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The Completion of Generalized 2-Inner Product Spaces

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Abstract

A complete metric space is a well-known concept. Kreyszig shows that every non-complete metric space W can be developed into a complete metric space $\widehat{W}$, referred to as completion of W.

We use the b-Cauchy sequence to form $\widehat{W}$ which "is the set of all b-Cauchy sequences equivalence classes". After that, we prove $\widehat{W}$ to be a 2-normed space. Then, we construct an isometric by defining the function from W to $\widehat{W}_0$; thus $\widehat{W}_0$ and W are isometric, where $\widehat{W}_0$ is the subset of $\widehat{W}$ composed of the equivalence classes that contains constant b-Cauchy sequences. Finally, we prove that $\widehat{W}_0$ is dense in $\widehat{W}$, $\widehat{W}$ is complete and the uniqueness of $\widehat{W}$ is up to isometrics.

Keywords: b-Cauchy Sequence, Equivalent Class, Metric space, Completion Generalized 2-Inner Product Space.

1. Introduction

Cho and Freese [3-4] introduced 2-normed space by: Let W be a real linear space with a dimension greater than 1. Suppose that $\|,\|$ is a real-valued function on $W \times W$ for all w, y, z in W and $\alpha \in \mathbb{R}$ satisfying the following requirements:

1. $\|w, y\| = 0$ if and only if w and y are linearly dependent.
2. $\|w, y\| = \|y, w\|$
3. $\|\alpha w, y\| = |\alpha|\|w, y\|$
4. $\|w + y, z\| \leq \|w, z\| + \|y, z\|$

Then $\|,\|$ is called a 2-norm on W and the pair $(W, \|,\|)$ is called a linear 2-normed space or 2-normed space. For more details, see [11-12]

Riys and Ravinderan [10] defines the generalized 2-inner product space as a complex vector space W, called a generalized 2-inner product space if there exists a complex valued function $\langle(,),(,)\rangle$ on $W^2 \times W^2$ such that $a, b, c, d \in W$, and $\alpha \in C$, as the following:

1. $\langle(a,b),(c,d)\rangle = \overline{\langle(c,d),(a,b)\rangle}$
2. If a and b are linear independent in W, then $\langle(a,b),(c,d)\rangle > 0$.
3. $\langle(a,b),(c,d)\rangle = -\langle(b,a),(c,d)\rangle$.
4. $\langle(\alpha a+e,b),(c,d)\rangle = \alpha\langle(a,b),(c,d)\rangle + \langle(e,b),(c,d)\rangle$. For more details, see [8][2][5].

Ghafoor [6], shows that the generalized 2-inner product space is a 2-normed space with $\|w,y\| = \langle(w,y),(w,y)\rangle^{\frac{1}{2}}$.

After many failures to define orthogonal vectors in 2-normed space, Riyas and Ravindran, in 2014, [10] defined orthogonal vectors in a 2-normed space by restriction space $W \times W$ to $W \times \langle b\rangle$, where $< b >$ is a non-zero subspace in W. Thus, the domain of the generalized 2-inner product restriction with space $W^2 \times \langle b\rangle^2$.

It is well-known that there are incomplete metric spaces. Kreyszig, in 1978 [7] discussed the strategy for completing every metric space by defining an equivalent relation on Cauchy sequences and the metric on it. In 2001, Cho and Freese [3] used the same strategy for the completion of a 2-normed space, but some difficulties appeared when defining a metric on it, and then, he had to give another condition on the space.

Despite that all generalized 2-inner product space is 2-normed space, but in this paper, we give a completion of the generalized 2-inner product space without need any condition using the b-Cauchy sequences.

This paper includes two sections. The first section discusses some of the properties of the b-Cauchy sequence in a generalized 2-inner product space. The second section proves the completion of the generalized 2-inner product.

We will abbreviate $\|w,b\|$by $\|w\|_b$ in the sequel.

2. b-Cauchy sequences.

This section discusses some of the properties of the b-Cauchy sequence in a generalized 2-inner product space. Mazaheri and Kazemi in [9] introduce a b-Cauchy sequence concept as follows

Definition (2.1)[9]: Let W be a generalized 2-inner product space, $0 \neq b \in W$, $\{w_n\}$ be a sequence in W, then, it is called a b-Cauchy sequence if $\lim\limits_{n,m\to\infty} \|w_n - w_m, b\| = 0$.

The following definition has been devised from [9]

Definition (2.2): An open ball of radius r and centered at y in a generalized 2-inner product space W is defined as: $B_r(y) \coloneqq \{w \in W: \|w-y\|_b < r\}$.

The following proposition is a characterization of b-Cauchy sequences in a generalized 2-inner product space. But first, we define a neighborhood of 0.

Definition (2.3): If w is a point in a generalized 2-inner product space W, then a neighborhood U of w is a set containing $B_r(w)$ for some $r > 0$, i.e, there exists $r > 0$, such that $w \in B_r(w) \subset U$.

Proposition (2.4): Let W be a generalized 2-inner product space, $\{w_n\}$ is a b-Cauchy sequence in W if and only if for any neighborhood U of 0; there is an integer $M(U)$ such that for all $n, m \geq M(U)$ implies that $w_n - w_m \in U$.

Proof: Let U be a neighborhood of 0, then there exists $\varepsilon > 0$ such that $B_0(\varepsilon) \subseteq U$. Since $\{w_n\}$ is a b-Cauchy sequence in W, thus $\lim_{n,m\to\infty} \|w_n - w_m\|_b = 0$. It implies that there exists $M(\varepsilon) > 0$ such that $\|w_n - w_m\|_b < \varepsilon;\ n, m \geq M(\varepsilon)$. Then, $w_n - w_m \in B_0(\varepsilon) \subseteq U$.

Conversely, let $\{w_n\}$ be a sequence in W such that for every neighborhood U of 0 there is an integer $M(U) > 0; w_n - w_m \in U$ where $n, m \geq M(U)$. Then, there exists $\delta(U) > 0$ such that $\|w_n - w_m\|_b < \delta$, where $n, m \geq M(U)$. It implies that for all $\varepsilon > 0$, there exists $M(\varepsilon)$ such that $\|w_n - w_m\|_b < \varepsilon$ for $n, m \geq M(\varepsilon)$, then , $\lim_{n,m\to\infty} \|w_n - w_m\|_b = 0$. Thus, $\{w_n\}$ is a b-Cauchy sequence in W.■

3. Completion of the generalized 2-inner product spaces.

Kreyszig [1] states few steps to prove that an arbitrary incomplete metric space can be completed. In this section, we follow Kreyszig strategy to prove the completion of the generalized 2-inner product space:

Step1: Forming $\widehat{W}$ is the set of all b-Cauchy sequence equivalence classes.

Definition (3.1): Two b-Cauchy sequences $\{w_n\}$ and $\{y_n\}$ in a generalized 2-inner product space W have a relation denoted by $\{w_n\} \sim \{y_n\}$, if for every neighborhood U of 0 there is an integer $M(U)$ such that $n \geq M(U)$ implies that $w_n - y_n \in U$.

It is clear that $\sim$ is a reflexive and symmetric relation. The following proposition shows that this relation is equivalent.

Proposition (3.2): The relation $\sim$ on the set of b-Cauchy sequences in W is an equivalent relation on W.

Proof: Let $\{w_n\} \sim \{y_n\}$ and $\{y_n\} \sim \{z_n\}$. Let, U is an arbitrary neighborhood of 0. There exists a neighborhood V of 0 such that $V + V \subset U$. By Definition (2.1) and for this V, there exists an integer M such that $w_n - y_n, y_n - z_n \in V$ for $n \geq M$. Hence , $w_n - z_n = (w_n - y_n) + (y_n - z_n)$ is an element of U for $n \geq M$. Therefore, $\{w_n\} \sim \{z_n\}$.■

Define $\widehat{W} := \{\widehat{w} : \widehat{w}$ is equivalent class of b-Cauchy sequences$\}$.

Step 2: Proof $\widehat{W}$ vector space.

Let $\widehat{w}, \widehat{y}$ in $\widehat{W}$. Define the terms addition and scalar multiplication. On $\widehat{W}$ where $\{w_n\} \in \widehat{w}$ and $\{y_n\} \in \widehat{y}$, as shown below:

- $\widehat{w} + \widehat{y} = \{w_n + y_n\}$
- $\alpha\widehat{w} = \{\alpha w_n\}$

The following proposition explains that the two operations defined on $\widehat{W}$ are well-defined because they are unaffected by the elements chosen from $\{\hat{w}_n\}$ and $\{\hat{y}_n\}$. But first, we need the following proposition:

Proposition (3.3): A b-Cauchy sequence $\{w_n\}$ is equivalent to $\{a_n\}$ in a generalized 2-inner product space W if and only if $\lim_{n\to\infty} \|w_n - a_n\|_b = 0$.

Proof: Let U be a neighborhood of zero, then, there exists $M(U) > 0$, such that $w_n - a_n \in U$ for $n \geq M(U)$. Hence, for all neighborhood U of 0, there exists $\varepsilon = \varepsilon(U) > 0$ such that $\|w_n - a_n\|_b < \varepsilon$; $n \geq M(U)$. Then, for every $\delta > 0$, there exists $M(\delta) > 0$ such that $\|w_n - a_n\|_b < \delta$ for all $n \geq M(\delta)$, therefore, $\lim_{n\to\infty} \|w_n - a_n\|_b = 0$ for $n \geq M(\delta)$.

Conversely, let $\{w_n\}, \{a_n\}$ be b-Cauchy sequences in W such that for every neighborhood U of 0, there exists $\varepsilon > 0$ such that $B_\varepsilon(0) \subset U$. By our hypothesis $\lim_{n\to\infty} \|w_n - a_n\|_b = 0$, then there exists $M(\varepsilon) > 0$ such that $\|w_n - a_n\|_b < \varepsilon$ for $n \geq M(\varepsilon)$. Hence, $w_n - a_n \in B_\varepsilon(0) \subset U$ for $n \geq M(\varepsilon)$, then $\{w_n\} \sim \{a_n\}$.■

Proposition (3.4): If $\{a_n\}$ and $\{b_n\}$ are equivalent to $\{w_n\}$ and $\{y_n\}$ in a generalized 2-inner product space W. Then, $\{a_n + b_n\}$ is equivalent to $\{w_n + y_n\}$ and $\{\alpha a_n\}$ is equivalent to $\{\alpha w_n\}$. Moreover, $\widehat{W}$ is a linear space.

Proof: Since $\{a_n\} \sim \{w_n\}$ and $\{b_n\} \sim \{y_n\}$ thus we get

$$\|(w_n + y_n) - (a_n + b_n)\|_b \leq \|w_n - a_n\|_b + \|y_n - b_n\|_b$$

Then

$$\lim_{n,m\to\infty} \|(w_n + y_n) - (a_n + b_n)\|_b = 0 \quad \dots (1)$$

On the other hand,

$$\lim_{n\to\infty} \|\alpha w_n - \alpha a_n\|_b = 0 \; \dots (2)$$

It implies that $\{a_n + b_n\} \sim \{w_n + y_n\}$ and $\{\alpha a_n\} \sim \{\alpha w_n\}$. Therefore, from (1), (2) and Proposition (2.3), $\widehat{W}$ is a linear space.■

Step3: Proof $\widehat{W}$ is a 2-normed space.

We will define a 2-norm function on the space $\widehat{W}$. as:

$$\|.\|_{\hat{b}} : \widehat{W} \times \langle \hat{b} \rangle \to R^+$$

is defined as:

$$\|\hat{w} - \hat{y}\|_{\hat{b}} = \lim_{n\to\infty} \|w_n - y_n\|_b \; \dots (3)$$

where $\{w_n\} \in \hat{w}, \{y_n\} \in \hat{y}$.

The function is well-defined as follows:

Proposition (3.5): If W is a generalized 2-inner product space, then for any two b-Cauchy sequences $\{w_n\}$ and $\{y_n\}$ in W:

1. $\lim_{n\to\infty} \|\mathrm{w_n - y_n}\|_\mathrm{b}$ exists.
2. For pairs of equivalent b-Cauchy sequences $\{\mathrm{a_n}\}\sim\{\mathrm{w_n}\}$ and $\{\mathrm{b_n}\}\sim\{\mathrm{y_n}\}$, $\lim_{n\to\infty}\|\mathrm{w_n - y_n}\|_\mathrm{b} = \lim_{n\to\infty}\|\mathrm{a_n - b_n}\|_\mathrm{b}$.

Proof:

1. Let $\{\mathrm{w_n}\} \in \widehat{\mathrm{w}}$, $\{\mathrm{y_n}\} \in \hat{\mathrm{y}}$ be any two b-Cauchy sequences, then

$$\|\mathrm{w_n - y_n}\|_\mathrm{b} \le \|\mathrm{w_n - w_m}\|_\mathrm{b} + \|\mathrm{w_m - y_m}\|_\mathrm{b} + \|\mathrm{y_m - y_n}\|_\mathrm{b}$$

Hence, $\|\mathrm{w_n - y_n}\|_\mathrm{b} - \|\mathrm{w_m - y_m}\|_\mathrm{b} \le \|\mathrm{w_n - w_m}\|_\mathrm{b} + \|\mathrm{y_m - y_n}\|_\mathrm{b}$

On the other hand, if we change m by n,

$$\|\mathrm{w_m - y_m}\|_\mathrm{b} - \|\mathrm{w_n - y_n}\|_\mathrm{b} \le \|\mathrm{w_m - w_n}\|_\mathrm{b} + \|\mathrm{y_n - y_m}\|_\mathrm{b}$$

It implies that

$$\big|\|\mathrm{w_n - y_n}\|_\mathrm{b} - \|\mathrm{w_m - y_m}\|_\mathrm{b}\big| \le \|\mathrm{w_n - w_m}\|_\mathrm{b} + \|\mathrm{y_n - y_m}\|_\mathrm{b} \dots (4)$$

Thus, by taking $\mathrm{n, m} \to \infty$ and Definition (1.1), it follows that

$$\lim_{n\to\infty}\big|\|\mathrm{w_n - y_n}\|_\mathrm{b} - \|\mathrm{w_m - y_m}\|_\mathrm{b}\big| = 0$$

Then, $\{\|\mathrm{w_n - y_n}\|_\mathrm{b}\}$ is a Cauchy sequence in R. But, R is complete, thus $\lim_{n\to\infty}\|\mathrm{w_n - y_n}\|_\mathrm{b}$ exists.

2. Let $\{\mathrm{a_n}\}\sim\{\mathrm{w_n}\}$ and $\{\mathrm{b_n}\}\sim\{\mathrm{y_n}\}$. By the same argument of (4), it implies that

$$\big|\|\mathrm{w_n - y_n}\|_\mathrm{b} - \|\mathrm{a_n - b_n}\|_\mathrm{b}\big| \le \|\mathrm{w_n - a_n}\|_\mathrm{b} + \|\mathrm{y_n - b_n}\|_\mathrm{b}$$

By taking $\mathrm{n, m} \to \infty$ and proposition (2.3), we get

$$\lim_{n\to\infty}\|\mathrm{w_n - y_n}\|_\mathrm{b} = \lim_{n\to\infty}\|\mathrm{a_n - b_n}\|_\mathrm{b} . \blacksquare$$

From equation (3) and Proposition (2.3), the conditions (1-3) of a 2-normed space are done.

Proposition (3.6): $(\widehat{\mathrm{W}}, \|.\|_{\hat{\mathrm{b}}})$ is a 2-normed space.

Proof: since $\|\widehat{\mathrm{w}} - \hat{\mathrm{z}}\|_{\hat{\mathrm{b}}} = \lim_{n\to\infty}\|\mathrm{w_n - z_n}\|_\mathrm{b} \le \lim_{n\to\infty}\|\mathrm{w_n - y_n}\|_\mathrm{b} + \lim_{n\to\infty}\|\mathrm{y_n - z_n}\|_\mathrm{b} = \|\widehat{\mathrm{w}} - \hat{\mathrm{y}}\|_{\hat{\mathrm{b}}} + \|\hat{\mathrm{y}} - \hat{\mathrm{z}}\|_{\hat{\mathrm{b}}}$, then $(\widehat{\mathrm{W}}, \|,\|_{\hat{\mathrm{b}}})$ is a 2-norm. $\blacksquare$

Step4: Construction of an isometric $T: W \to \widehat{W}_0 \subset \widehat{W}$.

Let $\widehat{\mathrm{W}}_0$ be the subset of $\widehat{\mathrm{W}}$ composed of the equivalence classes containing constant b-Cauchy sequences.

Define a function $\mathrm{T: W \to \widehat{W}_0 \subset \widehat{W}}$ by $\mathrm{T(w)} = \widehat{\mathrm{w}} = (\mathrm{w, w, \dots})$. It is clearly that T is a well-defined onto and one to one. In fact,

$$\|\mathrm{Tw - Ty}\|_\mathrm{b} = \|\widehat{\mathrm{w}} - \hat{\mathrm{y}}\|_{\hat{\mathrm{b}}} = \lim_{n\to\infty}\|\mathrm{w - y}\|_\mathrm{b} = \|\mathrm{w - y}\|_\mathrm{b},$$

Thus, $\widehat{\mathrm{W}}_0$ and W are isometric.

Step 5: Proof $\widehat{W}_0$ is dense in $\widehat{W}$.

Proposition (3.7): If W is a generalized 2-inner product space, then, $\widehat{W}_0$ is dense in $\widehat{W}$.

Proof: Let $\widehat{w} \in \widehat{W} - \widehat{W}_0$, then, there exists a b-Cauchy sequence $\{w_n\} \in \widehat{w}$ where $\{w_n\} = \{w_1, w_2, \ldots\}$. Define $\widehat{w}^m = \{w_m, w_m, \ldots\}$ for all $m \in N$, thus $\widehat{w}^m \in \widehat{X}_0$. Hence, by Definition (1.1)

$$\|\widehat{w}^m - \widehat{w}\|_{\hat{b}} = \lim_{n,m\to\infty} \|w_n - w_m\|_b = 0 .$$

Then, $\widehat{W}_0$ is dense in $\widehat{W}$.■

Step 6: Proof completeness of $\widehat{W}$.

Theorem (3.8): If W is a generalized 2-inner product space, then $\widehat{W}$ is complete.

Proof: Let $\{\widehat{w}_n\}$ be a b-Cauchy sequence in $\widehat{W}$. Since $\widehat{W}_0$ is dense in $\widehat{W}$, thus there exists $\{\hat{z}_n\} \in \widehat{W}_0$ such that $\|\widehat{w}_n - \hat{z}_n\|_{\hat{b}} = 0$. But

$$\|\hat{z}_n - \hat{z}_m\|_{\hat{b}} \le \|\hat{z}_n - \widehat{w}_n\|_{\hat{b}} + \|\widehat{w}_n - \widehat{w}_m\|_{\hat{b}} + \|\widehat{w}_m - \hat{z}_m\|_{\hat{b}}.$$

Then, by equation (3) and Definition (1.1) and if we take $n, m \to \infty$, we get $\lim_{n,m\to\infty} \|\hat{z}_n - \hat{z}_m\|_{\hat{b}} = 0$, it implies that $\{\hat{z}_n\}$ is a b-Cauchy sequence in $\widehat{W}_0$. But W and $\widehat{W}$ are isometric. Thus, there exists a b-Cauchy sequence $\{z_n\}$ in W which is contained in an equivalent class in $\widehat{W}$, say $\widehat{w}$.

Note that, $\|\widehat{w}_n - \widehat{w}\|_{\hat{b}} \le \|\widehat{w}_n - \hat{z}_n\|_{\hat{b}} + \|\hat{z}_n - \widehat{w}\|_{\hat{b}} = \|\widehat{w}_n - \hat{z}_n\|_{\hat{b}} + \|\hat{z}_n - \hat{z}_n\|_{\hat{b}}$. Thus, $\lim_{n\to\infty} \|\widehat{w}_n - \widehat{w}\|_{\hat{b}} = 0$. Therefore, $\widehat{W}$ is complete.■

Step7: Proof uniqueness of $\widehat{W}$ up to isometrics.

Theorem (3.9): The space $\widehat{W}$ is unique up to isometrics.

Proof: Let $\widehat{Y}$ be another completion to W with a dense subset $\widehat{Y}_0$ in $\widehat{Y}$. Then, there exists $S: W \to \widehat{Y}_0$ is isometric by step 4 defined by $S(w) = \hat{y} = (y, y, \ldots)$.

We will define $h: \widehat{W}_0 \to \widehat{Y}_0$ by $h(\widehat{w}) = ST^{-1}(\widehat{w})$. It implis that $\widehat{W}_0$ isometric to $\widehat{Y}_0$. For $\hat{y}_1$,$\hat{y}_2$ in $\widehat{Y}$ there exists b-Cauchy sequences $\{\hat{y}_{1n}\}$, $\{\hat{y}_{2n}\}$ in $\widehat{Y}_0$ such that $\hat{y}_{1n} \to \hat{y}_1$ and $\hat{y}_{2n} \to \hat{y}_2$. Thus, by equation (4)

$$\left| \|\hat{y}_1 - \hat{y}_2\|_{\hat{b}} - \|\hat{y}_{1n} - \hat{y}_{2n}\|_b \right| \le \|\hat{y}_1 - \hat{y}_{1n}\|_b - \|\hat{y}_2 - \hat{y}_{2n}\|_b \to 0$$

By taking $n \to \infty$

$$\|\hat{y}_1 - \hat{y}_2\|_{\hat{b}} = \lim_{n\to\infty} \|\hat{y}_{1n} - \hat{y}_{2n}\|_b \qquad (5)$$

by the same argument

$$\|\widehat{w}_1 - \widehat{w}_2\|_{\hat{b}} = \lim_{n\to\infty} \|\widehat{w}_{1n} - \widehat{w}_{2n}\|_b \qquad (6)$$

Thus, by (5) and (6) we get

$$\|\hat{y}_1 - \hat{y}_2\|_{\hat{b}} = \lim_{n\to\infty} \|\hat{y}_{1n} - \hat{y}_{2n}\|_b = \lim_{n\to\infty} \|\hat{w}_{1n} - \hat{w}_{2n}\|_b = \|\hat{w}_1 - \hat{w}_2\|_{\hat{b}}$$

It implies that $\widehat{\mathrm{W}}$ is isometric to $\widehat{\mathrm{Y}}$.■

4. Discussion and Conclusion

A complete metric space is a well-known concept. Every non-complete metric space W can be built into a complete metric space $\widehat{\mathrm{W}}$, which is known as a completion of W. In this paper, we construct equivalent classes of b-Cauchy sequences to complete a generalized 2-inner product space.

References

1.Anshul, R.; Ravinder, K., S., ; Sumit, C., Stability of Complex Functional Equations in 2-Banach Spaces. *Journal of Mathematical Physics, Analysis, Geometry* . **2021**, *17, 3,* 341–368.

2.Bahram, D. ; Mohammad, J., Atomic Systems in 2-inner Product Spaces, *Iranian Journal of Mathematical Sciences and Informatics*. **2018**, *13, 1*, 103-110.

3.Cho, Y., J ;Freese, R. W., Geometry of Linear 2-Normed Spaces, Nova Science Publishers, New York, **2001**.

4.Y. J. Cho, M. Matic, J. E., Pecaric, On Gram's Deteminant in 2-Inner Product Spaces, *J. Korean Math. Soc*., **2001,** *38(4),* 1125–1156.

5.Debnath, P., Saha, M., Categorization of n-inner product space. Asian Res. *J. Math*. **2018,***11(4),* 1–10.

6.Ghafoor, G., R. ; Jamil, Z., Z., Study of b-Hilbert Spaces and Some Classes of Operators, University of Baghdad, Baghdad, **2018**, 27-28.

7.Kreyszig, Erwin, Introductory Functional Analysis with Applications, *John Wiley and Sons, New York*, **1978**.

8.Prasenjit, G., Frame operator for K-frame in 2-inner product space, *International Journal of Mathematics Trends and Technology*. **2021,** *67.*

9.Mazaheri, H.; Kazemi, R., Some Results on 2-Inner Product Spaces, *Nove Sad. J.Math*. **2007,** *37*, 35-40.

10.Riyas, P., ; Ravindran, K., T., Riesz Theorems and ~Adjoint Operators On Generalized 2-Inner Product Spaces, *Global Journal mathematics*, **2015,** *3, 1, May 18*, 244-254.

11.Sibel, E. Ideal Strong Lacunary Quasi Cauchy Sequences in 2-normed spaces. AIP Conference Proceedings 2334, 040004 .**2021**.

12.Vijayakumar, S.; Baskaran B., A Characterization of 2-Inner Product Spaces. AIP Conference Proceedings 2282, 020040 .**2020**