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Improved Runge-Kutta Method for Oscillatory Problem Solution Using Trigonometric Fitting Approach

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Abstract

This paper provides a four-stage Trigonometrically Fitted Improved Runge-Kutta (TFIRK4) method of four orders to solve oscillatory problems, which contains an oscillatory character in the solutions. Compared to the traditional Runge-Kutta method, the Improved Runge-Kutta (IRK) method is a natural two-step method requiring fewer steps. The suggested method extends the fourth-order Improved Runge-Kutta (IRK4) method with trigonometric calculations. This approach is intended to integrate problems with particular initial value problems (IVPs) using the set functions $e^{\omega x}$ and $e^{-\omega x}$ for trigonometrically fitted. To improve the method's accuracy, the problem primary frequency $\omega \in R$ is used. The novel method is more accurate than the conventional Runge-Kutta method and IRK4. Several test problems for the system of first-order ordinary differential equations carry out numerically to demonstrate the effectiveness of this approach. The computational studies show that the TFIRK4 approach is more efficient than the existing Runge-Kutta methods.

Keywords: Improved Runge-Kutta Method, Trigonometrically-Fitted, Initial Value Problem, Oscillating Solution.

1. Introduction

This paper focuses on solving the system of the first-order ordinary differential equations (ODEs) of the form:

$$u'(t) = f(t,u),\ u(t_0) = u_0. \qquad (1)$$

These issues are frequently seen in various applied sciences, including quantum mechanics, electronics, chemical physics, and astronomy (see [1] and [2]). Traditionally, Runge-Kutta (RK) methods or two-step methods are used to solve equation (1) [3]. In 2013, [4] developed the Improved Runge-Kutta technique for solving first ODEs using new terms k_{-i} taking the values of $k_i, i \geq 2$ from earlier stages.. Another choice for coming up with approaches to solve the oscillatory ODEs is the trigonometrically-fitted approach. These techniques are an improved version of any earlier techniques. In 2019, [5] developed trigonometrically-fitted sixth-order two-derivative Runge-Kutta method for solving oscillatory problems. In 2021, [6] derived the fourth and fifth-order modified Runge-Kutta method to resolve oscillatory problems using phase-Lag properties. Recently, [7] derived the trigonometrically-fitted third-order Improved Runge-Kutta method for solving oscillatory problems.

This paper aims to develop the fourth-order IRK method called Trigonometrically Fitted Improved Runge-Kutta (TFIRK4) to solve a system of first-order ODEs with oscillatory solutions. Numerical experiments demonstrate the accuracy of the newly proposed method over other methods. A proposed TFIRK technique derivation is presented in Section 2. A numerical test and comparison with different approaches are discussed in Section 3 to show the efficiency of the TFIRK4 method. Section 4 contains the discussion and conclusion of this paper.

2. The Derivation of TFIRK4 Method

The Improved Runge-Kutta (IRK) method for solving equation (1) has the following form: [4]

$$u'_{n+1} = u_n + h(b_1 k_1 - b_{-1} k_{-1} + \sum_{i=2}^{s} b_i (k_i - k_{-i}), \tag{2}$$

$$k_1 = f(t_n, u_n), \tag{3}$$

$$k_{-1} = f(t_{n-1}, u_{n-1}), \tag{4}$$

$$k_i = f(t_n + c_i h, u_n + h \sum_{j=1}^{i-1} a_{ij} k_j), \tag{5}$$

$$k_{-i} = f(t_{n-1} + c_i h, u_{n-1} + h \sum_{j=1}^{i-1} a_{ij} k_{-j}). \tag{6}$$

Where c_i, b_i, b_{-1}, and a_{ij} are real numbers and $i, j = 1,2, \dots, s$. IRK method (2)-(6) can be expressed using the following Butcher Tableau (see **Table 1**):

Table 1. $s -$ Stages IRK method

0					
c_2	a_{21}				
c_3	a_{31}	a_{32}			
.	.	.	.		
.	.	.	.		
.	.	.	.		
c_s	a_{s1}	a_{s2}	...	a_{ss-1}	
b_{-1}	b_1	b_2	...	b_{s-1}	b_s

Following are the four stages of the IRK4 method in its general form:

$$u'_{n+1} = u_n + h(b_1 k_1 - b_{-1}k_{-1} + b_2(k_2 - k_{-2}) + b_3(k_3 - k_{-3}) + b_4(k_4 - k_{-4}), \quad (7)$$

$$k_1 = f(t_n, u_n), \quad (8)$$

$$k_{-1} = f(t_{n-1}, u_{n-1}), \quad (9)$$

$$k_2 = f(t_n + c_2 h, u_n + h\, a_{21} k_1), \quad (10)$$

$$k_{-2} = f(t_{n-1} + c_2 h, u_{n-1} + h\, a_{21} k_{-1}), \quad (11)$$

$$k_3 = f(t_n + c_3 h, u_n + h\,(a_{31} k_1 + a_{32} k_2)), \quad (12)$$

$$k_{-3} = f(t_{n-1} + c_3 h, u_{n-1} + h\,(a_{31} k_{-1} + a_{32} k_{-2})), \quad (13)$$

$$k_4 = f(t_n + c_4 h, u_n + h\,(a_{41} k_1 + a_{42} k_2 + a_{43} k_3)), \quad (14)$$

$$k_{-4} = f(t_n + c_4 h, u_{n-1} + h\,(a_{41} k_{-1} + a_{42} k_{-2} + a_{43} k_{-3})). \quad (15)$$

The coefficients of the IRK4 method in [4] are offered in **Table 2**:

Table 2. IRK4 method

0				
$\frac{1}{5}$	$\frac{1}{5}$			
$\frac{3}{5}$	0	$\frac{3}{5}$		
$\frac{4}{5}$	$\frac{2}{15}$	$\frac{4}{25}$	$\frac{38}{75}$	
$\frac{19}{288}$	$\frac{307}{288}$	$-\frac{25}{144}$	$\frac{25}{144}$	$\frac{125}{288}$

Applying the exponential function into the IRK4 method (7)-(15), the trigonometrically fitting approach is implemented by allow:

$$u_{\mathrm{n}} = u(t_{\mathrm{n}}) = \mathrm{e}^{\mathrm{i}\omega t_{\mathrm{n}}}, \quad (16)$$

$$u_{\mathrm{n+1}} = u(t_{\mathrm{n+1}}) = \mathrm{e}^{\mathrm{i}\omega(t_{\mathrm{n}}+h)}, \quad (17)$$

$$u_{\mathrm{n-1}} = u(t_{\mathrm{n-1}}) = \mathrm{e}^{\mathrm{i}\omega(t_{\mathrm{n}}-h)}. \quad (18)$$

Using Euler formula $e^{iv} = \cos(v) + isin(v)$, and substituting the equations (16)-(18) into equation IRK4 method (7)-(15), we get:

$$\begin{aligned} e^{iv} = \cos(v) + isin(v) = 1 - vb_{-1}\sin(v) - v^2 b_2\, a_{21} - vb_2 \sin(v) - v^2 b_3\, a_{31} \\ +v^2\, b_2 a_{21} \cos(v) - v^2 b_3\, a_{32} - vb_3 \sin(v) + v^2\, b_3 a_{31} \cos(v) - v^2 b_4\, a_{41} \\ +v^2\, b_3 a_{32} \cos(v) + v^3\, b_3 a_{32} a_{21} \sin(v) - v^2\, b_4 a_{42} - v^2\, b_4 a_{43} - vb_4 \sin(v) \\ +v^4\, b_4 a_{43} a_{32} a_{21} + v^2\, b_4 a_{41} \cos(v) + v^2\, b_4 a_{42} \cos(v) + v^3\, b_4 a_{42} a_{21} \sin(v) \end{aligned}$$

$$+v^2\, b_4 a_{43} \cos(v) + v^3\, b_4 a_{43} a_{31} \sin(v) + v^3\, b_4 a_{43} a_{32} \sin(v)$$

$$-v^3\, b_4 a_{43} a_{32}\, a_{21} \cos(v) + ivb_1 - ivb_{-1} \cos(v) + ivb_2 - ivb_2 \cos(v)$$

$$+ivb_3 - iv^2 b_2 a_{21} \sin(v) - iv^3\, b_3 a_{32} a_{21} - ivb_3 \cos(v) - iv^2 b_3 a_{31} \sin(v)$$

$$+ivb_4 + iv^3 b_3 a_{32} a_{21} \cos(v) - iv^3 b_4 a_{42} a_{21} - iv^3 b_4 a_{43} a_{31} - iv^3 b_4 a_{43} a_{32}$$

$$-ivb_4 \cos(v) - iv^2 b_4 a_{41} \sin(v) - iv^2 b_4 a_{42} \sin(v) - iv^2 b_4 a_{43} \sin(v)$$

$$+iv^3 b_4 a_{42} a_{21} \cos(v) + iv^3 b_4 a_{43} a_{31} \cos(v) + iv^3 b_4 a_{43} a_{32} \cos(v)$$

$$+iv^4 b_4 a_{43} a_{32} a_{21} \sin(v) - iv^2 b_3 a_{32} \sin(v). \qquad (19)$$

Where $v = \omega h$. We obtain the trigonometrically fitting order conditions by equating the real and imaginary parts:

$$\cos(v) = 1 - vb_{-1} \sin(v) - v^2 b_2\, a_{21} - vb_2 \sin(v) - v^2 b_3\, a_{31} + v^2\, b_2 a_{21} \cos(v)$$

$$-v^2 b_3\, a_{32} - vb_3 \sin(v) + v^2\, b_3 a_{31} \cos(v) - v^2 b_4\, a_{41} + v^2\, b_3 a_{32} \cos(v)$$

$$+v^3\, b_3 a_{32} a_{21} \sin(v) - v^2\, b_4 a_{42} - v^2\, b_4 a_{43} - vb_4 \sin(v) + v^4\, b_4 a_{43} a_{32} a_{21}$$

$$+v^2\, b_4 a_{41} \cos(v) + v^2\, b_4 a_{42} \cos(v) + v^3\, b_4 a_{42} a_{21} \sin(v) + v^2\, b_4 a_{43} \cos(v)$$

$$+v^3\, b_4 a_{43} a_{31} \sin(v) + v^3\, b_4 a_{43} a_{32} \sin(v) - v^3\, b_4 a_{43} a_{32}\, a_{21} \cos(v), \qquad (20)$$

$$\sin(v) = vb_1 - vb_{-1} \cos(v) + vb_2 - vb_2 \cos(v) - vb_3 \cos(v) - v^2 b_3 a_{31} \sin(v)$$

$$+vb_3 - v^2 b_2 a_{21} \sin(v) - v^3\, b_3 a_{32} a_{21} - v^3 b_4 a_{43} a_{31} - v^3 b_4 a_{43} a_{32}$$

$$+vb_4 + v^3 b_3 a_{32} a_{21} \cos(v) - v^3 b_4 a_{42} a_{21} - v^2 b_4 a_{42} \sin(v) - v^2 b_4 a_{43} \sin(v)$$

$$-vb_4 \cos(v) - v^2 b_4 a_{41} \sin(v) + v^3 b_4 a_{43} a_{31} \cos(v) + v^3 b_4 a_{43} a_{32} \cos(v)$$

$$+v^3 b_4 a_{42} a_{21} \cos(v) + v^4 b_4 a_{43} a_{32} a_{21} \sin(v) - v^2 b_3 a_{32} \sin(v). \qquad (21)$$

Solving equations (20) and (21) using the coefficients of the IRK4 method in Table2 for unknowns parameters a_{31} and a_{41} we obtain the following solution:

$$a_{31} = \frac{-72}{95\, v^3(\sin^2(v) + \cos^2(v) - 2\cos(v) + 1)} \left(-12\cos(v)\sin(v) + 12\sin(v)\right.$$

$$-3\, v \sin^2(v) + v^3 \sin^2(v) + 12\, v \cos(v) - 3\, v \cos^2(v) - 2\, v^3 \cos(v)$$

$$\left. -9v + v^3 \cos^2(v) + v^3\right), \qquad (22)$$

$$a_{41} = \frac{2}{35625\, v^3(\sin^2(v)+\cos^2(v)-2\cos(v)+1)} \left(-7560\, v - 9325\, v^3\right.$$

$$-17280 v \cos(v) + 24840\, v \cos^2(v) + 18650\, v^3 \cos(v)$$

$$+1083\, v^5 \cos^2(v) + 64800 \sin(v) - 57240\, v \sin^2(v) + 1083\, v^5$$

$$-64800 \cos(v)\sin(v) - 9325\, v^3 \sin^2(v) - 2166\, v^5 \cos(v)$$

$$\left. -9325\, v^3 \cos^2(v) + 41040 v^2 \sin(v) + 1083\, v^5 \sin^2(v)\right). \qquad (23)$$

It can be noticed that for $v \to 0$ using Maple package, the series expansions are given in following form:

$$a_{31} = -\frac{18}{475}v^2 + \frac{3}{3325}v^4 - \frac{1}{79800}v^6 + \frac{1}{8778000}v^8 - \frac{1}{1369368000}v^{10} + \frac{1}{28756728000}v^{12} - \frac{1}{78218300160000}v^{14} + \cdots, \quad (24)$$

$$a_{41} = \frac{2}{15} - \frac{276}{11875}v^2 + \frac{689}{249375}v^4 - \frac{77}{1425000}v^6 + \frac{1069}{1975050000}v^8 - \frac{61849}{10783773000000}v^{10} - \frac{1}{172540368000}v^{12} - \frac{77963}{87995587680000000}v^{14} + \cdots, \quad (25)$$

which lead to the new TFIRK4 method. As $v \to 0$, the newly obtained parameters a_{31} and a_{41} turn into the parameters of the original method.

3. Numerical Results

To evaluate the efficacy of the TFIRK4 method proposed, we apply them to oscillatory test problems and their numerical results are compared with the existing effective methods. The following numerical methods are applied in the comparison:

- Step size.
- **TFIRK4:** the Trigonometrically Fitted four-stage fourth-order IRK method presented here.
- **IRK4:** fourth-order IRK method derive in [4].
- **TFRK4:** Trigonometrically-Fitted RK method developed in [8].
- **RK4Z:** fifth-order RK method given in [1].
- **MRK4:** modified RK method of order four proposed in [9].
- **Max Error:** $\max(|u(x_n) - u_n|)$ This is the maximum between absolute errors of the exact solutions and the computed solutions.

Problem 1: [10] Inhomogeneous problem:

$$u_1'(t) = u_2(t), \quad u_1(0) = 1,$$
$$u_2'(t) = -100\, u_1(t) + 99\sin(t), \quad u_2(0) = 11.$$

Exact solution is:

$$u_1(t) = \cos(10t) + \sin(10t) + \sin(t),$$
$$u_2(t) = -10\sin(10t) + 10\cos(10t) + \cos(t).$$

Problem 2: [7] Inhomogeneous problem:

$$u_1'(t) = u_2(t), \quad u_1(0) = 1,$$
$$u_2'(t) = -u_1(t) + t, \quad u_2(0) = -2.$$

Exact solution is:

$$u_1(t) = \sin(t) + \cos(t) + t,$$
$$u_2(t) = \cos(t) - \sin(t) + 1$$

Problem 3: [11] Duffing problem:

$$u_1'(t) = u_2(t), \quad u_1(0) = 0.200426728067,$$
$$u_3'(t) = -u_1(t) - \left(u_1(t)\right)^3 + 0.002\cos(1.01\, t), \quad u_2(0) = 0,$$

Exact solution is:

$$u_1(t) = 0.200179477536\cos(1.01\, t) + 2.46946143 \times 10^{-4}\cos(3.03\, t)$$

$$+3.04014 \times 10^{-7} \cos(5.05\, t) + 3.74\, 10^{-10} \cos(7.07\, t),$$

$$u_2(t) = -0.2021812723 \sin(1.01\, t) - 7.482468133 \times 10^{-4} \sin(3.03\, t)$$

$$-1.53527070 \times 10^{-6} \sin(5.05\, t) - 2.64418 \times 10^{-9} \sin(7.07\, t).$$

Problem 4: [8]

$$u_1'(t) = u_3(t), \;\; u_1(0) = 1,$$

$$u_3'(t) = \frac{-u_1(t)}{\left(\sqrt{(u_1(t))^2 + (u_2(t))^2}\right)^3} \; u_3(0) = 0,$$

$$u_2'(t) = u_4(t), \;\; u_2(0) = 0,$$

$$u_4'(t) = \frac{-u_2(t)}{\left(\sqrt{(u_1(t))^2 + (u_2(t))^2}\right)^3} \; u_4(0) = 1.$$

Exact solution is:

$$u_1(t) = \cos(t),$$

$$u_1(t) = \sin(t),$$

$$u_3(t) = -\sin(t),$$

$$u_4(t) = \cos(t),$$

Problem 5: [11]

$$u_1'(t) = u_3(t), \;\; u_1(0) = 1,$$

$$u_3'(t) = -u_1(t) + 0.001 \cos(t), \; u_3(0) = 0,$$

$$u_2'(t) = u_4(t), \;\; u_2(0) = 0,$$

$$u_4'(t) = -u_2(t) + 0.001 \sin(t), \;\; u_4(0) = 0.9995.$$

Exact solution is:

$$u_1(t) = \cos(t) + 0.0005\, t \, \sin(t),$$

$$u_2(t) = -0.9995 \, \sin(t) + 0.0005\, t \, \cos(t),$$

$$u_3(t) = \sin(t) - 0.0005\, t \, \cos(t),$$

$$u_4(t) = 0.9995 \cos(t) + 0.0005\, t \, \sin(t).$$

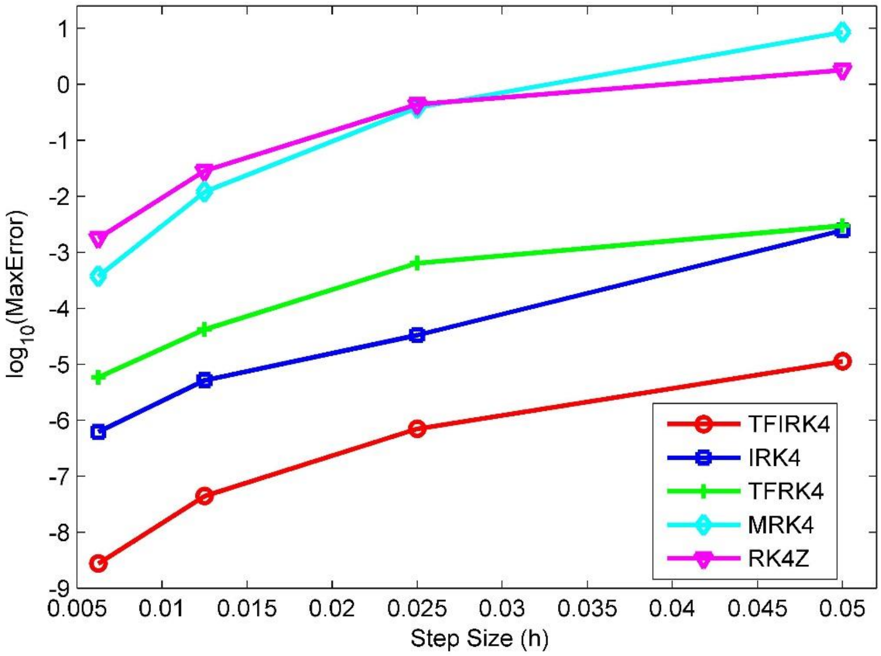

Figure 1. The curves comparisons for Problem 1 with step size $h = \frac{1}{2^r}, r = 0,1,2,4.$

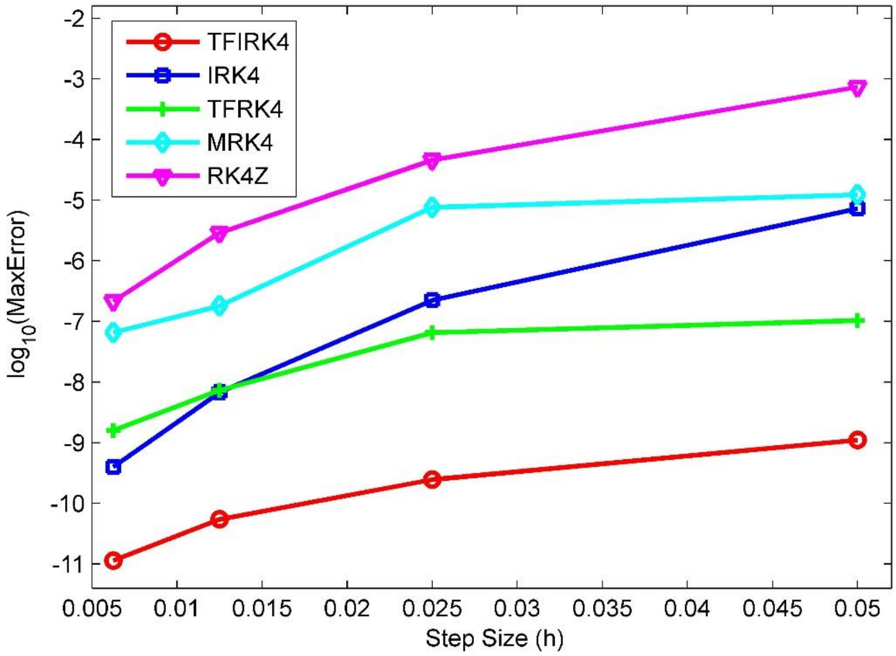

Figure 2. The curves comparisons for Problem 2 with step size $h = \frac{1}{2^r}, r = 0,1,2,4.$

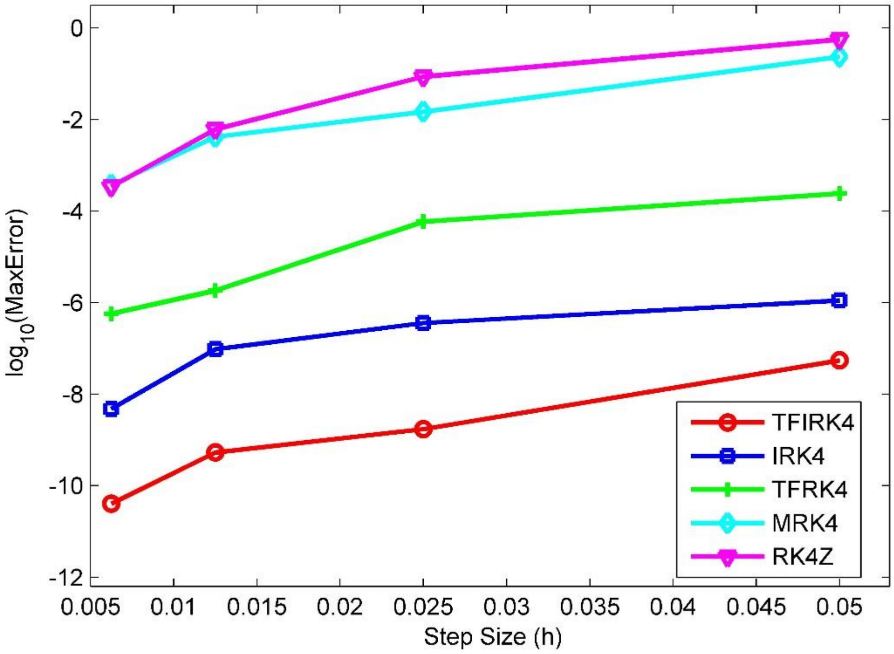

Figure 3. The curves comparisons for Problem 3 with step size $h = \frac{1}{2^r}, r = 0,1,2,4.$

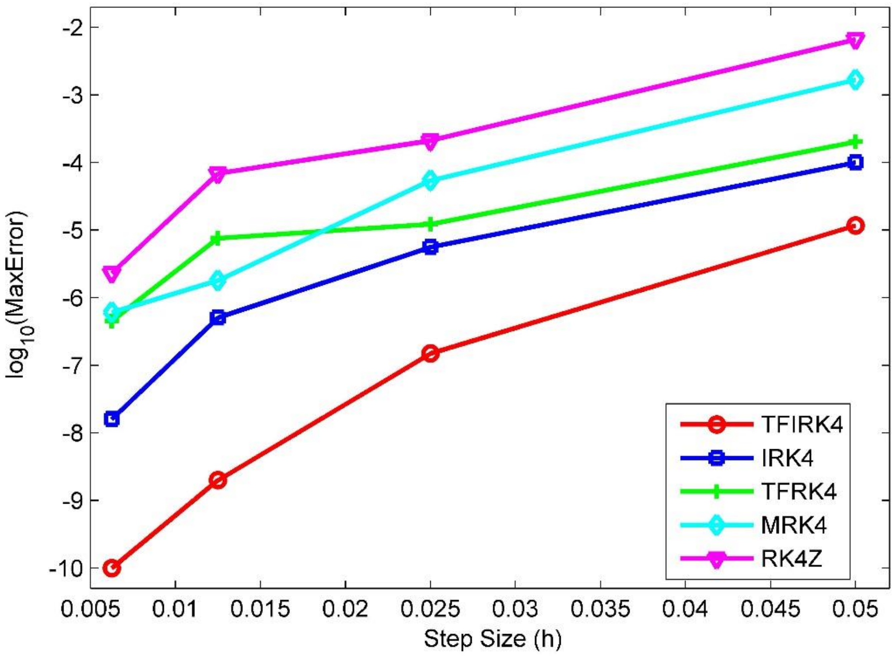

Figure 4. The curves comparisons for Problem 4 with step size $h = \frac{1}{2^r}, r = 0,1,2,4.$

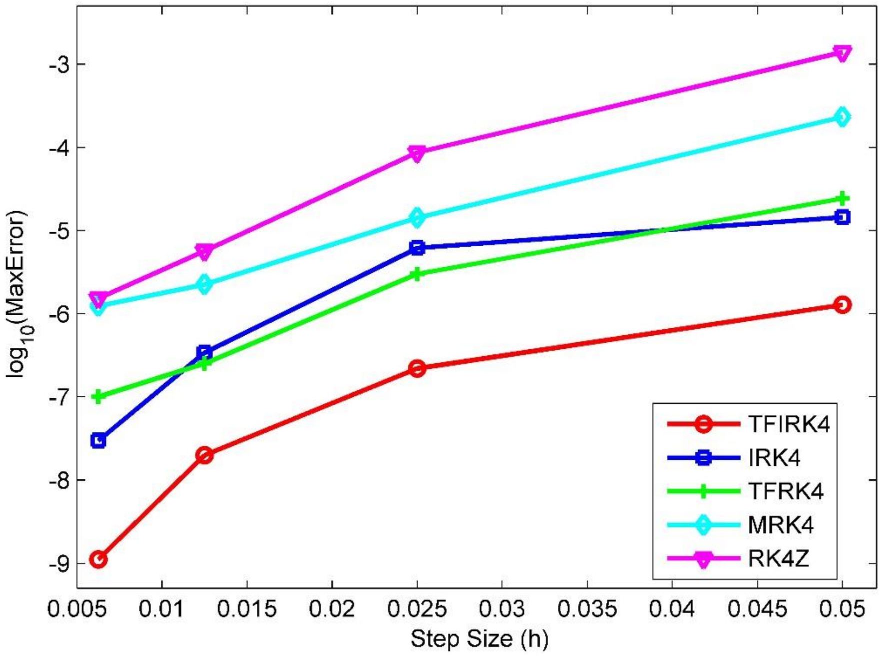

Figure 5. The curves comparisons for Problem 5 with step size $h = \frac{1}{2^r}, r = 0,1,2,4.$

To assess the method's accuracy, we utilize the absolute error criterion. The step size is $h = \frac{1}{2^r}, r = 0,1,2,4.$ and integration interval is [0, 1000] for all problems. The accuracy of the new TFIRK4 approach is depicted in **Figures 1–5** in terms of the greatest global absolute error versus the step sizes required by each method. Compared to other RK methods of the same order, the TFIRK4 approaches, as shown in **Figures 1–5**, have the smallest maximum global error per step. The TFIRK4 produces results that are more accurate than those of other research in the literature, as seen in **Figures 1–5**.

4. Conclusions

We derived the conditions of the Trigonometrically-Fitted IRK approach to solving oscillatory problems in this paper. As a result, we developed the TFIRK4 method, a four-stage, fourth-order IRK method that is trigonometrically fitted. The Figures show how the step size was used to calculate the common logarithm of the most significant global error during integration and computing cost. The numerical results made it clear that the TFIRK4 approach using a trigonometrically fitted strategy, had less global error than the existing methods used to solve oscillatory problems.

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