minimum.

Proof. Suppose $x^* = E(x^*)$ is not global minimum, then there exists $u \in A$ with $f(u) < f(x^*) = f(Ex^*)$. From the assumptions on f, we have for all $r, s \in [0,1]$ with $r^p + s^p = 1$ $\left(i.e., s = (1 - r^p)^{\frac{1}{p}}\right)$. Since $f(u) < f(x^*)$ and f is quasi semi (p, E)-convex then we have

$$f(rEu + sEx^*) \leq max\{f(u), f(x^*)\} = f(x^*) \tag{10}$$

Now, for sufficiently small, $r \in (0,1]$) then $rEu + (1 - r^p)^{\frac{1}{p}} x^*$ will be close enough to x^*. i.e., there exists $\delta > 0$ such that $r\varepsilon u + (1 - r^p)^{\frac{1}{p}} x^* \in B_\delta(x^*) \cap A$. From the local minimality of x^*, we have $f(x^*) \leq f(rEu + (1 - r^p)^{\frac{1}{p}} x^*)$ which contradicts (10). Thus, x^* is a global minimum. ■

Remark 4.3. The conclusions of Propositions 4.1 and 4.2 do not hold if the objective function f is not quasi semi (p, E)-convex (respectively, not pseudo semi (p, E)-convex) function as the following example confirms.

Example 4.4. Consider the optimization problem

$$f(x, y)$$
$$\text{such that } (x, y) \in A,$$

where $A = \{(x, y) \in R^2 : |x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} \leq 4\}$ and $f: R^2 \to R$ such that

$$f(x, y) = \begin{cases} (y-1)^2 & -2 \leq x \leq 2 \\ (y-1)^2 - (2-x)^2 & \text{o.w.} \end{cases}$$

Define $E: R^2 \to R^2$ as $E(x.y) = (0, y)$. Then A is $\left(\frac{1}{2}, E\right)$-convex set, f is not quasi semi $(\frac{1}{2}, E)$-convex and not pseudo semi $(\frac{1}{2}, E)$-convex on A. To show A is $(\frac{1}{2}, E)$-convex set, let $(x_1, y_1), (x_2, y_2) \in A$. Then, $|x_1|^{\frac{1}{2}} + |y_1|^{\frac{1}{2}} \leq 4$ and $|x_2|^{\frac{1}{2}} + |y_2|^{\frac{1}{2}} \leq 4$. Now, $rE(x_1, y_1) + sE(x_2, y_2) = (0, ry_2 + sy_2)$. Note that $|0|^{\frac{1}{2}} + |ry_1 + sy_2|^{\frac{1}{2}} \leq r|y_1|^{\frac{1}{2}} + s|y_2|^{\frac{1}{2}} \leq 4(r + s) \leq 4$. Hence, A is $(\frac{1}{2}, E)$-convex set. Next, we show that f is not quasi semi $(\frac{1}{2}, E)$- convex on A. Let $(2,1), (-2,1) \in A$, and $r = s = \frac{1}{4}$, $p = \frac{1}{2}$. Then

$$f\left(\frac{1}{4} E(2,1) + \frac{1}{4} E(-2,1)\right) = f\left(0, \frac{1}{2}\right) = \frac{1}{4} > max\{f(2,1), f(-2,1)\} = 0.$$

Also, f is not pseudo semi $(\frac{1}{2}, E)$- convex on A. To this end, let $x = \left(2, \frac{1}{2}\right)$, $y = (2,1) \in A$ and $r = s = \frac{1}{4}$, $p = \frac{1}{2}$ such that $f(x) = \frac{1}{4} < f(y) = 0$. Then,

$$f\left(\frac{1}{4} E\left(2, \frac{1}{2}\right) + \frac{1}{4} E(2,1)\right) = \left(\frac{1}{8} + \frac{1}{4} - 1\right)^2 = \frac{25}{64} > f(2,1) - \frac{1}{16} b(x, y)$$
$$= -\frac{1}{16} b(x, y),$$

for any strictly positive function $b(x, y)$. Thus, f is not pseudo semi $(\frac{1}{2}, E)$- convex on A as claimed. Now, take $x_0 = (0,1) \in A$ such that $E(0,1) = (0,1)$ and $f(0,1) = 0 \leq f(x, y)$ for all

$(x, y) \in A \cap [-2,2] \times R$. Hence, $x_0 = (0,1)$ is a local minimum. However, the conclusion of Propositions 4.1 and 4.2 does not satisfy, i.e., x_0 is not a global minimum. Indeed, take $= (3,1) \in A$. then
$f(3,1) = (1-1)^2 - (2-3)^2 = -1 < f(x_0) = 0.$

Proposition 4.5. Let f is strictly quasi semi (p, E)- convex on A. Then, the global minimum of problem (NLP) is singleton.
Proof. Let x^*, y^* be two different global minima of (NLP) then $f(x^*) = f(y^*) \leq f(x)$ for any $x \in A$. From the assumptions on f and A, we have $rEx^* + sEy^* \in A$ and

$$f(rEx^* + sEy^*) < max\{f(x^*), f(y^*)\} = f(x^*).$$

The above inequality yields that $rEx^* + sEy^*$ is a global minimum which is a contradiction. Hence, there is a unique global minimum. ■

Proposition 4.6. Let f is quasi semi (p, E)- convex on A. Then, the set of global minima of problem (NLP) is (p, E)- convex.
Proof. Let $x_1^*, x_2^* \in argmin_A f = \{x^* \in A: f(x^*) \leq f(x) \quad \forall x \in A\}$ the set of global minima of problem (NLP) we must prove $rEx_1^* + sEx_2^* \in argmin_A f$. since f is quasi semi (p, E)-convex on the (p, E)-convex set A, we have $rEx_1^* + sEx_2^* \in A$ and for each $x \in A$,
$f(rEx_1^* + sEx_2^*) \leq max\{f(x_1^*), f(x_2^*)\} \leq f(x)$. Therefore, $rEx_1^* + sEx_2^* \in argmin_A f$ as required. ■

5.Conclusion

In this paper, new generalized convex functions (quasi semi (p, E)-convex, and pseudo semi (p, E)-convex functions) are defined, and their various general and optimality properties are studied. These functions are a combination of p-convex and E-convex functions introduced in the literature. Different examples are established to illustrate these functions and to confirm some properties proved throughout the work.

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