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Nano S_C-Open Sets In Nano Topological Spaces

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Abstract

The objective of this paper is to define and introduce a new type of nano semi-open set which called nano S_C-open set as a strong form of nano semi-open set which is related to nano closed sets in nano topological spaces. In this paper, we find all forms of the family of nano S_C-open sets in term of upper and lower approximations of sets and we can easily find nano S_C-open sets and they are a gate to more study. Several types of nano open sets are known, so we study relationship between the nano S_C-open sets with the other known types of nano open sets in nano topological spaces. The Operators such as nano S_C-interior and nano S_C-closure are the part of this paper.

Keywords: nano closed sets, nano semi-open sets, nano S_C-open sets, nano S_C-interior, nano S_C-closure.

1.Introduction

The notion of nano topological space (briefly NTS) introduced by Thivagar and Carmel [1] with respect to a subset X of a universe U which is defined in terms of lower and upper approximations. Levine [2] introduced the notions of semi-open. Later, nano semi-open sets introduced by Thivagar Carmel [1], also nano S_β-open sets introduced by [4], and more nano open sets defined in [5-7]. In this paper, we introduce the concept nano S_C-open sets as a strong form of nano semi-open sets, since every nano S_C-open (briefly nS_C-oprn) sets is nano semi-open sets and the relationship with some class of nano near open sets. All forms of family of nano S_C-open sets under various cases of approximations idea also derived. Also, operators such as nano S_C-interior and nano S_C-closure are the part of this paper.

2. Preliminaries

Definition 2.1. [8] Let $\mathcal{W} \neq \phi$ denote the finite universe and the equivalence relation R on the universe W called the indiscernibility relation. The pair $(\mathcal{W}, R)$ is called the approximation space. Let $X \subseteq \mathcal{W}$:

i. The lower approximation defined by $L_R(X) = \bigcup_{x \in U} \{R(x); R(x) \subseteq X\}$, where $R(x)$ stands the equivalence class by x.
ii. The upper approximation defined by $U_R(X) = \bigcup_{x \in U} \{R(x); R(x) \cap X \neq \phi\}$.
iii. The boundary region defined by $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2. [1] Let $\mathcal{W}$ denote the universe and R be an equivalence relation on W and $\tau_R(X) = \{\phi, \mathcal{W}, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq \mathcal{W}$. Then the followings axioms hold for $\tau_R(X)$:

i. W and $\phi \in \tau_R(X)$
ii. $A, B \in \tau_R(X)$, then $A \cup B \tau_R(X)$
iii. The intersection of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

Then $\tau_R(X)$ forms a topology on $\mathcal{W}$ and called nano topology on $\mathcal{W}$ with respect to X. Also $(\mathcal{W}, \tau_R(X))$ is called the NTS and the members of $\tau_R(X)$ are called nano open sets.

Definition 2.3. Let $(\mathcal{W}, \tau_R(X))$ be a NTS and $K \subseteq \mathcal{W}$. The set K is called nano:

i. regular-open [1], if $K = nint(ncl(K))$.
ii. α-open [1], if $K \subseteq nint(ncl(nint((K)))$.
iii. semi-open [1], if $K \subseteq ncl(nint(K))$.
iv. β-open (nano semi pre-open) [3], if $K \subseteq ncl(nint(ncl(K)))$.
v. θ-open [1], if for each $x \in K$, there exists a nano open set G such that $x \in G \subseteq ncl(K) \subseteq K$.
vi. S_β-open [4], if K is nano semi-open and the union of nano β-closed sets.

The set of all nano regular-open (resp. nano α-open, nano semi-open, nano β-open, θ-open and nano S_β-open) sets denoted by $nRO(\mathcal{W}, X)$ (resp. $n\alpha O(\mathcal{W}, X)$, $nSO(\mathcal{W}, X)$, $n\beta O(\mathcal{W}, X)$, $n\theta O(\mathcal{W}, X)$ and $nS_\beta O(\mathcal{W}, X)$).

Theorem 2.4. [1] If $A, B \in nSO(\mathcal{W}, X)$, then $A \cup B \in nSO(\mathcal{W}, X)$.

Theorem 2.5. [1] Let $(\mathcal{W}, \tau_R(X))$ be a NTS, then:

i. If $U_R(X) = \mathcal{W}$ and $L_R(X) = \phi \Longrightarrow \tau_R^\theta(X) = \{\phi, \mathcal{W}\}$.
ii. If $U_R(X) = \mathcal{W}$ and $L_R(X) \neq \phi \Longrightarrow \tau_R(X) = \tau_R^\theta(X)$.
iii. If $U_R(X) = L_R(X) \neq \mathcal{W} \Longrightarrow \tau_R^\theta(X) = \{\phi, \mathcal{W}\}$.
iv. If $U_R(X) \neq \mathcal{W}$, $L_R(X) = \phi \Longrightarrow \tau_R^\theta(X) = \{\phi, \mathcal{W}\}$.
v. If $U_R(X) \neq L_R(X)$ where $U_R(X) \neq \mathcal{W}$ and $L_R(X) \neq \phi \Longrightarrow \tau_R^\theta(X) = \{\phi, \mathcal{W}\}$.

Theorem 2.6. [1] Let $(\mathcal{W}, \tau_R(X))$ be a NTS, then:

i. If $U_R(X) = \mathcal{W}$ and $L_R(X) = \phi \Longrightarrow nRO(\mathcal{W}, X) = \{\phi, \mathcal{W}\}$.
ii. If $U_R(X) = \mathcal{W}$ and $L_R(X) \neq \phi \Longrightarrow \tau_R(X) = nRO(\mathcal{W}, X)$.
iii. If $U_R(X) = L_R(X) \neq \mathcal{W} \Longrightarrow \tau_R(X) = nRO(\mathcal{W}, X)$.
iv. If $U_R(X) \neq \mathcal{W}$, $L_R(X) = \phi \Longrightarrow \tau_R(X) = nRO(\mathcal{W}, X)$.

v. If $U_R(X) \neq L_R(X)$ where $U_R(X) \neq \mathcal{W}$ and $L_R(X) \neq \phi \Longrightarrow nRO(\mathcal{W}, X) = \{\phi, \mathcal{W}, L_R(X), B_R(X)\}$.

Theorem 2.7. [1] Let $(\mathcal{W}, \tau_R(X))$ be a NTS, then:

i. If $U_R(X) = \mathcal{W}$ and $L_R(X) = \phi \Longrightarrow n\alpha O(\mathcal{W}, X) = \{\phi, \mathcal{W}\}$.
ii. If $U_R(X) = \mathcal{W}$ and $L_R(X) \neq \phi \Longrightarrow \tau_R(X) = n\alpha O(\mathcal{W}, X)$.
iii. If $U_R(X) = L_R(X) \neq \mathcal{W} \Longrightarrow \phi$ and those sets A for which $U_R(X) \subseteq A$ are the only $n\alpha$-open sets in $\mathcal{W}$.
iv. If $U_R(X) \neq \mathcal{W}$, $L_R(X) = \phi \Longrightarrow \phi$ and those sets A for which $U_R(X) \subseteq A$ are the only $n\alpha$-open sets in $\mathcal{W}$.
v. If $U_R(X) \neq L_R(X)$ where $U_R(X) \neq \mathcal{W}$ and $L_R(X) \neq \phi \Longrightarrow \phi$, $L_R(X), B_R(X)$, any set containing $U_R(X)$ are the only $n\alpha$-open sets in $\mathcal{W}$.

Theorem 2.8. [1] Let $(\mathcal{W}, \tau_R(X))$ be a NTS, then:

i. If $U_R(X) = \mathcal{W}$ and $L_R(X) = \phi \Longrightarrow nSO(\mathcal{W}, X) = \{\phi, \mathcal{W}\}$.
ii. If $U_R(X) = \mathcal{W}$ and $L_R(X) \neq \phi \Longrightarrow \tau_R(X) = nSO(\mathcal{W}, X)$.
iii. If $U_R(X) = L_R(X) \neq \mathcal{W} \Longrightarrow \phi$ and those sets A for which $U_R(X) \subseteq A$ are the only nS-open sets in $\mathcal{W}$.
iv. If $U_R(X) \neq \mathcal{W}$, $L_R(X) = \phi \Longrightarrow \phi$ and those sets A for which $U_R(X) \subseteq A$ are the only nS-open sets in $\mathcal{W}$.
v. If $U_R(X) \neq L_R(X)$ where $U_R(X) \neq \mathcal{W}$ and $L_R(X) \neq \phi \Longrightarrow \phi$, $L_R(X), B_R(X)$, any set containing $U_R(X), L_R(X) \cup B \text{ and } B_R(X) \cup B$ where $B \subseteq [U_R(X)]^c$ are the only nS-open sets in $\mathcal{W}$.

Theorem 2.9. [4] Let $(W, \tau_R(X))$ be a NTS, then:

i. If $U_R(X) = \mathcal{W}$ and $L_R(X) = \phi \Longrightarrow nS_\beta O(\mathcal{W}, X) = \{\phi, \mathcal{W}\}$.
ii. If $U_R(X) = \mathcal{W}$ and $L_R(X) \neq \phi \Longrightarrow \tau_R(X) = nS_\beta O(\mathcal{W}, X)$.
iii. If $U_R(X) = L_R(X) = \{x\}, x \in \mathcal{W}, \Longrightarrow \phi$ and those sets A for which $U_R(X) \subseteq A$ are the only nS_β-open sets in $\mathcal{W}$.
iv. If $U_R(X) = L_R(X) \neq \mathcal{W}$ and $U_R(X)$ containing more than one element of $U \Longrightarrow \phi$ and those sets A for which $U_R(X) \subseteq A$ are the only nS_β-open sets in $\mathcal{W}$.
v. If $U_R(X) \neq \mathcal{W}$, $L_R(X) = \phi$ and $U_R(X)$ containing more than one element of $\mathcal{W} \Longrightarrow \phi$ and those sets A for which $U_R(X) \subseteq A$ are the only nS_β-open sets in $\mathcal{W}$.
vi. If $U_R(X) \neq L_R(X)$ where $U_R(X) \neq \mathcal{W}$ and $L_R(X) \neq \phi \Longrightarrow \phi$, $L_R(X), B_R(X)$, any set containing $U_R(X), L_R(X) \cup B \text{ and } B_R(X) \cup B$ where $B \subseteq [U_R(X)]^c$ are the only nS_β-open sets in $\mathcal{W}$.

3. Nano S_C-open sets

Definition 3.1. A subset $A \in nSO(\mathcal{W}, X)$ is said to be nano S_C-open (briefly nS_C-open) sets in NTS $\mathcal{W}$ if for each $x \in A$, there exist a nano closed set F such that $x \in F \subseteq A$. The family of all nano S_C-open subsets of a NTS $\mathcal{W}$ denoted by $nS_CO(\mathcal{W}, X)$.

Definition 3.2. The complement of nS_C-open sets in a NTS $(\mathcal{W}, \tau_R(X))$ is said to ne nS_C-closed sets. The family of all nS_C-open sets denoted by $nS_CC(\mathcal{W}, X)$.

Remark 3.3. Every nS_C-open set is nS-open set, but the converse may not be true in general, as it shown in the next example.

Example 3.4. Let $\mathcal{W}=\{a,b,c,d\}$ with $\mathcal{W}/R=\{\{a,b\},\{c\},\{d\}\}$ and $X=\{b,c\}$, then $\tau_R(X)=\{\phi,\mathcal{W},\{a,b,c\},\{c\},\{a,b\}\}$ and $[\tau_R(X)]^C=\{\phi,\mathcal{W},\{d\},\{a,b,d\},\{c,d\}\}$. The $nSO(\mathcal{W},X)=\{\phi,\mathcal{W},\{a,b,c\},\{c\},\{a,b\},\{c,d\},\{a,b,d\}\}$. Then $nS_CO(\mathcal{W},X)=\{\phi,\mathcal{W},\{c,d\},\{a,b,d\}\}$ and it is clear that the subset $\{c\}$ is nS-open but not nS_C-open set in $\mathcal{W}$.

Proposition 3.5. A subset A of a NTS $(\mathcal{W},\tau_R(X))$ is nS_C-open if and only if A is nS-open and the union of nano closed sets.

Proof. Obvious.

Remark 3.6.

i. Nano open sets and nS_C-open sets are independent. In above example, the subset $\{c,d\}$ is nS_C-open but not nano open in U, also, the subset $\{c\}$ is nano open but not nS_C-open set in U.
ii. $n\alpha$-open sets and nS_C-open sets are independent. In above example, $\{a,b,c\}$ is $n\alpha$-open set but not nS_C-open, also $\{c,d\}$ is nS_C-open but not $n\alpha$-open.
iii. nR-open sets and nS_C-open sets are independent. In above example, $\{a,b\}$ is nR-open set but not nS_C-open, also $\{c,d\}$ is nS_C-open but not nR-open.
iv. The intersection of tow nS_C-open sets may not be nS_C-open. In above example, $\{c,d\}$ and $\{a,b,d\}$ are nS_C-open but $\{c,d\}\cap\{a,b,d\}=\{d\}$ which is not nS_C-open in U. So that, the family of nS_C-open sets forma supra topology.

Proposition 3.7. Let $\{A_\alpha:\alpha\in\Delta\}$ be a collection of nS_C-open sets in a NTS $(\mathcal{W},\tau_R(X))$. Then $\bigcup\{A_\alpha:\alpha\in\Delta\}$ is nS_C-open.

Proof. Let A_α be nS_C-open set for each α, then A_α is nS-open and hence by Theorem 5, $\cup\{A_\alpha:\alpha\in\Delta\}$ is nS-open. Let $x\in\bigcup\{A_\alpha:\alpha\in\Delta\}$, there exist $\alpha\in\Delta$ such that $x\in A_\alpha$. Since A_α is nS-open for each α, there exists a nano closed set F such that $x\in F\subseteq A_\alpha\subseteq\bigcup\{A_\alpha:\alpha\in\Delta\}$, so $x\in F\subseteq\bigcup\{A_\alpha:\alpha\in\Delta\}$. Therefore, $\cup\{A_\alpha:\alpha\in\Delta\}$ is nS_C-open set.

In the following results, we study all form of nS_C-open sets in term of upper and lower approximations in NTS.

Theorem 3.8. Let $(\mathcal{W},\tau_R(X))$ be a NTS, then:

i. If $U_R(X)=\mathcal{W}$ and $L_R(X)=\phi$, then $nS_CO(\mathcal{W},X)=\{\phi,\mathcal{W}\}$.
ii. If $U_R(X)=\mathcal{W}$ and $L_R(X)\neq\phi$, then $\tau_R(X)=nS_CO(\mathcal{W},X)$.
iii. If $U_R(X)=L_R(X)\neq\mathcal{W}$, then $nS_CO(W,X)=\{\phi,\mathcal{W}\}$.
iv. If $U_R(X)\neq W$, $L_R(X)=\phi$, then $nS_CO(\mathcal{W},X)=\{\phi,\mathcal{W}\}$.
v. If $U_R(X)\neq L_R(X)$ where $U_R(X)\neq\mathcal{W}$ and $L_R(X)\neq\phi$, then $nS_CO(\mathcal{W},X)=\{\phi,\mathcal{W},[B_R(X)\cup B],[L_R(X)\cup B]\}$, where $B=[U_R(X)]^C$.

Proof.

i. Follows form that $\tau_R(X)=[\tau_R(X)]^C=nSO(X,\mathcal{W})=\{\phi,\mathcal{W}\}$.
ii. Suppose that $U_R(X)=\mathcal{W}$ and $L_R(X)\neq\phi$, then $\tau_R(X)=\{\phi,\mathcal{W},L_R(X),B_R(X)\}=[\tau_R(X)]^C$. Then by Theorem 9, $nSO(\mathcal{W},X)=\tau_R(X)$. Therefore, $\tau_R(X)=nS_CO(\mathcal{W},X)$.
iii. Let $A\in nSO(\mathcal{W},X)$. By Theorem 9, ϕ, $\mathcal{W}$ and any subset A for which containing $U_R(X)$ are the only nS-open sets in $\mathcal{W}$. If $A=\phi$ or $A=\mathcal{W}$, the result is clear. Let $\phi,\mathcal{W}\neq A\in nSO(U,X)$, then A containing $U_R(X)$, but since $[U_R(X)]^C$ is the only non-empty proper

nano closed set in $\mathcal{W}$ and $[U_R(X)]^C \nsubseteq A$ for every $x \in U_R(X) \supseteq A$, hence $nS_CO(\mathcal{W},X) = \{\phi,\mathcal{W}\}$.

iv. The proof is similar to part (iii).

v. Let $A \in nSO(\mathcal{W},X)$. By Theorem 9, $\phi, L_R(X), B_R(X), U_R(X)$, any set $G \subseteq \mathcal{W}$ for which $U_R(X) \subseteq G$, $B_R(X) \cup B$ and $L_R(X) \cup B$ are the only nS-open sets in $\mathcal{W}$ where $B \subseteq [U_R(X)]^C$. It is clear ϕ and $\mathcal{W}$ are nS_C-open sets in $\mathcal{W}$. If $A = L_R(X)$, then A is not nS_C-open set, since every non-empty proper nano closed set containing $[U_R(X)]^C$ and $[U_R(X)]^C \nsubseteq A$. If $A = B_R(X)$, then A is not nS_C-open set, since every non-empty proper nano closed set containing $[U_R(X)]^C$ and $[U_R(X)]^C \nsubseteq A$. If $A = U_R(X)$, then A is not nS_C-open set, since every non-empty proper nano closed set containing $[U_R(X)]^C$ and $[U_R(X)]^C \nsubseteq A$. If A containing $U_R(X)$, then A is not nS_C-open set, since every non-empty proper nano closed set containing $[U_R(X)]^C$ and $[U_R(X)]^C \nsubseteq A \supseteq U_R(X)$. If $A = L_R(X) \cup B$ where $B \subset [U_R(X)]^C$, then A is not nS_C-open set, since every non-empty proper nano closed set containing $[U_R(X)]^C$ and $[U_R(X)]^C \nsubseteq A$. If $A = B_R(X) \cup B$ where $B \subset [U_R(X)]^C$, similarly A is not nS_C-open set. If $A = L_R(X) \cup B$ where $B = [U_R(X)]^C$, then A is nS_C-open set, since every non-empty proper nano closed set containing $[U_R(X)]^C$ and $[U_R(X)]^C \subseteq A$ for every $x \in A$. If $A = B_R(X) \cup B$ where $B = [U_R(X)]^C$, similarly A is nS_C-open set. Therefore, $nS_CO(\mathcal{W},X) = \{\phi, \mathcal{W}, [B_R(X) \cup B], [L_R(X) \cup B]\}$, where $B = [U_R(X)]^C$.

Proposition 3.9. Let $(\mathcal{W}, \tau_R(X))$ be a NTS and K be any subset of U:

i. If K is $n\theta$-open set, then K is nS_C-open.
ii. If K is nS_C-open set, then K is nS_β-open.
iii. If K is nS_C-open set, then K is $n\beta$-open.
iv. If K is nS_C-open set, then K is $n\lambda$-open.
v. If K is nS_C-open set, then K is $n\delta\beta$-open.

Proof. Obvious.

4. Nano S_C-Operators

Definition 4.1. A subset N of a NTS $(\mathcal{W}, \tau_R(X))$ is said to be a nS_C-neighborhood of a subset A of W, if there exists a nS_C-open set G such that $A \subseteq G \subseteq N$, and denoted by nS_C-neighborhood.

Definition 4.2. A point $x \in \mathcal{W}$ is called a nS_C-interior point of $A \subseteq W$, if $\exists$ a nS_C-open set G containing x such that $x \in G \subseteq A$. The set of all nS_C-interior points of A is called nS_C-interior of A and denoted by $nS_Cint(A)$.

Theorem 4.3. Let A be any subset of a NTS $(\mathcal{W}, \tau_R(X))$. If a point $x \in nS_Cint(A)$, then $\exists$ a nano closed set F containing x such that $F \subseteq A$.

Proof. Suppose that $x \in nS_Cint\ A$, then $\exists$ a nS_C-open set G containing x such that $G \subseteq A$. Since $G \in nS_CO(\mathcal{W},X)$, then $\exists$ a nano closed set F containing x such that $F \subseteq G \subseteq A$. Hence $x \in F \subseteq A$.

Theorem 4.4. Let A be any subset of a NTS $(\mathcal{W}, \tau_R(X))$, then:

i. $nS_Cint(A) \subseteq A$.
ii. $nS_Cint(A) = \cup \{G : G\ is\ nS_C$open and $G \subseteq A\}$
iii. A is nS_Copen if and only if $A = nS_Cint(A)$.

iv. $nS_C int\left(nS_\beta int(A)\right) = nS_C int(A)$.
v. $nS_C int(\phi) = \phi$ and $nS_C int(\mathcal{W}) = \mathcal{W}$.

Proof.

i. Follows form definition.

ii. $x \in nS_C int(A)$, then $G \subseteq A$ for some nS_C-open set G such that $x \in G$. Therefore, $x \in \cup \{G: G\ is\ nS_C$-open and $x \in G \subseteq A\}$. If $x \in \cup \{G: G\ is\ nS_C$-open and $x \in G \subseteq A\}$, then $x \in G$ for some nS_C-open set $G \subseteq A$. Therefore, $x \in nS_C int(A)$.

iii. If A is nS_C-open and $x \in A$, then $x \in \cup \{G: G\ is\ nS_C$-open and $G \subseteq A\}$. That is, $x \in nS_C int(A)$, hence $A \subseteq nS_C int(A)$, but since $nS_C int(A) \subseteq$ A. Therefore, $nS_C int(A) = A$. Conversely, if $nS_C int(A) =$ A, then A is nS_C-open in U since $nS_C int(A)$ is nS_C-open.

iv. Follows from part (iii).

v. Since ϕ and $\mathcal{W}$ are nS_C-open set, then by part (iii), $nS_C int(\phi) = \phi$ and $nS_C int(U) = \mathcal{W}$.

Theorem 4.5. Let A and B be any two subset of a $NTS\ \left(\mathcal{W}, \tau_R(X)\right)$, then:

i. If $A \subseteq B$, then $nS_C int(A) \subseteq nS_C int(B)$.
ii. $nS_C int(A) \cup nS_C int(B) \subseteq nS_C int(A \cup B)$.
iii. $nS_C int(A \cap B) \subseteq nS_C int(A) \cap nS_C int(B)$.

Proof.

i. If $A \subseteq B$ and $x \in nS_C int(A)$, then $G \subseteq A$ for some nS_C-open set G containing x. Hence $G \subseteq A$ and $x \in nS_C int(B)$. Therefore, $nS_C int(A) \subseteq nS_C int(B)$.

ii. Since, $A \subseteq A \cup B$, by (i), $nS_C int(A) \subseteq nS_C int(A \cup B)$. Again since $B \subseteq A \cup B$, $nS_C int(B) \subseteq nS_C int(A \cup B)$. Therefore, $nS_C int(A) \cup nS_C int(B) \subseteq nS_C int(A \cup B)$.

iii. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by part (i), $nS_C int(A \cap B) \subseteq nS_C int(A) \cap nS_C int(B)$.

The inclusion of parts (ii and iii) of above theorem cannot be replaced by equality in general, as it shown in the following example.

Example 4.6. Let $\mathcal{W} = \{a, b, c, d\}$ with $\mathcal{W}/R = \{\{a\}, \{b, c\}, \{d\}\}$ and $X = \{a, b\}$, then $\tau_R(X) = \{\phi, \mathcal{W}, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $nS_C O(\mathcal{W}, X) = \{\phi, \mathcal{W}, \{a, d\}, \{b, c, d\}\}$. For part (ii), take $A = \{b, d\}$ and $B = \{c, d\}$, then $nS_C int(A) \cup nS_C int(B) = \phi \cup \phi = \phi$ but $nS_C int\ (A \cup B) = \{b, c, d\}$. Therefore, $nS_C int(A) \cup nS_C int(B) \neq nS_C int(A \cup B)$. For part (iii), take $A = \{a, d\}$ and $B = \{b, c, d\}$, then $nS_C int(A \cap B) = nS_C int(\{d\}) = \phi$, but $nS_C int(A) \cap nS_C int(B) = \{a, d\} \cap \{b, c, d\} = \{d\}$. Therefore, $nS_C int(A \cap B) \neq nS_C int(A) \cap nS_C int(B)$.

Definition 4.7. A point $x \in \mathcal{W}$ of a $NTS\ \left(\mathcal{W}, \tau_R(X)\right)$ is said to be nS_C-cluster point of a subset A of U, if $A \cap G \neq \phi$ for every nS_C-open set G containing x.

Definition 4.8. The set of all nS_C-cluster points of a subset A of $\mathcal{W}$ is said to be nS_C-closure of A and denoted by $nS_C cl(A)$. Equivalently, The $nS_C cl(A)$ is the intersection of all nS_C-closed sets containing A.

Theorem 4.9. Let A be any subset of a $NTS\ (\mathcal{W}, \tau_R(X))$. A point $x \in nS_C cl(A)$ if and only if $A \cap H \neq \phi$ for every nS_C-open set H containing x.

Proof. Obvious.

Corollary 4.10. For any subset A of a $NTS\ (\mathcal{W}, \tau_R(X))$, the following statements are true.

i. $nS_C cl\ (\mathcal{W} - A) = \mathcal{W} - nS_C int(A)$.
ii. $nS_C int\ (\mathcal{W} - A) = \mathcal{W} - nS_C cl(A)$.

Proof.

i. Let $x \in\ nS_C cl\ (\mathcal{W} - A)$, then $G \cap (\mathcal{W} - A) \neq \phi$ for any nS_C-open set G containing x. Therefore, $G \nsubseteq A$ where G is nS_C-open set containing x. That is, $x \notin nS_C int(A)$. Therefore, $x \in \mathcal{W} - nS_C int(A)$. Thus, $nS_C cl(\mathcal{W} - A) \subseteq \mathcal{W} - nS_C int(A)$. Conversely, if $x \in \mathcal{W} - nS_C int(A)$, then $x \notin nS_C int(A)$, and this mean $G \nsubseteq A$ for every nS_C-open set G containing x. Therefore, $G \cap (\mathcal{W} - A) \neq \phi$ and so $x \in nS_C cl(\mathcal{W} - A)$. Hence, $\mathcal{W} - nS_C int(A) \subseteq nS_C cl(\mathcal{W} - A)$. Hence, $nS_C cl(\mathcal{W} - A) = \mathcal{W} - nS_C int(A)$.
ii. The proof is similar to part (i).

Theorem 4.11. For any subset A and B of a $NTS\ (\mathcal{W}, \tau_R(X))$, the following statements are true:

i. If $A \subseteq B$, then $nS_C cl(A) \subseteq nS_C cl(B)$.
ii. $nS_C cl(A) \cup nS_C cl(B) \subseteq nS_C cl\ (A \cup B)$.
iii. $nS_C cl(A \cap B) \subseteq nS_C cl(A) \cap nS_C cl(B)$.

Proof.

i. If $A \subseteq B$ and $x \in nS_C cl(A)$, then $G \cap A \neq \phi$ for every nS_C-open set G containing x.Since $G \cap A \subseteq G \cap B$, $G \cap B \neq \phi$ whenever G is nS_C-open set containing x. Therefore, $x \in nS_C cl(B)$. Hence, $nS_C cl(A) \subseteq nS_C cl(B)$.
ii. Since $A, B \subseteq A \cup B$, then by part (i), we get the result.
iii. Since $A \cap B \subseteq A, B$, then by part (i), we get the result.

The inclusion in (ii and iii) of above theorem cannot be replaced by quality in general, as it shown in the following two examples.

Example 4.12. Let $\mathcal{W} = \{a, b, c, d\}$ with $\mathcal{W}/R = \{\{a\}, \{b, c\}, \{d\}\}$ and $X = \{a, b\}$, then $\tau_R(\ X) = \{\phi, \mathcal{W}, \{a\}, \{b, c\}, \{a, b, c\}\}$, $nS_C O(\mathcal{W}, X) = \{\phi, \mathcal{W}, \{a, d\}, \{b, c, d\}\}$ and $nS_C C(\mathcal{W}, X) = \{\phi, \mathcal{W}, \{b, c\}, \{a\}\}$. For part (ii), take $F = \{a, d\}$ and $E = \{a, b\}$, then $nS_C cl(F \cap E) = nS_C cl(\{a\}) = \{a\}$, but $nS_C cl(F) \cap nS_C cl(E) = \mathcal{W}$. Therefore, $nS_C cl(F \cap E) \neq nS_C cl(F) \cap nS_C cl(E)$. For part (iii), take $F = \{b, c\}$ and $E = \{a\}$, then $nS_C cl(\{b, c\} \cup \{a\}) = \mathcal{W}$, but $nS_C cl(\{b, c\}) \cup nS_C cl(\{a\}) = \{a, b, c\}$. Therefore, $nS_C cl(F) \cup nS_C cl(E)\ \neq nS_C cl\ (F \cup E)$.

Theorem 4.13. Let A be any subset of a $NTS\ (\mathcal{W}, \tau_R(X))$, then the following statements are true:

i. $nS_C cl(\phi) = \phi$ and $nS_C cl(\mathcal{W}) = \mathcal{W}$.
ii. $A \subseteq nS_C cl(A)$.
iii. $A \in nS_C C(W, X)$ if and only if $A = nS_C cl(A)$.

iv. $nS_C cl\left(nS_\beta cl(A)\right) = nS_C cl(A)$

Proof.

i. Follows form the fact that ϕ and $\mathcal{W}$ are nS_β-closed set.

ii. By definition of nS_C-closure, $A \subseteq nS_C cl(A)$.

iii. Let A is nS_C-closed set, then A is smallest nS_C-closed set containing itself and hence $nS_C cl(A) = A$. Conversely, if $nS_C cl(A) = A$, then A is the smallest nS_C-closed set containing itself and hence A is nS_C-closed set in W.

iv. Since $nS_C cl(A)$ is nS_C-closed set, then the proof follows from part (iii).

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