IHJPAS. 35(4)2022

Ibn Al-Haitham Journal for Pure and Applied Sciences

Journal homepage: http://jih.uobaghdad.edu.iq/index.php/j/index

On the Double of the Emad - Falih Transformation and Its Properties with Applications

Saed M. Turq
Teacher at the Ministry of Education
Hebron, Palestine.
saedturq@gmail.com

Emad A. Kuffi
Al-Qadisiyah University, College of Engineering, Al-Qadisiyah, Iraq.
emad.abbas@qu.edu.iq

Article history: Received 20 July 2022, Accepted 21 August 2022, Published in October 2022.

Doi: 10.30526/35.4.2938

Abstract

In this paper, we have generalized the concept of one dimensional Emad - Falih integral transform into two dimensional, namely, a double Emad - Falih integral transform. Further, some main properties and theorems related to the double Emad - Falih transform are established. To show the proposed transform's efficiency, high accuracy, and applicability, we have implemented the new integral transform for solving partial differential equations. Many researchers have used double integral transformations in solving partial differential equations and their applications. One of the most important uses of double integral transformations is how to solve partial differential equations and turning them into simple algebraic ones. The most important partial differential equations are Laplace, Poisson, wave, heat, telegraph, and other equations. A new Double Emad -Falih integral transform denoted by the operator $\mathbf{D}_{EF}\{.\}$, the transform form is as follows: $\mathbf{D}_{EF}[f(x,t)] = \mathbf{T}(u,v) = \frac{1}{uv}\int_0^\infty \int_0^\infty f(x,t)e^{-(u^2x+v^2t)}dtdx$

Keywords: Emad - Falih Transform, Partial Differential Equation, Double Integral Transform.

1. Introduction

An integral transformation maps a function from its main function space into a new function space via integration, with properties of the main function that might be more characterized and manipulated than in the main function space. The transformed function can generally be mapped back to the main function space by applying the inverse integral transformation. Many researchers have applied double transforms for solving partial differential equations and their applications [1-4].

Many applications and problems in most applied nature science and engineering fields encounter double integral transforms or partial differential equations describing the physical phenomena. Solving such equations using single transforms is more complicated than applying the double integral transformation.

There are many types of integral transformations, such as: Fourier transformation [5], Laplace transformation [6], Sumudu transformation [7], Natural transformation [8], Elzaki tranformation [9] ,and so on. These types of transformations have wide variety applications in various areas in applied mathematics, physics, engineering, and in most of other sciences [10].

Emad - Falih integral transformation of a single variable is a new integral transform which has recently been introduced by Emad A. Kuffi and Sara F. in (2021) [11].

2. The Double Emad - Falih Integral Transform

Definition 2.1. Emad- Falih Integral Transform:

The Emad- Falih Integral Transform of $g(t)$ is defined as:

$$\mathbf{EF}[g(t)] = \mathbf{T}(v) = \frac{1}{v}\int_0^\infty g(t)e^{-v^2t}dt$$

3. The Double Emad- Falih Integral Transform of Some Famous Functions

This Section introduces the new double Emad- Falih integral transform of some famous functions:

1. Let $f(x,t) = 1$, then

$$\begin{aligned}\mathbf{D}_{EF}[1] &= \frac{1}{uv}\int_0^\infty\int_0^\infty 1e^{-(u^\alpha x+v^\alpha t)}dtdx\\ &= \frac{1}{u}\int_0^\infty 1e^{-u^2x}dx\cdot\frac{1}{v}\int_0^\infty 1e^{-v^2t}dt\\ &= \mathbf{EF}[1]\cdot\mathbf{EF}[1],\\ &= \frac{1}{u^3}\cdot\frac{1}{v^3}\\ &= \frac{1}{(uv)^3}\end{aligned}$$

2 Let $f(x,t) = x^n t^m$, then

$$\begin{aligned}\mathbf{D}_{EF}[x^n t^m] &= \frac{1}{uv}\int_0^\infty\int_0^\infty x^n t^m e^{-(u^2x+v^2t)}dtdx\\ &= \frac{1}{u}\int_0^\infty x^n e^{-u^2x}dx\cdot\frac{1}{v}\int_0^\infty t^m e^{-v^2t}dt\\ &= \mathbf{EF}[x^n]\cdot\mathbf{EF}[t^n]\\ &= \frac{n!}{u^{2n+3}}\cdot\frac{m!}{v^{2m+3}}\end{aligned}$$

3 Let $f(x,t) = e^{ax+bt}$, then

$$\mathbf{D}_{EF}[e^{ax+bt}] = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{ax+bt} e^{-(u^2x+v^2t)} dt dx$$
$$= \frac{1}{u}\int_0^\infty e^{ax} e^{-u^2x} dx \cdot \frac{1}{v}\int_0^\infty e^{bt} e^{-v^2t} dt$$
$$= \mathbf{EF}[e^{ax}] \cdot \mathbf{EF}[e^{bt}]$$
$$= \frac{1}{u(u^2-a)} \cdot \frac{1}{v(v^2-b)}$$
$$= \frac{1}{uv(u^2-a)(v^2-b)}$$

4 Let $f(x,t) = e^{-(ax+bt)}$, then

$$\mathbf{D}_{EF}\left[e^{-(ax+bt)}\right] = \frac{1}{(uv)}\int_0^\infty \int_0^\infty e^{-(ax+bt)} e^{-(u^2x+v^2t)} dt dx$$
$$= \frac{1}{u}\int_0^\infty e^{-ax} e^{-u^2x} dx \cdot \frac{1}{v}\int_0^\infty e^{-bt} e^{-v^2t} dt$$
$$= \mathbf{EF}[e^{-ax}] \cdot \mathbf{EF}[e^{-bt}]$$
$$= \frac{1}{u(u^2+a)} \cdot \frac{1}{v(v^2+b)}$$
$$= \frac{1}{uv(u^2+a)(v^2+b)}$$

5 Let $f(x,t) = e^{i(ax+bt)}$, then

$$\mathbf{D}_{EF}\left[e^{i(ax+bt)}\right] = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{i(ax+bt)} e^{-(u^2x+v^2t)} dt dx$$
$$= \frac{1}{u}\int_0^\infty e^{-x(u^2-ia)} dx \cdot \frac{1}{v}\int_0^\infty e^{-t(v^2-ib)} dt$$
$$= \frac{1}{u}\frac{1}{-(u^2-ia)} e^{-x(u^2-ia)}\Big|_0^\infty \frac{1}{v}\frac{1}{-(v^2-ib)} e^{-t(v^2-ib)}\Big|_0^\infty$$
$$= \frac{1}{u}\frac{1}{-(u^2-ia)}[0-1]\frac{1}{v}\frac{1}{-(v^2-ib)}[0-1]$$
$$= \frac{1}{u(u^2-ia)}\frac{1}{v(v^2-ib)},$$
$$= \frac{1}{uv(u^2-ia)(v^2-ib)}.$$

6 Let $f(x,t) = e^{-i(ax+bt)}$, then

$$\mathbf{D}_{EF}\left[e^{-i(ax+bt)}\right] = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-i(ax+bt)} e^{-(u^2x+v^2t)} dt dx$$
$$= \frac{1}{u}\int_0^\infty e^{-x(u^2+ia)} dx \cdot \frac{1}{v}\int_0^\infty e^{-t(v^2+ib)} dt$$
$$= \frac{1}{u(u^2+ia)}\frac{1}{v(v^2+ib)}$$
$$= \frac{1}{uv(u^2+ia)(v^2+ib)}$$

7 Let $f(x,t)=\sin(ax+bt)$, then

$$\mathbf{D}_{EF}[\sin(ax+bt)] = \frac{1}{uv}\int_0^\infty\int_0^\infty \sin(ax+bt)e^{-(u^2x+v^2t)}dtdx$$

$$= \frac{1}{uv}\int_0^\infty\int_0^\infty \left[\frac{e^{i(ax+bt)}-e^{-i(ax+bt)}}{2i}\right]e^{-(u^2x+v^2t)}dtdx$$

$$= \frac{1}{2i}\left[\begin{array}{l}\frac{1}{uv}\int_0^\infty\int_0^\infty e^{i(ax+bt)}e^{-(u^2x+v^2t)}dtdx \\ -\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-i(ax+bt)}e^{-(u^2x+v^2t)}dtdx\end{array}\right]$$

$$= \frac{1}{2i}\left[\mathbf{D}_{EF}\left[e^{i(ax+bt)}\right]-\mathbf{D}_{EF}\left[e^{-i(ax+bt)}\right]\right]$$

$$= \frac{1}{2i}\left[\frac{1}{uv(u^\alpha-ia)(v^2-ib)}-\frac{1}{uv(u^2+ia)(v^2+b)}\right]$$

$$= \frac{bu^2+av^2}{uv(u^4+a^2)(v^4+b^2)}.$$

8 Let $f(x,t)=\cos(ax+bt)$, then

$$\mathbf{D}_{EF}[\cos(ax+bt)] = \frac{1}{uv}\int_0^\infty\int_0^\infty \cos(ax+bt)e^{-(u^2x+v^2t)}dtdx$$

$$= \frac{1}{uv}\int_0^\infty\int_0^\infty \left[\frac{e^{i(ax+bt)}+e^{-i(ax+bt)}}{2}\right]e^{-(u^2x+v^2t)}dtdx$$

$$= \frac{1}{2}\left[\begin{array}{l}\frac{1}{uv}\int_0^\infty\int_0^\infty e^{i(ax+bt)}e^{-(u^2x+v^2t)}dtdx \\ +\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-i(ax+bt)}e^{-(u^2x+v^2t)}dtdx\end{array}\right]$$

$$= \frac{1}{2}\left[\mathbf{D}_{EF}\left[e^{i(ax+bt)}\right]+\mathbf{D}_{EF}\left[e^{-i(ax+bt)}\right]\right]$$

$$= \frac{1}{2}\left[\frac{1}{uv(u^\alpha-ia)(v^2-ib)}+\frac{1}{uv(u^2+ia)(v^2+b)}\right]$$

$$= \frac{u^2v^2-ab}{uv(u^4+a^2)(v^4+b^2)}.$$

9 Let $f(x,t)=\sinh(ax+bt)$, then

$$\mathbf{D}_{EF}[\sinh(ax+bt)] = \frac{1}{uv}\int_0^\infty\int_0^\infty \sinh(ax+bt)e^{-(u^2x+v^2t)}dtdx$$
$$= \frac{1}{uv}\int_0^\infty\int_0^\infty \left[\frac{e^{(ax+bt)}-e^{-(ax+bt)}}{2}\right]e^{-(u^2x+v^2t)}dtdx$$
$$= \frac{1}{2}\left[\begin{array}{l}\frac{1}{uv}\int_0^\infty\int_0^\infty e^{(ax+bt)}e^{-(u^2x+v^2t)}dtdx \\ -\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-(ax+bt)}e^{-(u^2x+v^2t)}dtdx\end{array}\right]$$
$$= \frac{1}{2}\left[\mathbf{D}_{EF}\left[e^{(ax+bt)}\right]-\mathbf{D}_{EF}\left[e^{-(ax+bt)}\right]\right]$$
$$= \frac{1}{2}\left[\frac{1}{uv(u^2-a)(v^2-b)}-\frac{1}{uv(u^2+a)(v^2+b)}\right]$$
$$= \frac{av^2+bu^2}{uv(u^4-a^2)(v^4-b^2)}$$

10 Let $f(x,t)=\cosh(ax+bt)$, then

$$\mathbf{D}_{EF}[\cosh(ax+bt)] = \frac{1}{uv}\int_0^\infty\int_0^\infty \cosh(ax+bt)e^{-(u^2x+v^2t)}dtdx$$
$$= \frac{1}{uv}\int_0^\infty\int_0^\infty \left[\frac{e^{(ax+bt)}+e^{-(ax+bt)}}{2}\right]e^{-(u^2x+v^2t)}dtdx$$
$$= \frac{1}{2}\left[\begin{array}{l}\frac{1}{uv}\int_0^\infty\int_0^\infty e^{(ax+bt)}e^{-(u^2x+v^2t)}dtdx \\ +\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-(ax+bt)}e^{-(u^2x+v^2t)}dtdx\end{array}\right]$$
$$= \frac{1}{2}\left[\mathbf{D}_{EF}\left[e^{(ax+bt)}\right]+\mathbf{D}_{EF}\left[e^{-(ax+bt)}\right]\right]$$
$$= \frac{1}{2}\left[\frac{1}{uv(u^2-a)(v^2-b)}+\frac{1}{uv(u^2+a)(v^2+b)}\right]$$
$$= \frac{u^2v^2+ab}{uv(u^4-a^2)(v^4-b^2)}.$$

4. Summarization

The new double Emad- Falih integral transform of some basic functions is shown in the following table:

Table 1: The new double Emad - Falih integral transform of some basic functions.

$f(x,t)$	$\mathbf{D}_{EF}[\text{Cosh}(ax+bt)]$
1	$\frac{1}{(uv)^3}$
x^nt^m	$\frac{n!}{u^{2n+3}}\cdot\frac{m!}{v^{2m+3}}$
e^{ax+bt}	$\frac{1}{uv(u^2-a)(v^2-b)}$
$e^{-(ax+bt)}$	$\frac{1}{uv(u^2+a)(v^2+b)}$
$e^{i(ax+bt)}$	$\frac{1}{uv(u^2-ia)(v^2-ib)}$

$e^{-i(ax+bt)}$	$\frac{1}{uv(u^2+ia)(v^2+ib)}$
$\mathrm{Sin}(ax+bt)$	$\frac{bu^2+av^2}{uv(u^4+a^2)(v^4+b^2)}$
$\mathrm{Cos}(ax+bt)$	$\frac{u^2v^2-ab}{uv(u^4+a^2)(v^4+b^2)}$
$\mathrm{Sinh}(ax+bt)$	$\frac{av^2+bu^2}{uv(u^4-a^2)(v^4-b^2)}$
$\mathrm{Cosh}(ax+bt)$	$\frac{u^2v^2+ab}{uv(u^4-a^2)(v^4-b^2)}$

5. Theorems And Proof

Theorem 5.1.

$$\mathbf{D}_{EF}\left[\frac{\partial f(x,t)}{\partial x}\right] = -\frac{1}{u}\mathbf{T}(0,v) + u^2\mathbf{T}(u,v)$$

Proof.

$$\begin{aligned}
\mathbf{D}_{EF}\left[\frac{\partial f(x,t)}{\partial x}\right] &= \frac{1}{uv}\int_0^\infty \int_0^\infty \frac{\partial f(x,t)}{\partial x} e^{-(u^\alpha x+v^\alpha t)}dtdx \\
&= \frac{1}{v}\int_0^\infty e^{-v^2t}\left[\frac{1}{u}\underbrace{\int_0^\infty \frac{\partial f(x,t)}{\partial x}e^{-u^2x}dx}_{Integrating\ by\ parts}\right]dt \\
&= \frac{1}{v}\int_0^\infty e^{-v^2t}\left[\frac{1}{u}\left(e^{-u^2x}f(x,t)\Big|_0^\infty + u^2\int_0^\infty f(x,t)e^{-u^2x}dx\right)\right]dt \\
&= \frac{1}{v}\int_0^\infty e^{-v^2t}\left[-\frac{f(0,t)}{u} + u\int_0^\infty f(x,t)e^{-u^2x}dx\right]dt, \\
&= -\frac{1}{u}\left[\frac{1}{v}\int_0^\infty e^{-v^2t}f(0,t)dt\right] + u^2\left[\frac{1}{uv}\int_0^\infty \int_0^\infty f(x,t)e^{-(u^2x+v^2t)}dtdx\right] \\
&= -\frac{1}{u}\mathbf{T}(0,v) + u^2\mathbf{T}(u,v).
\end{aligned}$$

Theorem 5.2.

$$\mathbf{D}_{EF}\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = -\frac{1}{u}\frac{\partial\mathbf{T}(0,v)}{\partial x} - u\mathbf{T}(0,v) + u^4\mathbf{T}(u,v)$$

Proof.

$$\mathbf{D}_{EF}\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right]=\frac{1}{uv}\int_0^\infty\int_0^\infty\frac{\partial^2 f(x,t)}{\partial x^2}e^{-(u^2x+v^2t)}dtdx$$

$$=\frac{1}{v}\int_0^\infty e^{-v^2t}[\underbrace{\frac{1}{u}\int_0^\infty\frac{\partial^2 f(x,t)}{\partial x^2}e^{-u^2x}dx}_{\text{Integration by parts}}]dt,$$

Let $\zeta=e^{-u^2x}\ d\eta=\frac{\partial^2 f(x,t)}{\partial x^2}dx$

$d\zeta=-u^2e^{-u^2x}\ \eta=\frac{\partial f(x,t)}{\partial x}$

$$=\frac{1}{v}\int_0^\infty e^{-v^2t}\left[\frac{1}{u}\left(e^{-u^2x}\frac{\partial f(x,t)}{\partial x}\Big|_0^\infty+u^2\int_0^\infty\frac{\partial f(x,t)}{\partial x}e^{-u^2x}dx\right)\right]dt,$$

$$=\frac{1}{v}\int_0^\infty e^{-v^2t}\left[\frac{1}{u}e^{-u^2x}\frac{\partial f(x,t)}{\partial x}\Big|_0^\infty+u\int_0^\infty\frac{\partial f(x,t)}{\partial x}e^{-u^2x}dx\right]dt$$

$$=-\frac{1}{u}\left[\frac{1}{v}\int_0^\infty e^{-v^2t}\frac{\partial f(0,t)}{\partial x}dt\right]+(u)^2\left[\frac{1}{uv}\int_0^\infty\int_0^\infty\frac{\partial f(x,t)}{\partial x}\right)e^{-(u^2x+v^2t)}dtdx\right],$$

$$=-\frac{1}{u}\frac{\partial\mathbf{T}(0,v)}{\partial x}+u^2\mathbf{D}_{EF}\left[\frac{\partial f(x,t)}{\partial x}\right].$$

$$=-\frac{1}{u}\frac{\partial\mathbf{T}(0,v)}{\partial x}+u^2\left[-\frac{1}{u}\mathbf{T}(0,v)+u^2\mathbf{T}(u,v)\right];$$

$$=-\frac{1}{u}\frac{\partial\mathbf{T}(0,v)}{\partial x}-u\mathbf{T}(0,v)+u^4\mathbf{T}(u,v).$$

In general

$$\mathbf{D}_{EF}\left[\frac{\partial^n f(x,t)}{\partial x^n}\right]=u^{2n}\mathbf{T}(u,v)-\left[u^{-1}\frac{\partial^{n-1}\mathbf{T}(0,v)}{\partial x^{n-1}}+u^1\frac{\partial^{n-2}\mathbf{T}(0,v)}{\partial x^{n-2}}+u^3\frac{\partial^{n-3}\mathbf{T}(0,v)}{\partial x^{n-3}}+\cdots+u^{2(n-1)-3}\frac{\partial\mathbf{T}(0,v)}{\partial x}+u^{2n-3}\mathbf{T}(0,v)\right]$$

$$\text{or }=u^{2n}\mathbf{T}(u,v)-\sum_{k=1}^{n}u^{2k-3}\frac{\partial^{n-k}\mathbf{T}(0,v)}{\partial x^{n-k}}$$

Theorem 5.3.

$$\mathbf{D}_{EF}\left[\frac{\partial f(x,t)}{\partial t}\right]=-\frac{1}{v}\mathbf{T}(u,0)+v^2\mathbf{T}(u,v)$$

Proof.

$$\mathbf{D}_{EF}\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{1}{uv}\int_0^\infty \int_0^\infty \frac{\partial f(x,t)}{\partial t} e^{-(u^\alpha x+v^\alpha t)} dtdx$$

$$= \frac{1}{u}\int_0^\infty e^{-u^2x} \underbrace{[\frac{1}{v}\int_0^\infty \frac{\partial f(x,t)}{\partial t} e^{-v^2t}dt\Big] dx}_{\text{Integration by parts}}$$

$$= \frac{1}{u}\int_0^\infty e^{-u^2x}\left[\frac{1}{v}\left(e^{-v^2t}f(x,t)\Big|_0^\infty + v^2\int_0^\infty f(x,t)e^{-v^2t}dt\right)\right]dx$$

$$= \frac{1}{u}\int_0^\infty e^{-u^2x}\left[-\frac{f(x,0)}{v} + v\int_0^\infty f(x,t)e^{-v^2t}dt\right]dx$$

$$= -\frac{1}{v}\left[\frac{1}{u}\int_0^\infty e^{-u^2x}f(x,0)dx\right] + v^2\left[\frac{1}{uv}\int_0^\infty \int_0^\infty f(x,t)e^{-(u^2x+v^2t)}dtdx\right]$$

$$= -\frac{1}{v}\mathbf{T}(u,0) + v^2\mathbf{T}(u,v).$$

Theorem 5.4.

$$\mathbf{D}_{EF}\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = -\frac{1}{v}\frac{\partial \mathbf{T}(u,0)}{\partial t} - v\mathbf{T}(0,v) + v^4\mathbf{T}(u,v)$$

Proof.

$$\mathbf{D}_{EF}\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{1}{uv}\int_0^\infty \int_0^\infty \frac{\partial^2 f(x,t)}{\partial t^2} e^{-(u^2x+v^2t)}dtdx$$

$$= \frac{1}{u}\int_0^\infty e^{-u^2x}\underbrace{[\frac{1}{v}\int_0^\infty \frac{\partial^2 f(x,t)}{\partial t^2}e^{-v^2t}dt]}_{\text{Integration by parts}}dx,$$

Let $\zeta = e^{-v^2t}$ $d\eta = \dfrac{\partial^2 f(x,t)}{\partial t^2}dt$

$d\zeta = -v^2e^{-v^2t}$ $\eta = \dfrac{\partial f(x,t)}{\partial t}$

$$= \frac{1}{u}\int_0^\infty e^{-u^2x}\left[\frac{1}{v}\left(e^{-v^2t}\frac{\partial f(x,t)}{\partial t}\Big|_0^\infty + v^2\int_0^\infty \frac{\partial f(x,t)}{\partial t}e^{-v^2t}dt\right)\right]dx$$

$$= \frac{1}{u}\int_0^\infty e^{-u^2x}\left[\frac{1}{v}e^{-v^2t}\frac{\partial f(x,t)}{\partial t}\Big|_0^\infty + v\int_0^\infty \frac{\partial f(x,t)}{\partial t}e^{-v^2t}dt\right]dx,$$

$$= -\frac{1}{v}\left[\frac{1}{u}\int_0^\infty e^{-u^2x}\frac{\partial f(x,0)}{\partial t}dt\right] + (v)^2\left[\frac{1}{uv}\int_0^\infty \int_0^\infty \frac{\partial f(x,t)}{\partial t}e^{-(u^2x+v^2t)}dtdx\right],$$

$$= -\frac{1}{v}\frac{\partial \mathbf{T}(u,0)}{\partial t} + v^2\mathbf{D}_{EF}\left[\frac{\partial f(x,t)}{\partial t}\right].$$

$$= -\frac{1}{v}\frac{\partial \mathbf{T}(u,0)}{\partial t} + v^2\left[-\frac{1}{v}\mathbf{T}(u,0) + v^2\mathbf{T}(u,v)\right];$$

$$= -\frac{1}{v}\frac{\partial \mathbf{T}(u,0)}{\partial t} - v\mathbf{T}(0,v) + v^4\mathbf{T}(u,v).$$

In general

$$\mathbf{D}_{EF}\left[\frac{\partial^n f(x,t)}{\partial t^n}\right] = v^{2n}\mathbf{T}(u,v) - \left[v^{-1}\frac{\partial^{n-1}\mathbf{T}(u,0)}{\partial t^{n-1}} + v^{1}\frac{\partial^{n-2}\mathbf{T}(u,0)}{\partial t^{n-2}} + v^{3}\frac{\partial^{n-3}\mathbf{T}(u,0)}{\partial t^{n-3}} + \cdots + v^{2(n-1)-3}\frac{\partial \mathbf{T}(u,0)}{\partial t} + u^{2n-3}\mathbf{T}(u,0)\right],$$

$$\text{or } = v^{2n}\mathbf{T}(u,v) - \sum_{k=1}^{n} v^{2k-3}\frac{\partial^{n-k}\mathbf{T}(u,0)}{\partial t^{n-k}}$$

Theorem 5. 5.

$$\mathbf{D}_{EF}\left[\frac{\partial^2 f(x,t)}{\partial t\,\partial x}\right] = -\frac{1}{v}\frac{\partial \mathbf{T}(u,0)}{\partial x} - \frac{v^2}{u}\mathbf{T}(0,v) + v^2u^2\mathbf{T}(u,v)$$

Proof.

$$\mathbf{D}_{EF}\left[\frac{\partial^2 f(x,t)}{\partial t\,\partial x}\right] = \frac{1}{uv}\int_0^\infty\int_0^\infty \frac{\partial^2 f(x,t)}{\partial t\,\partial x}e^{-(u^2x+v^2t)}dtdx,$$

$$= \frac{1}{u}\int_0^\infty e^{-u^2x}[\underbrace{\frac{1}{v}\int_0^\infty \frac{\partial^2 f(x,t)}{\partial t\,\partial x}e^{-v^2t}dt}_{\text{Integration by parts}}]dx,$$

$$\text{Let } \zeta = e^{-v^2t}\ d\eta = \frac{\partial^2 f(x,t)}{\partial t\,\partial x}dt$$

$$d\zeta = -v^2e^{-v^2t}\ \eta = \frac{\partial f(x,t)}{\partial x}$$

$$= \frac{1}{u}\int_0^\infty e^{-u^2x}\left[\frac{1}{v}\left(e^{-v^2t}\frac{\partial f(x,t)}{\partial x}\bigg|_0^\infty + v^2\int_0^\infty \frac{\partial f(x,t)}{\partial x}e^{-v^2t}dt\right)\right]dx,$$

$$= \frac{1}{u}\int_0^\infty e^{-u^2x}\left[\frac{1}{v}e^{-v^2t}\frac{\partial f(x,t)}{\partial x}\bigg|_0^\infty + v\int_0^\infty \frac{\partial f(x,t)}{\partial x}e^{-v^2t}dt\right]dx,$$

$$= \frac{1}{u}\int_0^\infty e^{-u^2x}\left[-\frac{1}{v}\frac{\partial f(x,0)}{\partial x} + v\int_0^\infty \frac{\partial f(x,t)}{\partial x}e^{-v^2t}dt\right]dx,$$

$$= -\frac{1}{v}\left[\frac{1}{u}\int_0^\infty e^{-u^2x}\frac{\partial f(x,0)}{\partial x}\right] + v^2\left[\frac{1}{uv}\int_0^\infty\int_0^\infty \frac{\partial f(x,t)}{\partial x}e^{-(u^2x+v^2t)}dtdx\right],$$

$$= -\frac{1}{v}\frac{\partial \mathbf{T}(u,0)}{\partial x} + v^2\mathbf{D}_{EF}\left[\frac{\partial f(x,t)}{\partial x}\right]$$

$$= -\frac{1}{v}\frac{\partial \mathbf{T}(u,0)}{\partial x} + v^2\left[-\frac{1}{u}\mathbf{T}(0,v) + u^2\mathbf{T}(u,v)\right],$$

$$= -\frac{1}{v}\frac{\partial \mathbf{T}(u,0)}{\partial x} - \frac{v^2}{u}\mathbf{T}(0,v) + v^2u^2\mathbf{T}(u,v)$$

Theorem 5. 6.

$$\mathbf{D}_{EF}\left[\frac{\partial^2 f(x,t)}{\partial x\,\partial t}\right] = -\frac{1}{u}\frac{\partial \mathbf{T}(0,v)}{\partial t} - \frac{u^2}{v}\mathbf{T}(0,v) + v^2u^2\mathbf{T}(u,v)$$

Proof.

$$\mathbf{D}_{EF}\left[\frac{\partial^2 f(x,t)}{\partial x\,\partial t}\right]=\frac{1}{uv}\int_0^\infty\int_0^\infty \frac{\partial^2 f(x,t)}{\partial x\,\partial t}e^{-(u^2x+v^2t)}dtdx,$$

$$=\frac{1}{v}\int_0^\infty e^{-v^2t}[\underbrace{\frac{1}{u}\int_0^\infty \frac{\partial^2 f(x,t)}{\partial x\,\partial t}e^{-u^2x}dx}_{\text{Integration by parts}}]dt,$$

Let $\zeta = e^{-u^2x}$ $d\eta=\frac{\partial^2 f(x,t)}{\partial x\,\partial t}dx$

$d\zeta=-u^2e^{-u^2x}$ $\eta=\frac{\partial f(x,t)}{\partial t}$

$$=\frac{1}{v}\int_0^\infty e^{-v^2t}\left[\frac{1}{u}\left(e^{-u^2x}\frac{\partial f(x,t)}{\partial t}\Big|_0^\infty+u^2\int_0^\infty\frac{\partial f(x,t)}{\partial t}e^{-u^2x}dt\right)\right]dt,$$

$$=\frac{1}{v}\int_0^\infty e^{-v^2t}\left[\frac{1}{u}e^{-u^2x}\frac{\partial f(x,t)}{\partial t}\Big|_0^\infty+u\int_0^\infty\frac{\partial f(x,t)}{\partial t}e^{-u^2x}dx\right]dt$$

$$=\frac{1}{v}\int_0^\infty e^{-v^2t}\left[-\frac{1}{u}\frac{\partial f(x,0)}{\partial t}+u\int_0^\infty\frac{\partial f(x,t)}{\partial t}e^{-u^2x}dx\right]dt,$$

$$=-\frac{1}{u}\left[\frac{1}{v}\int_0^\infty e^{-v^2t}\frac{\partial f(0,t)}{\partial t}\right]+u^2\left[\frac{1}{uv}\int_0^\infty\int_0^\infty\frac{\partial f(x,t)}{\partial x}e^{-(u^2x+v^2t)}dtdx\right],$$

$$=-\frac{1}{u}\frac{\partial \mathbf{T}(0,v)}{\partial t}+u^2\mathbf{D}_{EF}\left[\frac{\partial f(x,t)}{\partial t}\right],$$

$$=-\frac{1}{u}\frac{\partial \mathbf{T}(0,v)}{\partial t}+u^2\left[-\frac{1}{v}\mathbf{T}(u,0)+v^2\mathbf{T}(u,v)\right],$$

$$=-\frac{1}{u}\frac{\partial \mathbf{T}(0,v)}{\partial t}-\frac{u^2}{v}\mathbf{T}(0,v)+v^2u^2\mathbf{T}(u,v)$$

Table 2. Summarization

$f(x,t)$	$\mathbf{D}_{EF}[f(x,t)]=\mathbf{T}(u,v)$
$\frac{\partial f(x,t)}{\partial x}$	$-\frac{1}{u}\mathbf{T}(0,v)+u^2\mathbf{T}(u,v)$
$\frac{\partial^2 f(x,t)}{\partial x^2}$	$-\frac{1}{u}\frac{\partial \mathbf{T}(0,v)}{\partial x}-u\mathbf{T}(0,v)+u^4\mathbf{T}(u,v)$
$\frac{\partial^{\mathrm{n}} f(x,t)}{\partial x^{\mathrm{n}}}$	$u^{2n}\mathbf{T}(u,v)-\sum_{k=1}^{n}u^{2k-3}\frac{\partial^{n-k}\mathbf{T}(0,v)}{\partial x^{n-k}}$
$\frac{\partial f(x,t)}{\partial t}$	$-\frac{1}{v}\mathbf{T}(u,0)+v^2\mathbf{T}(u,v)$
$\frac{\partial^2 f(x,t)}{\partial t^2}$	$-\frac{1}{v}\frac{\partial \mathbf{T}(u,0)}{\partial t}-v\mathbf{T}(0,v)+v^4\mathbf{T}(u,v)$
$\frac{\partial^n f(x,t)}{\partial t^n}$	$v^{2n}\mathbf{T}(u,v)-\sum_{k=1}^{n}v^{2k-3}\frac{\partial^{n-k}\mathbf{T}(u,0)}{\partial t^{n-k}}$
$\frac{\partial^2 f(x,t)}{\partial t\,\partial x}$	$-\frac{1}{v}\frac{\partial \mathbf{T}(u,0)}{\partial x}-\frac{v^2}{u}\mathbf{T}(0,v)+v^2u^2\mathbf{T}(u,v)$
$\frac{\partial^2 f(x,t)}{\partial x\,\partial t}$	$-\frac{1}{u}\frac{\partial \mathbf{T}(0,v)}{\partial t}-\frac{u^2}{v}\mathbf{T}(0,v)+v^2u^2\mathbf{T}(u,v)$

Theorem 5.7. Let $\mathbf{F}^c(u,v)$ be the new double Emad- Falih transform of $f(x,t)(\mathbf{T}(u,v) = \mathbf{D}_{EF}[f(x,t)])$, then

$$\mathbf{D}_{EF}\left[\frac{\partial^n f(x,t)}{\partial x^n}\right] = u^{2n}\mathbf{T}(u,v) - \sum_{k=1}^{n} u^{2k-3}\frac{\partial^{n-k}\mathbf{T}(0,v)}{\partial x^{n-k}} \quad (1)$$

And

$$\mathbf{D}_{EF}\left[\frac{\partial^n f(x,t)}{\partial t^n}\right] = v^{2n}\mathbf{T}(u,v) - \sum_{k=1}^{n} v^{2k-3}\frac{\partial^{n-k}\mathbf{T}(u,0)}{\partial t^{n-k}} \quad (2)$$

Proof. Firstly, we prove (1) by the mathematical induction:

1 for $n = 1$

$$\begin{aligned}\mathbf{D}_{EF}\left[\frac{\partial f(x,t)}{\partial x}\right] &= u^{2(1)}\mathbf{T}(u,v) - \sum_{k=1}^{1} u^{2k-3}\frac{\partial^{1-k}\mathbf{T}(0,v)}{\partial x^{1-k}} \\ &= u^2\mathbf{T}(u,v) - \frac{1}{u}\mathbf{T}(0,v).\end{aligned}$$

Thus, true for $n = 1$

2 Assume true for $n = m$, to get:

$$\mathbf{D}_{EF}\left[\frac{\partial^m f(x,t)}{\partial x^m}\right] = u^{2m}\mathbf{T}(u,v) - \sum_{k=1}^{m} u^{2k-3}\frac{\partial^{m-k}\mathbf{T}(0,v)}{\partial x^{m-k}}.$$

3 We want to prove (1) for $n = m + 1$:

$$\mathbf{D}_{EF}\left[\frac{\partial^{m+1} f(x,t)}{\partial x^{m+1}}\right] = \mathbf{D}_{EF}\left[\frac{\partial}{\partial x}\left[\frac{\partial^{m} f(x,t)}{\partial x^{m}}\right]\right]$$

$$= u^2 \mathbf{D}_{EF}\left[\frac{\partial^{m} f(x,t)}{\partial x^{m}}\right] - \frac{1}{u}\frac{\partial^{m}\mathbf{T}(0,v)}{\partial x^{m}},$$

$$= u^2\left[u^{2m}\mathbf{T}(u,v) - \sum_{k=1}^{m} u^{2k-3}\frac{\partial^{m-k}\mathbf{T}(0,v)}{\partial x^{m-k}}\right] - \frac{1}{u}\frac{\partial^{m}\mathbf{T}(0,v)}{\partial x^{m}},$$

$$= u^{2(m+1)}\mathbf{T}(u,v) - u^2\sum_{k=1}^{m} u^{2k-3}\frac{\partial^{m-k}\mathbf{T}(0,v)}{\partial x^{m-k}} - \frac{1}{u}\frac{\partial^{m}\mathbf{T}(0,v)}{\partial x^{m}}$$

$$= u^{2(m+1)}\mathbf{T}(u,v) - \sum_{k=1}^{m} u^{2k-1}\frac{\partial^{m-k}\mathbf{T}(0,v)}{\partial x^{m-k}} - \frac{1}{u}\frac{\partial^{m}\mathbf{T}(0,v)}{\partial x^{m}},$$

$$= u^{2(m+1)}\mathbf{T}(u,v) - \sum_{k=0}^{m} u^{2k-1}\frac{\partial^{m-k}\mathbf{T}(0,v)}{\partial x^{m-k}},$$

$$= u^{2(m+1)}\mathbf{T}(u,v) - \sum_{k=1}^{m+1} u^{2(k-1)-1}\frac{\partial^{m-(k-1)}\mathbf{T}(0,v)}{\partial x^{m-(k-1)}},$$

$$= u^{2(m+1)}\mathbf{T}(u,v) - \sum_{k=1}^{m+1} u^{2k-3}\frac{\partial^{(m+1-k)}\mathbf{T}(0,v)}{\partial x^{(m+1-k)}},$$

$$= \mathbf{D}_{EF}\left[\frac{\partial^{m+1} f(x,t)}{\partial x^{m+1}}\right].$$

So theorem is true for $n \in \mathbb{N}$.

Finally, we want prove (2) by the mathematical induction:

1 for $n = 1$

$$\mathbf{D}_{EF}\left[\frac{\partial f(x,t)}{\partial t}\right] = v^{2(1)}\mathbf{T}(u,v) - \sum_{k=1}^{1} v^{2k-3}\frac{\partial^{1-k}\mathbf{T}(u,0)}{\partial t^{1-k}}$$

$$= v^2\mathbf{T}(u,v) - \frac{1}{v}\mathbf{T}(u,0).$$

Thus, true for $n = 1$

2 Assume true for $n = m$, to get:

$$\mathbf{D}_{EF}\left[\frac{\partial^{m} f(x,t)}{\partial t^{m}}\right] = v^{2m}\mathbf{T}(u,v) - \sum_{k=1}^{m} v^{2k-3}\frac{\partial^{m-k}\mathbf{T}(u,0)}{\partial t^{m-k}}.$$

3 We want to prove (1) for $n = m + 1$:

$$
\begin{aligned}
\mathbf{D}_{EF}\left[\frac{\partial^{m+1} f(x,t)}{\partial t^{m+1}}\right] &= \mathbf{D}_{EF}\left[\frac{\partial}{\partial x}\left[\frac{\partial^{m} f(x,t)}{\partial t^{m}}\right]\right] \\
&= v^2 \mathbf{D}_{EF}\left[\frac{\partial^{m} f(x,t)}{\partial t^{m}}\right] - \frac{1}{v}\frac{\partial^m \mathbf{T}(u,0)}{\partial t^m}, \\
&= v^2\left[v^{2m}\mathbf{T}(u,v) - \sum_{k=1}^{m} v^{2k-3}\frac{\partial^{m-k}\mathbf{T}(u,0)}{\partial t^{m-k}}\right] - \frac{1}{v}\frac{\partial^m \mathbf{T}(u,0)}{\partial t^m} \\
&= v^{2(m+1)}\mathbf{T}(u,v) - v^2\sum_{k=1}^{m} v^{2k-3}\frac{\partial^{m-k}\mathbf{T}(u,0)}{\partial t^{m-k}} - \frac{1}{v}\frac{\partial^m \mathbf{T}(u,0)}{\partial t^m} \\
&= v^{2(m+1)}\mathbf{T}(u,v) - \sum_{k=1}^{m} v^{2k-1}\frac{\partial^{m-k}\mathbf{T}(u,0)}{\partial t^{m-k}} - \frac{1}{v}\frac{\partial^m \mathbf{T}(u,0)}{\partial t^m}, \\
&= v^{2(m+1)}\mathbf{T}(u,v) - \sum_{k=0}^{m} v^{2k-1}\frac{\partial^{m-k}\mathbf{T}(u,0)}{\partial t^{m-k}}, \\
&= v^{2(m+1)}\mathbf{T}(u,v) - \sum_{k=1}^{m+1} v^{2(k-1)-1}\frac{\partial^{m-(k-1)}\mathbf{T}(u,0)}{\partial t^{m-(k-1)}}, \\
&= v^{2(m+1)}\mathbf{T}(u,v) - \sum_{k=1}^{m+1} v^{2k-3}\frac{\partial^{(m+1-k)}\mathbf{T}(u,0)}{\partial t^{(m+1-k)}}, \\
&= \mathbf{D}_{EF}\left[\frac{\partial^{m+1} f(x,t)}{\partial t^{m+1}}\right].
\end{aligned}
$$

So theorem is true for $n \in \mathbb{N}$.

6. Applications

In this section, we introduce the solution of the linear partial differential equation (Telegraph equation).

Application 6.1.

Consider the linear telegraph equation:

$$\frac{\partial^2 \zeta}{\partial x^2} = \frac{\partial^2 \zeta}{\partial t^2} + 2\frac{\partial \zeta}{\partial t} + \zeta.$$

Subject to initial conditions:

$$\zeta(x,0) = e^{x},\ \zeta(0,t) = e^{-2t}$$
$$\zeta_t(x,0) = -2e^{x},\ \zeta_x(0,t) = e^{-2t}$$

Solution:

Applying Emad- Falih transform,

$$\mathbf{D}_{EF}\left[\frac{\partial^2 \zeta}{\partial x^2} = \frac{\partial^2 \zeta}{\partial t^2} + 2\frac{\partial \zeta}{\partial t} + \zeta\right],$$

to obtain the following:

$$-\frac{1}{u}\frac{\partial \mathbf{T}(0,v)}{\partial x}-u\mathbf{T}(0,v)+u^4\mathbf{T}(u,v)=-\frac{1}{v}\frac{\partial \mathbf{T}(u,0)}{\partial t}-v\mathbf{T}(0,v)+v^4\mathbf{T}(u,v)$$
$$+2\left[-\frac{1}{v}\mathbf{T}(u,0)+v^2\mathbf{T}(u,v)\right]+T(u,v).$$

Applying the Emad- Falih transform to the initial conditions, we have:

$$-\frac{1}{u}\left(\frac{1}{v(v^2+2)}\right)-u\left(\frac{1}{v(v^2+2)}\right)+u^4\mathbf{T}(u,v)=-\frac{1}{v}\left(\frac{-2}{u(u^2-1)}\right)-v\left(\frac{1}{u(u^2-1)}\right)+v^4\mathbf{T}(u,v)$$
$$-2\frac{1}{v}\left(\frac{1}{u(u^2-1)}\right)+2v^2\mathbf{T}(u,v)+T(u,v).$$

We get:

$$(u^4-v^4-2v^2-1)\mathbf{T}(u,v)=\frac{u^4-v^4-2v^2-1}{uv(v^2+2)(u^2-1)},$$
$$\mathbf{T}(u,v)=\frac{1}{uv(v^2+2)(u^2-1)}.$$

Then

$$\zeta(x,t)=e^{x-2t}$$

7.Conclusions

In this research, we have extended the double Emad - Falih integral transform. First, we have proved the fundamental properties and theorems using the new double Emad - Falih integral transform which have also been introduced and presented. Second, the new double integral transform to solve the telegraph partial differential equation is applied.

References

1. Dhunde, R. R. ; Waghmare, G. L. Double Laplace Transform Method in mathematical Physics. International Journal of Theoretical and Mathematical Physics, 2017,7(1),14–20.
2. Eltayeb, H. ; Kilic¸man. A. On Double Sumudu Transform And Double Laplace Transform. Malaysian Journal of Mathematical Sciences, 2010, 4(1),17–30.
3. Hassan, M. A. Partial Differential Equation. Journal of Applied Mathematics and Physics, 2020,8(8), 1463-1471.
4. Meddahi M. ; Jafari, H. ; Yang, X. J. Towards New General Double Integral Transform And Its Applications to Differential Equations. Mathematical Methods in the Applied Sciences, 2021,*45(4)*,1–18,
5. Bracewell, R. N. The Fourier Transform And Its Applications. McGraw-Hill, New York, 1986.
6. Widder, D. V. The Laplace Transform. Princetion University Press, Princetion, 1946.
7. Watugala, G. K. Sumudu Transform: a New Integral Transform to Solve Differential Equations And Control Engineering Problems. International Journal of Mathematical Education in Science and Technology, 1993, 24(1),35-43.
8. Al-Omari, S. K. Q. On the Application of Natural Transforms. International Journal of Pure and Applied Mathematics, 2013,85(4),729-744.

9. Elzaki, T. M. The New Integral Transform "Elzaki Transform". Global Journal of Pure and Applied Mathematics, 2011,7(1),57-64.
10. Kuffi, E. ; Maktoof, S. F. Emad-Falih Transform a New Integral Transform. Journal of Interdisciplinary Mathematics, 2021, 24(8), 2381-2390.
11. Mansour, E. A. ; Kuffi, E. A. ; Mehdi, S. A. The New Integral Transform See Transform And Its Applications. Periodicals of Engineering and Natural Sciences, June 2021,9(2): $1016 - 1029$.